10. Bounded Linear Functionals in L^2

In the following, $(\Omega, \mathcal{F}, \mu)$ is a measure space.

Definition 78 *We call* **subsequence** *of a sequence $(x_n)_{n\geq 1}$, any sequence of the form $(x_{\phi(n)})_{n\geq 1}$ where $\phi : \mathbf{N}^* \to \mathbf{N}^*$ is a strictly increasing map.*

Exercise 1. Let (E, d) be a metric space, with metric topology $\mathcal{T}$. Let $(x_n)_{n\geq 1}$ be a sequence in E. For all $n \geq 1$, let F_n be the closure of the set $\{x_k : k \geq n\}$.

1. Show that for all $x \in E$, $x_n \xrightarrow{\mathcal{T}} x$ is equivalent to:

$$\forall \epsilon > 0 \ , \ \exists n_0 \geq 1 \ , \ n \geq n_0 \ \Rightarrow \ d(x_n, x) \leq \epsilon$$

2. Show that $(F_n)_{n\geq 1}$ is a decreasing sequence of closed sets in E.
3. Show that if $F_n \downarrow \emptyset$, then $(F_n^c)_{n\geq 1}$ is an open covering of E.
4. Show that if $(E, \mathcal{T})$ is compact then $\cap_{n=1}^{+\infty} F_n \neq \emptyset$.
5. Show that if $(E, \mathcal{T})$ is compact, there exists $x \in E$ such that for all $n \geq 1$ and $\epsilon > 0$, we have $B(x, \epsilon) \cap \{x_k \ , \ k \geq n\} \neq \emptyset$.
6. By induction, construct a subsequence $(x_{n_p})_{p\geq 1}$ of $(x_n)_{n\geq 1}$ such that $x_{n_p} \in B(x, 1/p)$ for all $p \geq 1$.
7. Conclude that if $(E, \mathcal{T})$ is compact, any sequence $(x_n)_{n\geq 1}$ in E has a convergent subsequence.

Exercise 2. Let (E, d) be a metric space, with metric topology $\mathcal{T}$. We assume that any sequence $(x_n)_{n\geq 1}$ in E has a convergent subsequence. Let $(V_i)_{i\in I}$ be an open covering of E. For $x \in E$, let:

$$r(x) \stackrel{\triangle}{=} \sup\{r > 0 \ : \ B(x, r) \subseteq V_i \ , \ \text{for some } i \in I\}$$

1. Show that $\forall x \in E$, $\exists i \in I$, $\exists r > 0$, such that $B(x, r) \subseteq V_i$.
2. Show that $\forall x \in E$, $r(x) > 0$.

Exercise 3. Further to ex. (2), suppose $\inf_{x\in E} r(x) = 0$.

1. Show that for all $n \geq 1$, there is $x_n \in E$ such that $r(x_n) < 1/n$.
2. Extract a subsequence $(x_{n_k})_{k\geq 1}$ of $(x_n)_{n\geq 1}$ converging to some $x^* \in E$. Let $r^* > 0$ and $i \in I$ be such that $B(x^*, r^*) \subseteq V_i$. Show that we can find some $k_0 \geq 1$, such that $d(x^*, x_{n_{k_0}}) < r^*/2$ and $r(x_{n_{k_0}}) \leq r^*/4$.
3. Show that $d(x^*, x_{n_{k_0}}) < r^*/2$ implies that $B(x_{n_{k_0}}, r^*/2) \subseteq V_i$. Show that this contradicts $r(x_{n_{k_0}}) \leq r^*/4$, and conclude that $\inf_{x\in E} r(x) > 0$.

EXERCISE 4. Further to ex. (3), Let r_0 with $0 < r_0 < \inf_{x\in E} r(x)$. Suppose that E cannot be covered by a finite number of open balls with radius r_0.

1. Show the existence of a sequence $(x_n)_{n\geq 1}$ in E, such that for all $n \geq 1$, $x_{n+1} \notin B(x_1, r_0) \cup \ldots \cup B(x_n, r_0)$.
2. Show that for all $n > m$ we have $d(x_n, x_m) \geq r_0$.
3. Show that $(x_n)_{n\geq 1}$ cannot have a convergent subsequence.
4. Conclude that there exists a finite subset $\{x_1, \ldots, x_n\}$ of E such that $E = B(x_1, r_0) \cup \ldots \cup B(x_n, r_0)$.
5. Show that for all $x \in E$, we have $B(x, r_0) \subseteq V_i$ for some $i \in I$.
6. Conclude that $(E, \mathcal{T})$ is compact.
7. Prove the following:

Theorem 47 *A metrizable topological space $(E, \mathcal{T})$ is compact, if and only if for every sequence $(x_n)_{n\geq 1}$ in E, there exists a subsequence $(x_{n_k})_{k\geq 1}$ of $(x_n)_{n\geq 1}$ and some $x \in E$, such that $x_{n_k} \stackrel{\mathcal{T}}{\rightarrow} x$.*

EXERCISE 5. Let $a, b \in \mathbf{R}$, $a < b$ and $(x_n)_{n\geq 1}$ be a sequence in $]a, b[$.

1. Show that $(x_n)_{n\geq 1}$ has a convergent subsequence.
2. Can we conclude that $]a, b[$ is a compact subset of $\mathbf{R}$?

EXERCISE 6. Let $E = [-M, M] \times \ldots \times [-M, M] \subseteq \mathbf{R}^n$, where $n \geq 1$ and $M \in \mathbf{R}^+$. Let $\mathcal{T}_{\mathbf{R}^n}$ be the usual product topology on $\mathbf{R}^n$, and $\mathcal{T}_E = (\mathcal{T}_{\mathbf{R}^n})_{|E}$ be the induced topology on E.

1. Let $(x_p)_{p\geq 1}$ be a sequence in E. Let $x \in E$. Show that $x_p \stackrel{\mathcal{T}_E}{\rightarrow} x$ is equivalent to $x_p \stackrel{\mathcal{T}_{\mathbf{R}^n}}{\rightarrow} x$.
2. Propose a metric on $\mathbf{R}^n$, inducing the topology $\mathcal{T}_{\mathbf{R}^n}$.
3. Let $(x_p)_{p\geq 1}$ be a sequence in $\mathbf{R}^n$. Let $x \in \mathbf{R}^n$. Show that $x_p \stackrel{\mathcal{T}_{\mathbf{R}^n}}{\rightarrow} x$ if and only if, $x_p^i \stackrel{\mathcal{T}_{\mathbf{R}}}{\rightarrow} x^i$ for all $i \in \mathbf{N}_n$.

EXERCISE 7. Further to ex. (6), suppose $(x_p)_{p\geq 1}$ is a sequence in E.

1. Show the existence of a subsequence $(x_{\phi(p)})_{p\geq 1}$ of $(x_p)_{p\geq 1}$, such that $x^1_{\phi(p)} \stackrel{\mathcal{T}_{[-M,M]}}{\rightarrow} x^1$ for some $x^1 \in [-M, M]$.
2. Explain why the above convergence is equivalent to $x^1_{\phi(p)} \stackrel{\mathcal{T}_{\mathbf{R}}}{\rightarrow} x^1$.

3. Suppose that $1 \leq k \leq n-1$ and $(y_p)_{p\geq 1} = (x_{\phi(p)})_{p\geq 1}$ is a subsequence of $(x_p)_{p\geq 1}$ such that:

$$\forall j = 1, \ldots, k \ , \ x^j_{\phi(p)} \stackrel{\mathcal{T}_{\mathbf{R}}}{\to} x^j \text{ for some } x^j \in [-M, M]$$

Show the existence of a subsequence $(y_{\psi(p)})_{p\geq 1}$ of $(y_p)_{p\geq 1}$ such that $y^{k+1}_{\psi(p)} \stackrel{\mathcal{T}_{\mathbf{R}}}{\to} x^{k+1}$ for some $x^{k+1} \in [-M, M]$.

4. Show that $\phi \circ \psi : \mathbf{N}^* \to \mathbf{N}^*$ is strictly increasing.

5. Show that $(x_{\phi\circ\psi(p)})_{p\geq 1}$ is a subsequence of $(x_p)_{p\geq 1}$ such that:

$$\forall j = 1, \ldots, k+1 \ , \ x^j_{\phi\circ\psi(p)} \stackrel{\mathcal{T}_{\mathbf{R}}}{\to} x^j \in [-M, M]$$

6. Show the existence of a subsequence $(x_{\phi(p)})_{p\geq 1}$ of $(x_p)_{p\geq 1}$, and $x \in E$, such that $x_{\phi(p)} \stackrel{\mathcal{T}_E}{\to} x$

7. Show that $(E, \mathcal{T}_E)$ is a compact topological space.

Exercise 8. Let A be a closed subset of $\mathbf{R}^n$, $n \geq 1$, which is bounded with respect to the usual metric of $\mathbf{R}^n$.

1. Show that $A \subseteq E = [-M, M] \times \ldots \times [-M, M]$, for some $M \in \mathbf{R}^+$.

2. Show from $E \setminus A = E \cap A^c$ that A is closed in E.

3. Show $(A, (\mathcal{T}_{\mathbf{R}^n})_{|A})$ is a compact topological space.

4. Conversely, let A is a compact subset of $\mathbf{R}^n$. Show that A is closed and bounded.

Theorem 48 *A subset of $\mathbf{R}^n$ is compact if and only if it is closed and bounded with respect to its usual metric.*

Exercise 9. Let $n \geq 1$. Consider the map:

$$\phi : \left\{ \begin{array}{ccc} \mathbf{C}^n & \to & \mathbf{R}^{2n} \\ (a_1 + ib_1, \ldots, a_n + ib_n) & \to & (a_1, b_1, \ldots, a_n, b_n) \end{array} \right.$$

1. Recall the expressions of the usual metrics $d_{\mathbf{C}^n}$ and $d_{\mathbf{R}^{2n}}$ of $\mathbf{C}^n$ and $\mathbf{R}^{2n}$ respectively.

2. Show that for all $z, z' \in \mathbf{C}^n$, $d_{\mathbf{C}^n}(z, z') = d_{\mathbf{R}^{2n}}(\phi(z), \phi(z'))$.

3. Show that ϕ is a homeomorphism from $\mathbf{C}^n$ to $\mathbf{R}^{2n}$.

4. Show that a subset K of $\mathbf{C}^n$ is compact, if and only if $\phi(K)$ is a compact subset of $\mathbf{R}^{2n}$.

5. Show that K is closed, if and only if $\phi(K)$ is closed.

6. Show that K is bounded, if and only if $\phi(K)$ is bounded.

7. Show that a subset K of $\mathbf{C}^n$ is compact, if and only if it is closed and bounded with respect to its usual metric.

Definition 79 *Let (E,d) be a metric space. A sequence $(x_n)_{n\geq 1}$ in E is said to be a* **Cauchy sequence** *with respect to the metric d, if and only if for all $\epsilon > 0$, there exists $n_0 \geq 1$ such that:*

$$n, m \geq n_0 \;\Rightarrow\; d(x_n, x_m) \leq \epsilon$$

Definition 80 *We say that a metric space (E,d) is* **complete***, if and only if for any Cauchy sequence $(x_n)_{n\geq 1}$ in E, there exists $x \in E$ such that $(x_n)_{n\geq 1}$ converges to x.*

Exercise 10.

1. Explain why strictly speaking, given $p \in [1, +\infty]$, definition (77) of Cauchy sequences in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is not a covered by definition (79).

2. Explain why $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is not a complete metric space, despite theorem (46) and definition (80).

Exercise 11. Let $(z_k)_{k\geq 1}$ be a Cauchy sequence in $\mathbf{C}^n$, $n \geq 1$, with respect to the usual metric $d(z, z') = \|z - z'\|$, where:

$$\|z\| \stackrel{\triangle}{=} \sqrt{\sum_{i=1}^{n} |z_i|^2}$$

1. Show that the sequence $(z_k)_{k\geq 1}$ is bounded, i.e. that there exists $M \in \mathbf{R}^+$ such that $\|z_k\| \leq M$, for all $k \geq 1$.

2. Define $B = \{z \in \mathbf{C}^n \;,\; \|z\| \leq M\}$. Show that $\delta(B) < +\infty$, and that B is closed in $\mathbf{C}^n$.

3. Show the existence of a subsequence $(z_{k_p})_{p\geq 1}$ of $(z_k)_{k\geq 1}$ such that $z_{k_p} \stackrel{\mathcal{T}_{\mathbf{C}^n}}{\rightarrow} z$ for some $z \in B$.

4. Show that for all $\epsilon > 0$, there exists $p_0 \geq 1$ and $n_0 \geq 1$ such that $d(z, z_{k_{p_0}}) \leq \epsilon/2$ and:

$$k \geq n_0 \;\Rightarrow\; d(z_k, z_{k_{p_0}}) \leq \epsilon/2$$

5. Show that $z_k \stackrel{\mathcal{T}_{\mathbf{C}^n}}{\rightarrow} z$.

6. Conclude that $\mathbf{C}^n$ is complete with respect to its usual metric.

7. For which theorem of Tutorial 9 was the completeness of $\mathbf{C}$ used?

EXERCISE 12. Let $(x_k)_{k\geq 1}$ be a sequence in $\mathbf{R}^n$ such that $x_k \stackrel{\mathcal{T}_{\mathbf{C}^n}}{\to} z$, for some $z \in \mathbf{C}^n$.

1. Show that $z \in \mathbf{R}^n$.
2. Show that $\mathbf{R}^n$ is complete with respect to its usual metric.

Theorem 49 *$\mathbf{C}^n$ and $\mathbf{R}^n$ are complete w.r. to their usual metrics.*

EXERCISE 13. Let (E, d) be a metric space, with metric topology $\mathcal{T}$. Let $F \subseteq E$, and $\bar{F}$ denote the closure of F.

1. Explain why, for all $x \in \bar{F}$ and $n \geq 1$, we have $F \cap B(x, 1/n) \neq \emptyset$.
2. Show that for all $x \in \bar{F}$, there exists a sequence $(x_n)_{n\geq 1}$ in F, such that $x_n \stackrel{\mathcal{T}}{\to} x$.
3. Show conversely that if there is a sequence $(x_n)_{n\geq 1}$ in F with $x_n \stackrel{\mathcal{T}}{\to} x$, then $x \in \bar{F}$.
4. Show that F is closed if and only if for all sequence $(x_n)_{n\geq 1}$ in F such that $x_n \stackrel{\mathcal{T}}{\to} x$ for some $x \in E$, we have $x \in F$.
5. Explain why $(F, \mathcal{T}_{|F})$ is metrizable.
6. Show that if F is complete with respect to the metric $d_{|F\times F}$, then F is closed in E.
7. Let $d_{\bar{\mathbf{R}}}$ be a metric on $\bar{\mathbf{R}}$, inducing the usual topology $\mathcal{T}_{\bar{\mathbf{R}}}$. Show that $d' = (d_{\bar{\mathbf{R}}})_{|\mathbf{R}\times\mathbf{R}}$ is a metric on $\mathbf{R}$, inducing the topology $\mathcal{T}_{\mathbf{R}}$.
8. Find a metric on $[-1, 1]$ which induces its usual topology.
9. Show that $\{-1, 1\}$ is not open in $[-1, 1]$.
10. Show that $\{-\infty, +\infty\}$ is not open in $\bar{\mathbf{R}}$.
11. Show that $\mathbf{R}$ is not closed in $\bar{\mathbf{R}}$.
12. Let $d_{\mathbf{R}}$ be the usual metric of $\mathbf{R}$. Show that $d' = (d_{\bar{\mathbf{R}}})_{|\mathbf{R}\times\mathbf{R}}$ and $d_{\mathbf{R}}$ induce the same topology on $\mathbf{R}$, but that however, $\mathbf{R}$ is complete with respect to $d_{\mathbf{R}}$, whereas it cannot be complete with respect to d'.

Definition 81 *Let $\mathcal{H}$ be a $\mathbf{K}$-vector space, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. We call* **inner-product** *on $\mathcal{H}$, any map $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbf{K}$ with the following properties:*

$$
\begin{array}{ll}
(i) & \forall x, y \in \mathcal{H}\ ,\ \langle x, y\rangle = \overline{\langle y, x\rangle} \\
(ii) & \forall x, y, z \in \mathcal{H}\ ,\ \langle x + z, y\rangle = \langle x, y\rangle + \langle z, y\rangle \\
(iii) & \forall x, y \in \mathcal{H}, \forall \alpha \in \mathbf{K}\ ,\ \langle \alpha x, y\rangle = \alpha\langle x, y\rangle \\
(iv) & \forall x \in \mathcal{H}\ ,\ \langle x, x\rangle \geq 0 \\
(v) & \forall x \in \mathcal{H}\ ,\ (\langle x, x\rangle = 0 \ \Leftrightarrow\ x = 0)
\end{array}
$$

where for all $z \in \mathbf{C}$, $\bar{z}$ denotes the complex conjugate of z. For all $x \in \mathcal{H}$, we call **norm** *of x, denoted $\|x\|$, the number defined by:*

$$\|x\| \stackrel{\triangle}{=} \sqrt{\langle x, x\rangle}$$

EXERCISE 14. Let $\langle\cdot,\cdot\rangle$ be an inner-product on a $\mathbf{K}$-vector space $\mathcal{H}$.

1. Show that for all $y \in \mathcal{H}$, the map $x \to \langle x, y\rangle$ is linear.
2. Show that for all $x \in \mathcal{H}$, the map $y \to \langle x, y\rangle$ is linear if $\mathbf{K} = \mathbf{R}$, and conjugate-linear if $\mathbf{K} = \mathbf{C}$.

EXERCISE 15. Let $\langle\cdot,\cdot\rangle$ be an inner-product on a $\mathbf{K}$-vector space $\mathcal{H}$. Let $x, y \in \mathcal{H}$. Let $A = \|x\|^2$, $B = |\langle x, y\rangle|$ and $C = \|y\|^2$. let $\alpha \in \mathbf{K}$ be such that $|\alpha| = 1$ and:

$$B = \alpha\overline{\langle x, y\rangle}$$

1. Show that $A, B, C \in \mathbf{R}^+$.
2. For all $t \in \mathbf{R}$, show that $\langle x - t\alpha y, x - t\alpha y\rangle = A - 2tB + t^2C$.
3. Show that if $C = 0$ then $B^2 \leq AC$.
4. Suppose that $C \neq 0$. Show that $P(t) = A - 2tB + t^2C$ has a minimal value which is in $\mathbf{R}^+$, and conclude that $B^2 \leq AC$.
5. Conclude with the following:

Theorem 50 (Cauchy-Schwarz inequality:second) *Let $\mathcal{H}$ be a $\mathbf{K}$-vector space, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, and $\langle\cdot,\cdot\rangle$ be an inner-product on $\mathcal{H}$. Then, for all $x, y \in \mathcal{H}$, we have:*

$$|\langle x, y\rangle| \leq \|x\|.\|y\|$$

EXERCISE 16. For all $f, g \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we define:

$$\langle f, g\rangle \stackrel{\triangle}{=} \int_\Omega f\bar{g}d\mu$$

1. Use the first Cauchy-Schwarz inequality (42) to prove that for all $f, g \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we have $f\bar{g} \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Conclude that $\langle f, g\rangle$ is a well-defined complex number.
2. Show that for all $f \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we have $\|f\|_2 = \sqrt{\langle f, f\rangle}$.
3. Make another use of the first Cauchy-Schwarz inequality to show that for all $f, g \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we have:

$$|\langle f, g\rangle| \leq \|f\|_2.\|g\|_2 \tag{1}$$

4. Go through definition (81), and indicate which of the properties $(i)-(v)$ fails to be satisfied by $\langle\cdot,\cdot\rangle$. Conclude that $\langle\cdot,\cdot\rangle$ is not an inner-product on $L^2_{\mathbf{C}}(\Omega,\mathcal{F},\mu)$, and therefore that inequality (*) is not a particular case of the second Cauchy-Schwarz inequality (50).

5. Let $f,g\in L^2_{\mathbf{C}}(\Omega,\mathcal{F},\mu)$. By considering $\int(|f|+t|g|)^2d\mu$ for $t\in\mathbf{R}$, imitate the proof of the second Cauchy-Schwarz inequality to show that:
$$\int_\Omega |fg|d\mu \leq \left(\int_\Omega |f|^2 d\mu\right)^{\frac{1}{2}}\left(\int_\Omega |g|^2 d\mu\right)^{\frac{1}{2}}$$

6. Let $f,g:(\Omega,\mathcal{F})\to[0,+\infty]$ non-negative and measurable. Show that if $\int f^2d\mu$ and $\int g^2d\mu$ are finite, then f and g are μ-almost surely equal to elements of $L^2_{\mathbf{C}}(\Omega,\mathcal{F},\mu)$. Deduce from 5. a new proof of the first Cauchy-Schwarz inequality:
$$\int_\Omega fg d\mu \leq \left(\int_\Omega f^2 d\mu\right)^{\frac{1}{2}}\left(\int_\Omega g^2 d\mu\right)^{\frac{1}{2}}$$

EXERCISE 17. Let $\langle\cdot,\cdot\rangle$ be an inner-product on a $\mathbf{K}$-vector space $\mathcal{H}$.

1. Show that for all $x,y\in\mathcal{H}$, we have:
$$\|x+y\|^2=\|x\|^2+\|y\|^2+\langle x,y\rangle+\overline{\langle x,y\rangle}$$

2. Using the second Cauchy-Schwarz inequality (50), show that:
$$\|x+y\|\leq\|x\|+\|y\|$$

3. Show that $d_{\langle\cdot,\cdot\rangle}(x,y)=\|x-y\|$ defines a metric on $\mathcal{H}$.

Definition 82 *Let $\mathcal{H}$ be a $\mathbf{K}$-vector space, where $\mathbf{K}=\mathbf{R}$ or $\mathbf{C}$, and $\langle\cdot,\cdot\rangle$ be an inner-product on $\mathcal{H}$. We call* **norm topology** *on $\mathcal{H}$, denoted $\mathcal{T}_{\langle\cdot,\cdot\rangle}$, the metric topology associated with $d_{\langle\cdot,\cdot\rangle}(x,y)=\|x-y\|$.*

Definition 83 *We call* **Hilbert space** *over $\mathbf{K}$ where $\mathbf{K}=\mathbf{R}$ or $\mathbf{C}$, any ordered pair $(\mathcal{H},\langle\cdot,\cdot\rangle)$ where $\langle\cdot,\cdot\rangle$ is an inner-product on a $\mathbf{K}$-vector space $\mathcal{H}$, which is complete w.r. to $d_{\langle\cdot,\cdot\rangle}(x,y)=\|x-y\|$.*

EXERCISE 18. Let $(\mathcal{H},\langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbf{K}$ and let $\mathcal{M}$ be a closed linear subspace of $\mathcal{H}$, (closed with respect to the norm topology $\mathcal{T}_{\langle\cdot,\cdot\rangle}$). Define $[\cdot,\cdot]=\langle\cdot,\cdot\rangle_{|\mathcal{M}\times\mathcal{M}}$.

1. Show that $[\cdot,\cdot]$ is an inner-product on the $\mathbf{K}$-vector space $\mathcal{M}$.

2. With obvious notations, show that $d_{[\cdot,\cdot]}=(d_{\langle\cdot,\cdot\rangle})_{|\mathcal{M}\times\mathcal{M}}$.

3. Deduce that $\mathcal{T}_{[\cdot,\cdot]}=(\mathcal{T}_{\langle\cdot,\cdot\rangle})_{|\mathcal{M}}$.

EXERCISE 19. Further to ex. (18), Let $(x_n)_{n\geq 1}$ be a Cauchy sequence in $\mathcal{M}$, with respect to the metric $d_{[\cdot,\cdot]}$.

1. Show that $(x_n)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{H}$.
2. Explain why there exists $x \in \mathcal{H}$ such that $x_n \stackrel{\mathcal{T}_{\langle\cdot,\cdot\rangle}}{\rightarrow} x$.
3. Explain why $x \in \mathcal{M}$.
4. Explain why we also have $x_n \stackrel{\mathcal{T}_{[\cdot,\cdot]}}{\rightarrow} x$.
5. Explain why $(\mathcal{M}, \langle\cdot,\cdot\rangle_{|\mathcal{M}\times\mathcal{M}})$ is a Hilbert space over $\mathbf{K}$.

EXERCISE 20. For all $z, z' \in \mathbf{C}^n$, $n \geq 1$, we define:

$$\langle z, z'\rangle \stackrel{\triangle}{=} \sum_{i=1}^n z_i \bar{z}_i'$$

1. Show that $\langle\cdot,\cdot\rangle$ is an inner-product on $\mathbf{C}^n$.
2. Show that the metric $d_{\langle\cdot,\cdot\rangle}$ is equal to the usual metric of $\mathbf{C}^n$.
3. Conclude that $(\mathbf{C}^n, \langle\cdot,\cdot\rangle)$ is a Hilbert space over $\mathbf{C}$.
4. Show that $\mathbf{R}^n$ is a closed subset of $\mathbf{C}^n$.
5. Show however that $\mathbf{R}^n$ is not a linear subspace of $\mathbf{C}^n$.
6. Show that $(\mathbf{R}^n, \langle\cdot,\cdot\rangle_{|\mathbf{R}^n\times\mathbf{R}^n})$ is a Hilbert space over $\mathbf{R}$.

Definition 84 *We call* **usual inner-product** *in* $\mathbf{K}^n$*, where* $\mathbf{K} = \mathbf{R}$ *or* $\mathbf{C}$*, the inner-product denoted* $\langle\cdot,\cdot\rangle$ *and defined by:*

$$\forall x, y \in \mathbf{K}^n \ , \ \langle x, y\rangle = \sum_{i=1}^n x_i \bar{y}_i$$

Theorem 51 $\mathbf{C}^n$ *and* $\mathbf{R}^n$ *together with their usual inner-products, are Hilbert spaces over* $\mathbf{C}$ *and* $\mathbf{R}$ *respectively.*

Definition 85 *Let* $\mathcal{H}$ *be a* $\mathbf{K}$*-vector space, where* $\mathbf{K} = \mathbf{R}$ *or* $\mathbf{C}$*. Let* $\mathcal{C} \subseteq \mathcal{H}$*. We say that* $\mathcal{C}$ *is a* **convex subset** *or* $\mathcal{H}$*, if and only if, for all* $x, y \in \mathcal{C}$ *and* $t \in [0, 1]$*, we have* $tx + (1-t)y \in \mathcal{C}$*.*

EXERCISE 21. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbf{K}$. Let $\mathcal{C} \subseteq \mathcal{H}$ be a non-empty closed convex subset of $\mathcal{H}$. Let $x_0 \in \mathcal{H}$. Define:

$$\delta_{\min} \stackrel{\triangle}{=} \inf\{\|x - x_0\| \ : \ x \in \mathcal{C}\}$$

1. Show the existence of a sequence $(x_n)_{n\geq 1}$ in $\mathcal{C}$ such that $\|x_n - x_0\| \to \delta_{\min}$.

2. Show that for all $x, y \in \mathcal{H}$, we have:
$$\|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 - 4\left\|\frac{x+y}{2}\right\|^2$$

3. Explain why for all $n, m \geq 1$, we have:
$$\delta_{\min} \leq \left\|\frac{x_n + x_m}{2} - x_0\right\|$$

4. Show that for all $n, m \geq 1$, we have:
$$\|x_n - x_m\|^2 \leq 2\|x_n - x_0\|^2 + 2\|x_m - x_0\|^2 - 4\delta_{\min}^2$$

5. Show the existence of some $x^* \in \mathcal{H}$, such that $x_n \stackrel{\mathcal{T}_{\langle\cdot,\cdot\rangle}}{\to} x^*$.

6. Explain why $x^* \in \mathcal{C}$

7. Show that for all $x, y \in \mathcal{H}$, we have $|\ \|x\| - \|y\|\ | \leq \|x - y\|$.

8. Show that $\|x_n - x_0\| \to \|x^* - x_0\|$.

9. Conclude that we have found $x^* \in \mathcal{C}$ such that:
$$\|x^* - x_0\| = \inf\{\|x - x_0\|\ :\ x \in \mathcal{C}\}$$

10. Let y^* be another element of $\mathcal{C}$ with such property. Show that:
$$\|x^* - y^*\|^2 \leq 2\|x^* - x_0\|^2 + 2\|y^* - x_0\|^2 - 4\delta_{\min}^2$$

11. Conclude that $x^* = y^*$.

Theorem 52 *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let $\mathcal{C}$ be a non-empty, closed and convex subset of $\mathcal{H}$. For all $x_0 \in \mathcal{H}$, there exists a unique $x^* \in \mathcal{C}$ such that:*
$$\|x^* - x_0\| = \inf\{\|x - x_0\|\ :\ x \in \mathcal{C}\}$$

Definition 86 *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let $\mathcal{G} \subseteq \mathcal{H}$. We call* **orthogonal** *of $\mathcal{G}$, the subset of $\mathcal{H}$ denoted $\mathcal{G}^\perp$ and defined by:*
$$\mathcal{G}^\perp \stackrel{\triangle}{=} \{\ x \in \mathcal{H}\ :\ \langle x, y\rangle = 0\ ,\ \forall y \in \mathcal{G}\ \}$$

Exercise 22. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbf{K}$ and $\mathcal{G} \subseteq \mathcal{H}$.

1. Show that $\mathcal{G}^\perp$ is a linear subspace of $\mathcal{H}$, even if $\mathcal{G}$ isn't.

2. Show that $\phi_y : \mathcal{H} \to K$ defined by $\phi_y(x) = \langle x, y \rangle$ is continuous.
3. Show that $\mathcal{G}^\perp = \cap_{y\in\mathcal{G}} \phi_y^{-1}(\{0\})$.
4. Show that $\mathcal{G}^\perp$ is a closed subset of $\mathcal{H}$, even if $\mathcal{G}$ isn't.
5. Show that $\emptyset^\perp = \{0\}^\perp = \mathcal{H}$.
6. Show that $\mathcal{H}^\perp = \{0\}$.

Exercise 23. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbf{K}$. Let $\mathcal{M}$ be a closed linear subspace of $\mathcal{H}$, and $x_0 \in \mathcal{H}$.

1. Explain why there exists $x^* \in \mathcal{M}$ such that:
$$\|x^* - x_0\| = \inf\{\ \|x - x_0\|\ :\ x \in \mathcal{M}\ \}$$
2. Define $y^* = x_0 - x^* \in \mathcal{H}$. Show that for all $y \in \mathcal{M}$ and $\alpha \in \mathbf{K}$:
$$\|y^*\|^2 \le \|y^* - \alpha y\|^2$$
3. Show that for all $y \in \mathcal{M}$ and $\alpha \in \mathbf{K}$, we have:
$$0 \le -\alpha\langle y, y^*\rangle - \overline{\alpha\langle y, y^*\rangle} + |\alpha|^2.\|y\|^2$$
4. For all $y \in \mathcal{M} \setminus \{0\}$, taking $\alpha = \overline{\langle y, y^*\rangle}/\|y\|^2$, show that:
$$0 \le -\frac{|\langle y, y^*\rangle|^2}{\|y\|^2}$$
5. Conclude that $x^* \in \mathcal{M}$, $y^* \in \mathcal{M}^\perp$ and $x_0 = x^* + y^*$.
6. Show that $\mathcal{M} \cap \mathcal{M}^\perp = \{0\}$
7. Show that $x^* \in \mathcal{M}$ and $y^* \in \mathcal{M}^\perp$ with $x_0 = x^* + y^*$, are unique.

Theorem 53 *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let $\mathcal{M}$ be a closed linear subspace of $\mathcal{H}$. Then, for all $x_0 \in \mathcal{H}$, there is a unique decomposition:*
$$x_0 = x^* + y^*$$
where $x^* \in \mathcal{M}$ and $y^* \in \mathcal{M}^\perp$.

Definition 87 *Let $\mathcal{H}$ be a $\mathbf{K}$-vector space, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. We call* **linear functional**, *any map $\lambda : \mathcal{H} \to \mathbf{K}$, such that for all $x, y \in \mathcal{H}$ and $\alpha \in \mathbf{K}$:*
$$\lambda(x + \alpha y) = \lambda(x) + \alpha\lambda(y)$$

Exercise 24. Let λ be a linear functional on a $\mathbf{K}$-Hilbert[1] space $\mathcal{H}$.

[1] Norm vector spaces are introduced later in these tutorials.

1. Suppose that λ is continuous at some point $x_0 \in \mathcal{H}$. Show the existence of $\eta > 0$ such that:

$$\forall x \in \mathcal{H} \ , \ \|x - x_0\| \leq \eta \ \Rightarrow \ |\lambda(x) - \lambda(x_0)| \leq 1$$

   Show that for all $x \in \mathcal{H}$ with $x \neq 0$, we have $|\lambda(\eta x / \|x\|)| \leq 1$.

2. Show that if λ is continuous at x_0, there exits $M \in \mathbf{R}^+$, with:

$$\forall x \in \mathcal{H} \ , \ |\lambda(x)| \leq M\|x\| \tag{2}$$

3. Show conversely that if (2) holds, λ is continuous everywhere.

Definition 88 *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert*[2] *space over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let λ be a linear functional on $\mathcal{H}$. Then, the following are equivalent:*

$$\begin{array}{ll} (i) & \lambda : (\mathcal{H}, \mathcal{T}_{\langle \cdot, \cdot \rangle}) \to (K, \mathcal{T}_{\mathbf{K}}) \ \ \textit{is continuous} \\ (ii) & \exists M \in \mathbf{R}^+ \ , \ \forall x \in \mathcal{H} \ , \ |\lambda(x)| \leq M.\|x\| \end{array}$$

In which case, we say that λ is a **bounded linear functional**.

Exercise 25. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbf{K}$. Let λ be a bounded linear functional on $\mathcal{H}$, such that $\lambda(x) \neq 0$ for some $x \in \mathcal{H}$, and define $\mathcal{M} = \lambda^{-1}(\{0\})$.

1. Show the existence of $x_0 \in \mathcal{H}$, such that $x_0 \notin \mathcal{M}$.

2. Show the existence of $x^* \in \mathcal{M}$ and $y^* \in \mathcal{M}^\perp$ with $x_0 = x^* + y^*$.

3. Deduce the existence of some $z \in \mathcal{M}^\perp$ such that $\|z\| = 1$.

4. Show that for all $\alpha \in \mathbf{K} \setminus \{0\}$ and $x \in \mathcal{H}$, we have:

$$\frac{\lambda(x)}{\bar{\alpha}} \langle z, \alpha z \rangle = \lambda(x)$$

5. Show that in order to have:

$$\forall x \in \mathcal{H} \ , \ \lambda(x) = \langle x, \alpha z \rangle$$

   it is sufficient to choose $\alpha \in \mathbf{K} \setminus \{0\}$ such that:

$$\forall x \in \mathcal{H} \ , \ \frac{\lambda(x) z}{\bar{\alpha}} - x \in \mathcal{M}$$

6. Show the existence of $y \in \mathcal{H}$ such that:

$$\forall x \in \mathcal{H} \ , \ \lambda(x) = \langle x, y \rangle$$

7. Show the uniqueness of such $y \in \mathcal{H}$.

[2] Norm vector spaces are introduced later in these tutorials.

Theorem 54 *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let λ be a bounded linear functional on $\mathcal{H}$. Then, there exists a unique $y \in \mathcal{H}$ such that: $\forall x \in \mathcal{H}\ ,\ \lambda(x) = \langle x, y\rangle$.*

Definition 89 *Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. We call K-**vector space**, any set $\mathcal{H}$, together with operators $\oplus$ and $\otimes$ for which there exits an element $0_{\mathcal{H}} \in \mathcal{H}$ such that for all $x, y, z \in \mathcal{H}$ and $\alpha, \beta \in \mathbf{K}$, we have:*

$$\begin{array}{rl} (i) & 0_{\mathcal{H}} \oplus x = x \\ (ii) & \exists(-x) \in \mathcal{H}\ ,\ (-x) \oplus x = 0_{\mathcal{H}} \\ (iii) & x \oplus (y \oplus z) = (x \oplus y) \oplus z \\ (iv) & x \oplus y = y \oplus x \\ (v) & 1 \otimes x = x \\ (vi) & \alpha \otimes (\beta \otimes x) = (\alpha\beta) \otimes x \\ (vii) & (\alpha + \beta) \otimes x = (\alpha \otimes x) \oplus (\beta \otimes x) \\ (viii) & \alpha \otimes (x \oplus y) = (\alpha \otimes x) \oplus (\alpha \otimes y) \end{array}$$

Exercise 26. For all $f \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$, define:

$$\mathcal{H} \stackrel{\triangle}{=} \{\ [f]\ :\ f \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)\ \}$$

where $[f] = \{g \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) : g = f, \mu\text{-a.s.}\}$. Let $0_{\mathcal{H}} = [0]$, and for all $[f], [g] \in \mathcal{H}$, and $\alpha \in \mathbf{K}$, we define:

$$\begin{array}{rcl} [f] \oplus [g] & \stackrel{\triangle}{=} & [f + g] \\ \alpha \otimes [f] & \stackrel{\triangle}{=} & [\alpha f] \end{array}$$

We assume f, f', g and g' are elements of $L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$.

1. Show that for $f = g$ μ-a.s. is equivalent to $[f] = [g]$.
2. Show that if $[f] = [f']$ and $[g] = [g']$, then $[f + g] = [f' + g']$.
3. Conclude that $\oplus$ is well-defined.
4. Show that $\otimes$ is also well-defined.
5. Show that $(\mathcal{H}, \oplus, \otimes)$ is a $\mathbf{K}$-vector space.

Exercise 27. Further to ex. (26), we define for all $[f], [g] \in \mathcal{H}$:

$$\langle [f], [g]\rangle_{\mathcal{H}} \stackrel{\triangle}{=} \int_\Omega f\bar{g}d\mu$$

1. Show that $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is well-defined.
2. Show that $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is an inner-product on $\mathcal{H}$.

3. Show that $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ is a Hilbert space over $\mathbf{K}$.

4. Why is $\langle f, g \rangle \stackrel{\triangle}{=} \int_\Omega f\bar{g}d\mu$ not an inner-product on $L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$?

Exercise 28. Further to ex. (27), Let $\lambda : L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \to \mathbf{K}$ be a continuous linear functional[3]. Define $\Lambda : \mathcal{H} \to \mathbf{K}$ by $\Lambda([f]) = \lambda(f)$.

1. Show the existence of $M \in \mathbf{R}^+$ such that:

$$\forall f \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \ , \ |\lambda(f)| \leq M.\|f\|_2$$

2. Show that if $[f] = [g]$ then $\lambda(f) = \lambda(g)$.

3. Show that Λ is a well defined bounded linear functional on $\mathcal{H}$.

4. Conclude with the following:

Theorem 55 *Let $\lambda : L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \to \mathbf{K}$ be a continuous linear functional, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. There exists $g \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$ such that:*

$$\forall f \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \ , \ \lambda(f) = \int_\Omega f\bar{g}d\mu$$

[3]As defined in these tutorials, $L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$ is not a Hilbert space (not even a norm vector space). However, both $L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$ and $\mathbf{K}$ have natural topologies and it is therefore meaningful to speak of *continuous linear functional*. Note however that we are slightly outside the framework of definition (88).

Solutions to Exercises

Exercise 1.

1. Let $(x_n)_{n\geq 1}$ be a sequence in E, and $x \in E$. Suppose that $x_n \stackrel{\mathcal{T}}{\rightarrow} x$. Let $\epsilon > 0$. The open ball $B(x,\epsilon)$ being open in E, there exists $n_0 \geq 1$, such that $n \geq n_0 \;\Rightarrow\; x_n \in B(x,\epsilon)$. In other words, we have:
$$n \geq n_0 \;\Rightarrow\; d(x_n, x) \leq \epsilon \tag{3}$$
Conversely, suppose that for all $\epsilon > 0$, there exists $n_0 \geq 1$ such that (3) holds. Let U be open in E, with $x \in U$. By definition (30) of the metric topology, there exists $\epsilon > 0$ such that $B(x,\epsilon) \subseteq U$. Since, there exists $n_0 \geq 1$ such that (3) holds, we have found $n_0 \geq 1$ such that:
$$n \geq n_0 \;\Rightarrow\; x_n \in U$$
This proves that $x_n \stackrel{\mathcal{T}}{\rightarrow} x$.

2. $F_n = \overline{\{x_k : k \geq n\}}$. So $F_{n+1} \subseteq F_n$ for all $n \geq 1$. Being the closure of some subset of E, for all $n \geq 1$, F_n is a closed subset of E, (see definition (37) and following exercise). It follows that $(F_n)_{n\geq 1}$ is a decreasing sequence of closed subsets of E.

3. Suppose $F_n \downarrow \emptyset$, i.e. $F_{n+1} \subseteq F_n$ with $\cap_{n\geq 1} F_n = \emptyset$. Then:
$$E = \emptyset^c = \left(\bigcap_{n=1}^{+\infty} F_n\right)^c = \bigcup_{n=1}^{+\infty} F_n^c$$
Since each F_n is closed in E, each F_n^c is an open subset of E. We conclude that $(F_n^c)_{n\geq 1}$ is an open covering of E.

4. Suppose $(E,\mathcal{T})$ is compact. If $\cap_{n\geq 1} F_n = \emptyset$, then from 3. $(F_n^c)_{n\geq 1}$ is an open covering of E, of which we can extract a finite sub-covering (see definition (65)). There exists a finite subset $\{n_1,\ldots,n_p\}$ of $\mathbf{N}^*$ such that:
$$E = F_{n_1}^c \cup \ldots \cup F_{n_p}^c$$
and therefore $F_{n_1} \cap \ldots \cap F_{n_p} = \emptyset$. However, without loss of generality, we can assume that $n_p \geq n_k$ for all $k = 1,\ldots,p$. Since $F_{n+1} \subseteq F_n$ for all $n \geq 1$, it follows that:
$$F_{n_p} = F_{n_1} \cap \ldots \cap F_{n_p} = \emptyset$$
This is a contradiction since F_{n_p} contains all x_k's for $k \geq n_p$. We conclude that if $(E,\mathcal{T})$ is a compact, then $\cap_{n\geq 1} F_n \neq \emptyset$.

5. Suppose $(E,\mathcal{T})$ is compact. From 4., there exists $x \in \cap_{n\geq 1} F_n$. Then, for all $n \geq 1$, we have $x \in F_n = \overline{\{x_k : k \geq n\}}$, i.e. x lies in the closure of $\{x_k : k \geq n\}$. It follows that for all $\epsilon > 0$:
$$\{x_k : k \geq n\} \cap B(x,\epsilon) \neq \emptyset \tag{4}$$

We have proved the existence of $x \in E$, such that (4) holds for all $n \geq 1$ and $\epsilon > 0$.

6. Let $x \in E$ be as in 5. Take $n = 1$ and $\epsilon = 1$. Then, we have $\{x_k : k \geq 1\} \cap B(x, 1) \neq \emptyset$. There exists $n_1 \geq 1$, such that $x_{n_1} \in B(x, 1)$. Suppose we have found $n_1 < \ldots < n_p$ $(p \geq 1)$, such that $x_{n_k} \in B(x, 1/k)$ for all $k \in \mathbf{N}_p$. Take $n = n_p + 1$ and $\epsilon = 1/(p+1)$ in 5. We have:
$$\{x_k : k \geq n_p + 1\} \cap B(x, 1/(p+1)) \neq \emptyset$$
So there exists $n_{p+1} > n_p$, such that $x_{n_{p+1}} \in B(x, 1/(p+1))$. Following this induction argument, we can construct a subsequence $(x_{n_p})_{p\geq 1}$ of $(x_n)_{n\geq 1}$, such that $x_{n_p} \in B(x, 1/p)$ for all $p \geq 1$.

7. If $(E, \mathcal{T})$ is compact, then from 5. and 6., given a sequence $(x_n)_{n\geq 1}$ in E, there exists $x \in E$ and a subsequence $(x_{n_p})_{p\geq 1}$ such that $d(x, x_{n_p}) < 1/p$ for all $p \geq 1$. From 1., it follows that $x_{n_p} \xrightarrow{\mathcal{T}} x$ as $p \to +\infty$, and we have proved that any sequence in a compact metric space, has a convergent subsequence.

Exercise 1

Exercise 2.

1. Let $x \in E$. By assumption, $(V_i)_{i \in I}$ is an open covering of E, so in particular $E = \cup_{i\in I} V_i$. There exists $i \in I$, such that $x \in V_i$. Furthermore, V_i is open with respect to the metric topology on E. There exists $r > 0$, such that $B(x, r) \subseteq V_i$. We have proved that for all $x \in E$, there exists $i \in I$ and $r > 0$, such that $B(x, r) \subseteq V_i$.

2. Let $x \in E$. Then $r(x) = \sup A(x)$, where:
$$A(x) \stackrel{\triangle}{=} \{r > 0 : \exists i \in I\ ,\ B(x, r) \subseteq V_i\}$$
From 1., the set $A(x)$ is non-empty. There exists $r > 0$ such that $r \in A(x)$. $r(x)$ being an upper-bound of $A(x)$, we have $r \leq r(x)$. In particular, $r(x) > 0$. We have proved that for all $x \in E$, $r(x) > 0$.

Exercise 2

Exercise 3.

1. Let $\alpha = \inf_{x\in E} r(x)$. We assume that $\alpha = 0$. Let $n \geq 1$. Then $\alpha < 1/n$. α being the greatest lower bound of all $r(x)'s$ for $x \in E$, $1/n$ cannot be such lower bound. There exists $x_n \in E$, such that $r(x_n) < 1/n$.

2. From 1., we have constructed a sequence $(x_n)_{n\geq 1}$ in E, such that $r(x_n) < 1/n$ for all $n \geq 1$. By assumption (see previous exercise (2)), the metric space (E, d) is such that any sequence has a convergent sub-sequence. Let $(x_{n_k})_{k\geq 1}$ be a sub-sequence of $(x_n)_{n\geq 1}$ and let $x^* \in E$, be such that $x_{n_k} \xrightarrow{\mathcal{T}}$

x^*. From exercise (2), there exists $r^* > 0$ and $i \in I$, with $B(x^*, r^*) \subseteq V_i$. Since $r^* > 0$ and $x_{n_k} \xrightarrow{\mathcal{T}} x^*$, there exists $k'_0 \geq 1$, such that:

$$k \geq k'_0 \ \Rightarrow \ d(x^*, x_{n_k}) < r^*/2$$

Since $n_k \uparrow +\infty$ as $k \to +\infty$, there exists $k''_0 \geq 1$, such that:

$$k \geq k''_0 \ \Rightarrow \ \frac{1}{n_k} \leq r^*/4$$

It follows that for all $k \geq k''_0$, we have $r(x_{n_k}) \leq 1/n_k \leq r^*/4$. Take $k_0 = \max(k'_0, k''_0)$. We have both $d(x^*, x_{n_{k_0}}) < r^*/2$ and $r(x_{n_{k_0}}) \leq r^*/4$.

3. From 2., we have found $k_0 \geq 1$, such that $d(x^*, x_{n_{k_0}}) < r^*/2$. Let $y \in B(x_{n_{k_0}}, r^*/2)$. Then, from the triangle inequality:

$$d(x^*, y) \leq d(x^*, x_{n_{k_0}}) + d(x_{n_{k_0}}, y) < \frac{r^*}{2} + \frac{r^*}{2} = r^*$$

So $y \in B(x^*, r^*)$. Hence, we see that $B(x_{n_{k_0}}, r^*/2) \subseteq B(x^*, r^*)$. However, from 2., $B(x^*, r^*) \subseteq V_i$. So $B(x_{n_{k_0}}, r^*/2) \subseteq V_i$. It follows that $r^*/2$ belongs to the set:

$$A(x_{n_{k_0}}) = \{r > 0 : \exists i \in I \ , \ B(x_{n_{k_0}}, r) \subseteq V_i\}$$

and consequently, $r^*/2 \leq r(x_{n_{k_0}}) = \sup A(x_{n_{k_0}})$. This contradicts the fact that $r(x_{n_{k_0}}) \leq r^*/4$, as obtained in 2. We conclude that our initial hypothesis of $\alpha = \inf_{x \in E} r(x) = 0$ is absurd, and we have proved that $\inf_{x \in E} r(x) > 0$.

Exercise 3

Exercise 4.

1. Let $r_0 > 0$ be such that $0 < r_0 < \inf_{x \in E} r(x)$. We assume that E cannot be covered by a finite number of open balls with radius r_0. Let x_1 be an arbitrary element of E. Then, by assumption, $B(x_1, r_0)$ cannot cover the whole of E. There exists $x_2 \in E$, such that $x_2 \notin B(x_1, r_0)$. By assumption, $B(x_1, r_0) \cup B(x_2, r_0)$ cannot cover the whole of E. There exists $x_3 \in E$, such that $x_3 \notin B(x_1, r_0) \cup B(x_2, r_0)$. By induction, we can construct a sequence $(x_n)_{n \geq 1}$ in E, such that for all $n \geq 1$:

$$x_{n+1} \notin B(x_1, r_0) \cup \ldots \cup B(x_n, r_0)$$

2. Let $n > m$. Then $x_n \notin B(x_m, r_0)$. So $d(x_n, x_m) \geq r_0$.

3. Suppose $(x_n)_{n \geq 1}$ has a convergent sub-sequence, There exists $x^* \in E$, and a sub-sequence $(x_{n_k})_{k \geq 1}$ such that $x_{n_k} \xrightarrow{\mathcal{T}} x^*$. Take $\epsilon = r_0/4 > 0$. There exists $k_0 \geq 1$, such that:

$$k \geq k_0 \ \Rightarrow \ d(x^*, x_{n_k}) < r_0/4$$

It follows that for all $k, k' \geq k_0$, we have:

$$d(x_{n_k}, x_{n_{k'}}) \leq d(x^*, x_{n_k}) + d(x^*, x_{n_{k'}}) < r_0/2$$

This contradicts 2., since $d(x_{n_k}, x_{n_{k'}}) \geq r_0$ for $k \neq k'$. So $(x_n)_{n\geq 1}$ cannot have a convergent sub-sequence.

4. From 3., $(x_n)_{n\geq 1}$ cannot have a convergent sub-sequence. This is a contradiction to our initial assumption (see exercise (2)), that any sequence in E should have a convergent sub-sequence. It follows that the hypothesis in 1. is absurd, and we conclude that E can indeed be covered by a finite number of open balls of radius r_0. In other words, there exists a finite subset $\{x_1, \ldots, x_n\}$ of E, such that $E = B(x_1, r_0) \cup \ldots \cup B(x_n, r_0)$.

5. Let $x \in E$. By assumption, $r_0 < \inf_{x\in E} r(x)$. In particular, we have $r_0 < r(x) = \sup A(x)$, where:

$$A(x) = \{r > 0 : \exists i \in I \ , \ B(x, r) \subseteq V_i\}$$

$r(x)$ being the smallest upper-bound of $A(x)$, it follows that r_0 cannot be such upper bound. There exists $r > 0$, $r \in A(x)$, such that $r_0 < r$. Since $r \in A(x)$, there exists $i \in I$, such that $B(x, r) \subseteq V_i$. But from $r_0 < r$, we conclude that $B(x, r_0) \subseteq V_i$. We have proved that for all $x \in E$, there exists $i \in I$, such that $B(x, r_0) \subseteq V_i$.

6. From 4., we have $E = B(x_1, r_0) \cup \ldots \cup B(x_n, r_0)$. However, from 5., for all $k \in \mathbf{N}_n$, there exists $i_k \in I$, such that $B(x_k, r_0) \subseteq V_{i_k}$. It follows that:

$$E = V_{i_1} \cup \ldots \cup V_{i_n} \tag{5}$$

Given a family of open sets $(V_i)_{i\in I}$ such that $E = \cup_{i\in I} V_i$ (see exercise (2)), we have been able to find a finite subset $\{i_1, \ldots, i_n\}$ of I, such that (5) holds. We conclude that the metrizable space $(E, \mathcal{T})$ is a compact topological space.

7. Let $(E, \mathcal{T})$ be a metrizable topological space. If $(E, \mathcal{T})$ is compact, then from exercise (1), any sequence in E has a convergent sub-sequence. Conversely, if E is such that any sequence in E has a convergent sub-sequence, then as proved in 6., $(E, \mathcal{T})$ is a compact topological space. This proves the difficult and very important theorem (47).

Exercise 4

Exercise 5.

1. Let $a, b \in \mathbf{R}$, $a < b$. Let $(x_n)_{n\geq 1}$ be a sequence in $]a, b[$. In particular, $(x_n)_{n\geq 1}$ is a sequence in $[a, b]$. From theorem (34), $[a, b]$ is a compact subset of $\mathbf{R}$. Applying theorem (47), there exists a subsequence $(x_{n_k})_{k\geq 1}$ of $(x_n)_{n\geq 1}$, and $x \in [a, b]$, such that $x_{n_k} \to x$[4]. So $(x_n)_{n\geq 1}$ has a convergent subsequence.

---

[4]In a clear context, we shall omit notations such as $x_{n_k} \stackrel{\mathcal{T}_{\mathbf{R}}}{\to} x$ or $x_{n_k} \stackrel{\mathcal{T}_{[a,b]}}{\to} x$.

2. No. One cannot conclude that $]a,b[$ is compact. In fact, $\mathbf{R}$ being Hausdorff, from theorem (35), if $]a,b[$ was compact, it would be closed, and $]-\infty,a]\cup[b,+\infty[$ would be open in $\mathbf{R}$... One has to be careful with the phrase *having a convergent subsequence*. As proved in 1., any sequence in $]a,b[$ *has a convergent subsequence*, but the limit of such subsequence may not lie in $]a,b[$ itself (we only know for sure it lies in $[a,b]$). This is why, one cannot apply theorem (47) to conclude that $]a,b[$ is compact.

Exercise 5

Exercise 6.

1. The equivalence between $x_p \stackrel{\mathcal{T}_E}{\rightarrow} x$ and $x_p \stackrel{\mathcal{T}_{\mathbf{R}^n}}{\rightarrow} x$ has already been proved in exercise (7) of the previous tutorial. Since the topology $\mathcal{T}_E$ is induced by the topology $\mathcal{T}_{\mathbf{R}^n}$ on E, whether we regard $(x_p)_{p\geq 1}$ and x as belonging to E or $\mathbf{R}^n$, is irrelevant as far as the convergence $x_p \to x$ is concerned. Note however that it is important to have $x_p \in E$ for all $p \geq 1$, and $x \in E$.

2. As seen in exercise (14) of Tutorial 6, various metrics will induce the product topology $\mathcal{T}_{\mathbf{R}^n}$ on $\mathbf{R}^n$. The most common, viewed as the *usual* metric on $\mathbf{R}^n$, is:

$$d_2(x,y) \stackrel{\triangle}{=} \sqrt{\sum_{i=1}^n (x^i - y^i)^2}$$

Other possible metrics are:

$$d_1(x,y) \stackrel{\triangle}{=} \sum_{i=1}^n |x^i - y^i|$$

or:

$$d_\infty(x,y) \stackrel{\triangle}{=} \max_{i\in\mathbf{N}^n} |x^i - y^i|$$

3. Let $(x_p)_{p\geq 1}$ be a sequence in $\mathbf{R}^n$ and $x \in \mathbf{R}^n$. Suppose that $x_p \to x$[5]. Then for all $\epsilon > 0$, there exists $p_0 \geq 1$, such that:

$$p \geq p_0 \ \Rightarrow\ d_1(x,x_p) = \sum_{i=1}^n |x^i - x_p^i| \leq \epsilon$$

In particular, for all $i \in \mathbf{N}_n$, we have:

$$p \geq p_0 \Rightarrow\ |x^i - x_p^i| \leq \epsilon$$

So $x_p^i \to x^i$[6] for all $i \in \mathbf{N}_n$. Conversely, suppose $x_p^i \to x^i$ for all i's. Given $\epsilon > 0$, for all $i \in \mathbf{N}_n$, there exists $p_i \geq 1$, such that:

$$p \geq p_i \ \Rightarrow\ |x^i - x_p^i| \leq \epsilon/n$$

[5] i.e. $x_p \stackrel{\mathcal{T}_{\mathbf{R}^n}}{\rightarrow} x$, as should be clear from context.

[6] i.e. $x_p^i \stackrel{\mathcal{T}_{\mathbf{R}}}{\rightarrow} x^i$, as should be clear from context.

Taking $p_0 = \max(p_1, \ldots, p_n)$, we obtain:

$$p \geq p_0 \;\Rightarrow\; d_1(x, x_p) = \sum_{i=1}^{n} |x^i - x_p^i| \leq \epsilon$$

So $x_p \to x$, which is equivalent to $[x_p^i \to x^i$ for all $i \in \mathbf{N}_n]$. Note that although we used the metric structure of $\mathbf{R}$ and $\mathbf{R}^n$ to prove this equivalence, we had no need to do so. In fact, any sequence with values in an arbitrary product, even uncountable, of topological spaces, even non-metrizable, will converge if and only if each coordinate sequence itself converges. For those interested in this small digression, here is a quick proof: let $(x_p)_{p\geq 1}$ be a sequence in the product $\Pi_{i\in I}\Omega_i$. Let x be an element of $\Pi_{i\in I}\Omega_i$. Suppose $x_p \to x$, with respect to the product topology. Let $i \in I$ and U be an arbitrary open set in Ω_i containing x^i. Then $U \times \Pi_{j\neq i}\Omega_j$ is an open set in $\Pi_{i\in I}\Omega_i$ containing x. Since $x_p \to x$, x_p is eventually[7] in $U \times \Pi_{j\neq i}\Omega_j$. It follows that x_p^i is eventually in U, and we see that $x_p^i \to x^i$. Conversely, suppose $x_p^i \to x^i$ for all $i \in I$. Let U be an open set in $\Pi_{i\in I}\Omega_i$ containing x. There exists a rectangle $V = \Pi_{i\in I}A_i$ such that $x \in V \subseteq U$. The set $J = \{i \in I : A_i \neq \Omega_i\}$ is finite, and for all $j \in J$, A_j is an open set in Ω_j containing x^j. From $x_p^j \to x^j$ we see that x_p^j is eventually in A_j. J being finite, it follows that x_p is eventually in $(\Pi_{j\in J}A_j) \times (\Pi_{i\notin J}\Omega_i) = V$. Since $V \subseteq U$, x_p is eventually in U, and we have proved that $x_p \to x$.

Exercise 6

Exercise 7.

1. Let $(x_p)_{p\geq 1}$ be a sequence in E. Then $(x_p^1)_{p\geq 1}$ is a sequence in $[-M, M]$, which is a compact subset of $\mathbf{R}$. From theorem (47), we can extract a subsequence of $(x_p^1)_{p\geq 1}$, converging to some $x^1 \in [-M, M]$. In other words, from definition (78), there exists a strictly increasing map $\phi : \mathbf{N}^* \to \mathbf{N}^*$, and $x^1 \in [-M, M]$ such that[8] $x_{\phi(p)}^1 \to x^1$. Hence, we have found a subsequence $(x_{\phi(p)})_{p\geq 1}$ such that $x_{\phi(p)}^1 \to x^1$, for some $x^1 \in [-M, M]$.

2. The topology on $[-M, M]$ being induced by the topology on $\mathbf{R}$, the convergence $x_{\phi(p)}^1 \to x^1$ is independent of the particular topology (that of $\mathbf{R}$ or $[-M, M]$) with respect to which, it is being considered.

3. Let $1 \leq k \leq n-1$. Let $(y_p)_{p\geq 1} = (x_{\phi(p)})_{p\geq 1}$ be a subsequence of $(x_p)_{p\geq 1}$, with the property that for all $j \in \mathbf{N}_k$, we have $y_p^j \to x^j$ for some $x^j \in [-M, M]$. Then, $(y_p^{k+1})_{p\geq 1}$ is a sequence in the compact interval $[-M, M]$. From theorem (47), there exists a strictly increasing map $\psi : \mathbf{N}^* \to \mathbf{N}^*$ such that $y_{\psi(p)}^{k+1} \to x^{k+1}$, for some $x^{k+1} \in [-M, M]$.

[7] there exists $p_0 \geq 1$ such that $p \geq p_0 \Rightarrow x_p \in U \times \Pi_{j\neq i}\Omega_j$.

[8] i.e. $x_{\phi(p)}^1 \stackrel{\mathcal{T}_{[-M,M]}}{\to} x^1$, which is the same as $x_{\phi(p)}^1 \stackrel{\mathcal{T}_{\mathbf{R}}}{\to} x^1$.

4. If both $\phi, \psi : \mathbf{N}^* \to \mathbf{N}^*$ are strictly increasing, so is $\phi \circ \psi$.

5. Since $\phi \circ \psi$ is strictly increasing, $(x_{\phi\circ\psi(p)})_{p\geq 1}$ is indeed a subsequence of $(x_p)_{p\geq 1}$, which furthermore coincides with $(y_{\psi(p)})_{p\geq 1}$, as defined in 3. It follows that $x^{k+1}_{\phi\circ\psi(p)} \to x^{k+1}$. Furthermore, from 3. the subsequence $(y_p)_{p\geq 1}$ is assumed to be such that $y^j_p \to x^j$ for all $j \in \mathbf{N}_k$. Hence, we also have $y^j_{\psi(p)} \to x^j$, i.e. $x^j_{\phi\circ\psi(p)} \to x^j$ for all $j \in \mathbf{N}_k$. We conclude that $(x_{\phi\circ\psi(p)})_{p\geq 1}$ is a subsequence of $(x_p)_{p\geq 1}$ such that $x^j_{\phi\circ\psi(p)} \to x^j$ for all $j \in \mathbf{N}_{k+1}$.

6. From 1., given a sequence $(x_p)_{p\geq 1}$ in E, we can extract a subsequence $(x_{\phi(p)})_{p\geq 1}$ of $(x_p)_{p\geq 1}$ such that $x^1_{\phi(p)} \to x^1$ for some $x^1 \in [-M, M]$. Given $1 \leq k \leq n-1$, and a subsequence $(x_{\phi(p)})_{p\geq 1}$ of $(x_p)_{p\geq 1}$, such that for all $j \in \mathbf{N}_k$, $x^j_{\phi(p)} \to x^j$ for some $x^j \in [-M, M]$, we showed in 5. that we could extract a further subsequence $(x_{\phi\circ\psi(p)})_{p\geq 1}$ having a similar property for all $j \in \mathbf{N}_{k+1}$. By induction, it follows that there exists a subsequence $(x_{\phi(p)})_{p\geq 1}$ of $(x_p)_{p\geq 1}$, such that for all $j \in \mathbf{N}_n$, we have $x^j_{\phi(p)} \to x^j$ for some $x^j \in [-M, M]$. Hence, taking $x = (x^1, \ldots, x^n)$, we see that $x_{\phi(p)} \to x$[9].

7. Let $(x_p)_{p\geq 1}$ be a sequence in E. From 6., there exists $x \in E$, and a subsequence $(x_{\phi(p)})_{p\geq 1}$ of $(x_p)_{p\geq 1}$, with $x_{\phi(p)} \to x$. From theorem (47), we conclude that $(E, \mathcal{T}_E)$ is a compact topological space, or equivalently, that E is a compact subset of $\mathbf{R}^n$. The purpose of this exercise is to prove that the n-dimensional product $[-M, M] \times \ldots \times [-M, M]$ is compact[10].

Exercise 7

Exercise 8.

1. If $A = \emptyset$ then $A \subseteq [-M, M] \times \ldots \times [-M, M]$, for all $M \in \mathbf{R}^+$. We assume that $A \neq \emptyset$. Let $\delta(A)$ be the diameter of A (see definition (68)) with respect to the usual metric:

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x^i - y^i)^2}$$

i.e. $\delta(A) = \sup\{d(x, y) : x, y \in A\}$. Since $A \neq \emptyset$, $\delta(A) \geq 0$. Furthermore, A being assumed to be bounded with respect to the usual metric of $\mathbf{R}^n$, we have $\delta(A) < +\infty$. So $\delta(A) \in \mathbf{R}^+$. Let y be an arbitrary element of A. For all $x \in A$, we have:

$$|x^i - y^i| \leq d(x, y) \leq \delta(A)$$

So $|x^i| \leq |y^i| + \delta(A)$, and taking $M = \max(|y^1|, \ldots, |y^n|) + \delta(A)$, we conclude that $A \subseteq [-M, M] \times \ldots \times [-M, M]$, with $M \in \mathbf{R}^+$.

[9]Both with respect to $\mathcal{T}_E$ and $\mathcal{T}_{\mathbf{R}^n}$.

[10]Tychonoff theorem will hopefully be the subject of some future tutorial :-)

2. By assumption, A is a closed subset of $\mathbf{R}^n$. So A^c is open. It follows that $E \setminus A = E \cap A^c$ is an element of the topology induced on E, by the topology on $\mathbf{R}^n$. In other words, $E \setminus A$ is an open subset of E. We conclude that A is a closed subset of E.

3. From ex. (7), $(E, \mathcal{T}_E)$ is a compact topological space. From 2., A is a closed subset of E. Using exercise (2)[6.] of Tutorial 8, we conclude that A is a compact subset of E. In other words, $(A, (\mathcal{T}_E)_{|A})$ is a compact topological space. However, the topology $\mathcal{T}_E$ is induced by $\mathcal{T}_{\mathbf{R}^n}$, i.e. $\mathcal{T}_E = (\mathcal{T}_{\mathbf{R}^n})_{|E}$. It follows that $(\mathcal{T}_E)_{|A} = (\mathcal{T}_{\mathbf{R}^n})_{|A}$. So $(A, (\mathcal{T}_{\mathbf{R}^n})_{|A})$ is a compact topological space, or equivalently, A is a compact subset of $\mathbf{R}^n$.

4. Let A be a compact subset of $\mathbf{R}^n$. From theorem (35), $\mathbf{R}^n$ being Hausdorff, A is closed in $\mathbf{R}^n$. From exercise (6)[4.] of Tutorial 8, A is bounded with respect to any metric inducing the usual topology of $\mathbf{R}^n$. This proves theorem (48).

Exercise 8

Exercise 9.

1. $d_{\mathbf{C}^n}$ and $d_{\mathbf{R}^{2n}}$ are defined by:

$$\begin{aligned} d_{\mathbf{C}^n}(z, z') &= \sqrt{\sum_{i=1}^{n} |z_i - z'_i|^2} \\ d_{\mathbf{R}^{2n}}(x, x') &= \sqrt{\sum_{i=1}^{2n} (x_i - x'_i)^2} \end{aligned}$$

for all $z, z' \in \mathbf{C}^n$ and $x, x' \in \mathbf{R}^{2n}$.

2. Given $z, z' \in \mathbf{C}^n$, we have:

$$d_{\mathbf{C}^n}(z, z') = \sqrt{\sum_{i=1}^{n} (Re(z_i) - Re(z'_i))^2 + \sum_{i=1}^{n} (Im(z_i) - Im(z'_i))^2}$$

It follows that $d_{\mathbf{C}^n}(z, z') = d_{\mathbf{R}^{2n}}(\phi(z), \phi(z'))$.

3. ϕ is clearly a bijection between $\mathbf{C}^n$ and $\mathbf{R}^{2n}$. From 2., we see that ϕ is continuous, and furthermore that:

$$d_{\mathbf{C}^n}(\phi^{-1}(x), \phi^{-1}(x')) = d_{\mathbf{R}^{2n}}(x, x')$$

for all $x, x' \in \mathbf{R}^{2n}$. So ϕ^{-1} is also continuous. From definition (31), ϕ is a homeomorphism from $\mathbf{C}^n$ to $\mathbf{R}^{2n}$.

4. Let $K \subseteq \mathbf{C}^n$. Suppose K is a compact subset of $\mathbf{C}^n$. Then, $(K, (\mathcal{T}_{\mathbf{C}^n})_{|K})$ is a compact topological space. ϕ being continuous, its restriction $\phi_{|K}$

is also continuous.[11] Using exercise (8) of Tutorial 8., the direct image $\phi_{|K}(K)$ is a compact subset of $\mathbf{R}^{2n}$. In other words, $\phi(K)$ is a compact subset of $\mathbf{R}^{2n}$. Conversely, suppose $\phi(K)$ is a compact subset of $\mathbf{R}^{2n}$. Since K can be written as the direct image $K = \phi^{-1}(\phi(K))$ of $\phi(K)$ by ϕ^{-1}, and ϕ^{-1} is continuous, we conclude similarly that K is a compact subset of $\mathbf{C}^n$. We have proved that for all $K \subseteq \mathbf{C}^n$, K is compact if and only if $\phi(K)$ is compact.

5. Let $K \subseteq \mathbf{C}^n$. Suppose K is a closed subset of $\mathbf{C}^n$. Since $\phi(K)$ can be written as the inverse image $\phi(K) = (\phi^{-1})^{-1}(K)$ of K by ϕ^{-1}, and ϕ^{-1} is continuous, we see that $\phi(K)$ is a closed subset of $\mathbf{R}^{2n}$. Conversely, suppose $\phi(K)$ is a closed subset of $\mathbf{R}^{2n}$. Since K can be written as the inverse image $K = \phi^{-1}(\phi(K))$ of $\phi(K)$ by ϕ, and ϕ is continuous, we see that K is a closed subset of $\mathbf{C}^n$. We have proved that for all $K \subseteq \mathbf{C}^n$, K is closed if and only if $\phi(K)$ is closed.

6. Let $K \subseteq \mathbf{C}^n$ and $\delta(\phi(K))$ be the diameter of $\phi(K)$ in $\mathbf{R}^{2n}$:
$$\begin{aligned}\delta(\phi(K)) &= \sup\{d_{\mathbf{R}^{2n}}(x, x') : x, x' \in \phi(K)\} \\ &= \sup\{d_{\mathbf{R}^{2n}}(\phi(z), \phi(z')) : z, z' \in K\} \\ &= \sup\{d_{\mathbf{C}^n}(z, z') : z, z' \in K\}\end{aligned}$$
i.e. $\delta(\phi(K)) = \delta(K)$, where $\delta(K)$ is the diameter of K in $\mathbf{C}^n$. It follows that $\delta(K) < +\infty$ is equivalent to $\delta(\phi(K)) < +\infty$. we have proved that for all $K \subseteq \mathbf{C}^n$, K is bounded if and only if $\phi(K)$ is bounded.

7. Let $K \subseteq \mathbf{C}^n$. From 4., K is compact, if and only if $\phi(K)$ is compact. From theorem (48), $\phi(K)$ being a subset of $\mathbf{R}^{2n}$, it is compact if and only if, it is closed and bounded. From 5. and 6., this in turn is equivalent to K being itself closed and bounded. We have proved that for all $K \subseteq \mathbf{C}^n$, K is compact if and only if K is closed and bounded.

Exercise 9

Exercise 10.

1. Definition (79) defines the notion of Cauchy sequences in a metric space. In contrast, definition (77) defines the notion of Cauchy sequences in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Since that latter was defined in (73) as a set of functions, as opposed to a set of μ-almost sure equivalence classes, strictly speaking $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is not a metric space. So definition (77) is not a particular case of definition (79).

2. Definition (80) defines the notion of complete metric space, as a metric space where all Cauchy sequences converge.[12] Theorem (46) does state that all Cauchy sequences in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ converge. However, since $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is not strictly speaking a metric space, it cannot be said to be a *complete metric space*.

[11] If uneasy with $K = \emptyset$ and $\phi_{|K} = \emptyset$, consider the case separately.
[12] to a limit belonging to that same metric space...

Exercise 10

Exercise 11.

1. Let $(z_k)_{k\geq 1}$ be a Cauchy sequence in $\mathbf{C}^n$. Taking $\epsilon = 1$, there exists $k_0 \geq 1$, such that:
$$k, k' \geq k_0 \;\Rightarrow\; \|z_k - z_{k'}\| \leq 1$$
Since $|\,\|z\| - \|z'\|\,| \leq \|z - z'\|$ for all $z, z' \in \mathbf{C}^n$, we have:
$$k \geq k_0 \;\Rightarrow\; \|z_k\| \leq 1 + \|z_{k_0}\|$$
Taking $M = \max(1 + \|z_{k_0}\|, \|z_1\|, \ldots, \|z_{k_0-1}\|)$, we see that $\|z_k\| \leq M$ for all $k \geq 1$. We have proved that $(z_k)_{k\geq 1}$ is a bounded sequence in $\mathbf{C}^n$.

2. Let $B = \{z \in \mathbf{C}^n : \|z\| \leq M\}$. For all $z, z' \in B$, we have $\|z - z'\| \leq \|z\| + \|z'\| \leq 2M$. It follows that $\delta(B) \leq 2M$, where $\delta(B)$ is the diameter of B in $\mathbf{C}^n$. So $\delta(B) < +\infty$, i.e. B is a bounded subset of $\mathbf{C}^n$. Let $z_0 \in B^c$. Then $M < \|z_0\|$. Let $\epsilon = \|z_0\| - M > 0$, and $z \in \mathbf{C}^n$ with $\|z - z_0\| < \epsilon$. Then, we have $\|z_0\| - \|z\| \leq \|z - z_0\| < \epsilon = \|z_0\| - M$, and consequently $M < \|z\|$, i.e. $z \in B^c$. So $B(z_0, \epsilon) \subseteq B^c$. For all $z_0 \in B^c$, we have found $\epsilon > 0$, such that $B(z_0, \epsilon) \subseteq B^c$. This proves that B^c is open with respect to the (metric) topology of $\mathbf{C}^n$. So B is a closed subset of $\mathbf{C}^n$.

3. From 2., B is a closed and bounded subset of $\mathbf{C}^n$. From exercise (9), it follows that B is a compact subset of $\mathbf{C}^n$. In other words, $(B, (\mathcal{T}_{\mathbf{C}^n})_{|B})$ is a compact topological space. However, from 1., $(z_k)_{k\geq 1}$ is a sequence of elements of B. Using theorem (47), $(z_k)_{k\geq 1}$ has a convergent subsequence, i.e. there exists $z \in B$, and a subsequence $(z_{k_p})_{p\geq 1}$, such that $z_{k_p} \to z$.[13]

4. $(z_k)_{k\geq 1}$ being Cauchy, given $\epsilon > 0$, there exist $n_0 \geq 1$, such that:
$$k, k' \geq n_0 \;\Rightarrow\; d(z_k, z_{k'}) \leq \epsilon/2$$
Furthermore, since $z_{k_p} \to z$, there exists $p'_0 \geq 1$, such that:
$$p \geq p'_0 \;\Rightarrow\; d(z, z_{k_p}) \leq \epsilon/2$$
Moreover, since $k_p \uparrow +\infty$ as $p \to +\infty$, there exists $p''_0 \geq 1$, such that $p \geq p''_0 \Rightarrow k_p \geq n_0$. Take $p_0 = \max(p'_0, p''_0)$. Then, $d(z, z_{k_{p_0}}) \leq \epsilon/2$, and we have:
$$k \geq n_0 \;\Rightarrow\; d(z_k, z_{k_{p_0}}) \leq \epsilon/2$$

5. From 4., we have found $n_0 \geq 1$, such that:
$$k \geq n_0 \;\Rightarrow\; d(z, z_k) \leq \epsilon$$
It follows that $z_k \to z$.

[13]Both with respect to $\mathcal{T}_{\mathbf{C}^n}$ and the induced topology $(\mathcal{T}_{\mathbf{C}^n})_{|B}$.

6. From 5., we see that every Cauchy sequence $(z_k)_{k\geq 1}$ in $\mathbf{C}^n$, converges to some limit $z \in \mathbf{C}^n$. From definition (80), we conclude that $\mathbf{C}^n$ is complete metric space.

7. The completeness of $\mathbf{C}$ was used in exercise (12)[6.] of Tutorial 9, leading to theorem (44) where we proved that any sequence $(f_n)_{n\geq 1}$ in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that:
$$\sum_{k=1}^{+\infty} \|f_{k+1} - f_k\|_p < +\infty$$
converges to some $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. This, in turn, was crucially important in proving theorem (46), where $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is shown to be complete.

Exercise 11

Exercise 12.

1. Let $(x_k)_{k\geq 1}$ be a sequence in $\mathbf{R}^n$, such that $x_k \to z$, for some $z \in \mathbf{C}^n$. For all $k \geq 1$ and $i \in \mathbf{N}_n$, we have:
$$|Im(z^i)| = |Im(z^i) - Im(x_k^i)| \leq \|z - x_k\|$$
Taking the limit as $k \to +\infty$, we obtain $Im(z^i) = 0$. This being true for all $i \in \mathbf{N}_n$, we have proved that $z \in \mathbf{R}^n$.

2. Let $(x_k)_{k\geq 1}$ be a Cauchy sequence in $\mathbf{R}^n$. In particular, it is a Cauchy sequence in $\mathbf{C}^n$. From exercise (11), $\mathbf{C}^n$ is a complete metric space. Hence, there exists $z \in \mathbf{C}^n$, such that $x_k \to z$. From 1., z is in fact an element of $\mathbf{R}^n$. We have proved that any Cauchy sequence $(x_k)_{k\geq 1}$ in $\mathbf{R}^n$, converges to some $z \in \mathbf{R}^n$. From definition (80), we conclude that $\mathbf{R}^n$ is a complete metric space. This, together with exercise (11), proves theorem (49).

Exercise 12

Exercise 13.

1. Let $x \in \bar{F}$. From definition (37), if U is an open set with $x \in U$, then $F \cap U \neq \emptyset$. Given $n \geq 1$, the open ball $B(x, 1/n)$ is an open set with $x \in B(x, 1/n)$. So $F \cap B(x, 1/n) \neq \emptyset$.

2. Let $x \in \bar{F}$. From 1., for all $n \geq 1$, we can choose an arbitrary element $x_n \in F \cap B(x, 1/n)$. This defines a sequence $(x_n)_{n\geq 1}$ of elements of F, such that $d(x, x_n) < 1/n$ for all $n \geq 1$. So $x_n \to x$.

3. Let $x \in E$. We assume that there exists a sequence $(x_n)_{n\geq 1}$ of elements of F, with $x_n \to x$. Let U be an open set containing x. Since $x_n \to x$, there exists $n_0 \geq 1$, such that:
$$n \geq n_0 \;\Rightarrow\; x_n \in U$$
In particular, $x_{n_0} \in U$. But x_{n_0} is also an element of F. So $x_{n_0} \in F \cap U$. We have proved that for all open set U containing x, we have $F \cap U \neq \emptyset$. From definition (37), we conclude that $x \in \bar{F}$.

4. Suppose that F is closed, and let $(x_n)_{n\geq 1}$ be a sequence in F such that $x_n \to x$ for some $x \in E$. From 3. we have $x \in \bar{F}$. However from exercise (21) of Tutorial 4, we have $F = \bar{F}$. So $x \in F$. Conversely, suppose that for any sequence $(x_n)_{n\geq 1}$ in F such that $x_n \to x$ for some $x \in E$, we have $x \in F$. We claim that F is closed. From exercise (21) of Tutorial 4., it is sufficient to show that $\bar{F} = F$, or equivalently that $\bar{F} \subseteq F$. So let $x \in \bar{F}$. From 2. there exists a sequence $(x_n)_{n\geq 1}$ in F such that $x_n \to x$. By assumption, this implies that $x \in F$. It follows that $\bar{F} \subseteq F$.

5. The fact that the induced topological space $(F, \mathcal{T}_{|F})$ is metrizable, is a consequence of theorem (12). The induced topology $\mathcal{T}_{|F}$ is nothing but the metric topology associated with the induced metric $d_{|F} = d_{|F\times F}$.

6. Suppose F is complete with respect to the induced metric $d_{|F}$. Let $x \in E$ and $(x_n)_{n\geq 1}$ be a sequence of elements of F, with $x_n \to x$. In particular, $(x_n)_{n\geq 1}$ is a Cauchy sequence with respect to the metric d. $(x_n)_{n\geq 1}$ being a sequence of elements of F, it is also a Cauchy sequence with respect to the induced metric $d_{|F}$. F being complete, there exists $y \in F$, such that $x_n \to y$. This convergence, with respect to $\mathcal{T}_{|F}$, is also valid with respect $\mathcal{T}$. Since we also have $x_n \to x$, we see that $x = y$. It follows that $x \in F$. Given $x \in E$, and a sequence $(x_n)_{n\geq 1}$ of elements of F such that $x_n \to x$, we have proved that $x \in F$. From 4., this shows that F is a closed subset of E. We conclude that if F is complete (with respect to its natural metric $d_{|F}$), then it is a closed subset of E.

7. From theorem (12), the induced metric $d' = (d_{\bar{\mathbf{R}}})_{|\mathbf{R}}$ induces the induced topology $(\mathcal{T}_{\bar{\mathbf{R}}})_{|\mathbf{R}}$. Such topology is nothing but the usual topology on $\mathbf{R}$. It follows that d' induces $\mathcal{T}_{\mathbf{R}}$.

8. Let $d_{\mathbf{R}}$ be the usual metric on $\mathbf{R}$. From theorem (12), the induced metric $(d_{\mathbf{R}})_{|[-1,1]}$ induces the induced topology on $[-1,1]$. Such topology is nothing but the usual topology on $[-1,1]$.

9. From 8., if $\{-1,1\}$ was open in $[-1,1]$, there would exists $\epsilon > 0$, such that $]1-\epsilon, 1] \subseteq \{-1,1\}$, which is absurd.

10. If $\{-\infty,+\infty\}$ was open in $\bar{\mathbf{R}}$, then $\{-1,1\}$ would be open in $[-1,1]$, since one is the inverse image of the other, by a strictly increasing homeomorphism.

11. If $\mathbf{R}$ was closed in $\bar{\mathbf{R}}$, then $\{-\infty,+\infty\}$ would be open in $\bar{\mathbf{R}}$.

12. Let $d_{\mathbf{R}}$ be the usual metric on $\mathbf{R}$. Then $d_{\mathbf{R}}$ induces the usual topology on $\mathbf{R}$. However, from 7., the metric d' also induces the usual topology on $\mathbf{R}$. It follows that $d_{\mathbf{R}}$ and d' both induce the same topology. From theorem (49), $\mathbf{R}$ is complete with respect to its usual metric $d_{\mathbf{R}}$. If $\mathbf{R}$ was complete with respect to $d' = (d_{\bar{\mathbf{R}}})_{|\mathbf{R}}$, then from 6., $\mathbf{R}$ would be a closed subset of $\bar{\mathbf{R}}$, contradicting 11. So $\mathbf{R}$ is not complete with respect to d'.

We conclude that although the two metric spaces $(\mathbf{R}, d_{\mathbf{R}})$ and $(\mathbf{R}, d')$ are identical in the topological sense, one is complete whereas the other is not.

Exercise 13

Exercise 14.

1. Let $y \in \mathcal{H}$. For all $x, x' \in \mathcal{H}$ and $\alpha \in \mathbf{K}$, using (ii) and (iii) of definition (81), we obtain:
$$\langle x + \alpha x', y\rangle = \langle x, y\rangle + \alpha\langle x', y\rangle$$
We conclude that $x \to \langle x, y\rangle$ is linear for all $y \in \mathcal{H}$.

2. Let $x \in \mathcal{H}$. For all $y, y' \in \mathcal{H}$ and $\alpha \in \mathbf{K}$, using (i), (ii) and (iii) of definition (81), we obtain:
$$\langle x, y + \alpha y'\rangle = \langle x, y\rangle + \bar{\alpha}\langle x, y'\rangle$$
where $\bar{\alpha}$ is the complex conjugate of α. Hence, $y \to \langle x, y\rangle$ is conjugate-linear for all $x \in \mathcal{H}$. In the case when $\mathbf{K} = \mathbf{R}$, it is in fact linear.

Exercise 14

Exercise 15.

1. The inner-product $\langle\cdot,\cdot\rangle$ has values in $\mathbf{K}$. From (iv) of definition (81), $\langle x, x\rangle \geq 0$ for all $x \in \mathcal{H}$. It follows that $\|x\| = \sqrt{\langle x, x\rangle}$ is a well-defined element of $\mathbf{R}^+$, for all $x \in \mathcal{H}$. Hence, we see that $A = \|x\|^2$ and $C = \|y\|^2$ are both well-defined elements of $\mathbf{R}^+$. Furthermore, $B = |\langle x, y\rangle|$ being the modulus of an element of $\mathbf{K}$, is a well-defined element of $\mathbf{R}^+$.

2. Let $t \in \mathbf{R}$. Using the linearity properties of exercise (14):
$$\langle x - t\alpha y, x - t\alpha y\rangle = \langle x, x\rangle - t\alpha\overline{\langle x, y\rangle} - t\bar{\alpha}\langle x, y\rangle + t^2\alpha\bar{\alpha}\langle y, y\rangle$$
Since $B = \bar{B} = \alpha\overline{\langle x, y\rangle}$ and $\alpha\bar{\alpha} = 1$, we conclude that:
$$\langle x - t\alpha y, x - t\alpha y\rangle = A - 2tB + t^2C$$

3. Suppose $C = 0$. Then $\langle y, y\rangle = 0$. From (v) of definition (81), we see that $y = 0$. From the conjugate linearity of $y' \to \langle x, y'\rangle$, we have $\langle x, 0\rangle = 0$ for all $x \in \mathcal{H}$, and consequently $\langle x, y\rangle = 0$. So $B = 0$, and finally $B^2 \leq AC$.

4. Suppose $C \neq 0$. Let $P(t) = A - 2tB + t^2C$ for all $t \in \mathbf{R}$. Since $C > 0$ and $P'(t) = 2tC - 2B$, the second degree polynomial P has a minimum value at $t = B/C$. From 2., for all $t \in \mathbf{R}$:
$$P(t) = \langle x - t\alpha y, x - t\alpha y\rangle \geq 0$$
In particular, $P(B/C) \geq 0$. It follows that $B^2 \leq AC$.

5. From $B^2 \leq AC$, since $A, B, C \in \mathbf{R}^+$, we obtain $B \leq \sqrt{AC}$, i.e.
$$|\langle x, y\rangle| \leq \|x\|.\|y\|$$
This proves theorem (50).

Exercise 15

Exercise 16.

1. Let $f, g \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Then, $f\bar{g}$ is a complex-valued and measurable map. Furthermore, from theorem (42):

$$\int |f||g|d\mu \leq \left(\int |f|^2 d\mu\right)^{\frac{1}{2}} \left(\int |g|^2 d\mu\right)^{\frac{1}{2}}$$

So $\int |f\bar{g}|d\mu < +\infty$ and $f\bar{g} \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. It follows that $\langle f, g\rangle = \int f\bar{g}d\mu$ is a well-defined complex number.

2. Let $f \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. From definition (73), $\|f\|_2$ is defined as $\|f\|_2 = (\int |f|^2 d\mu)^{1/2}$. It follows that:

$$\|f\|_2 = \left(\int f\bar{f}d\mu\right)^{\frac{1}{2}} = \sqrt{\langle f, f\rangle}$$

3. Let $f, g \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. From theorems (24) and (42) , we have:

$$|\langle f, g\rangle| = \left|\int f\bar{g}d\mu\right| \leq \int |f||g|d\mu \leq \|f\|_2.\|g\|_2$$

4. Among properties $(i) - (v)$ of definition (81), only (v) fails to be satisfied. Indeed, although $f = 0$ does imply that $\langle f, f\rangle = \int |f|^2 d\mu = 0$, the converse is not true. Having $\int |f|^2 d\mu = 0$ only guarantees that $f = 0$ μ-almost surely, and not necessarily everywhere. We conclude that $\langle \cdot, \cdot\rangle$ is not strictly speaking an inner-product on $L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, as defined by definition (81). It follows that equation (1) which we proved in 3., cannot be viewed as a consequence of theorem (50).

5. Let $f, g \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Let $P(t) = \int (|f| + t|g|)^2 d\mu$ for all $t \in \mathbf{R}$. Then, $P(t) \geq 0$ for all $t \in \mathbf{R}$, and furthermore:

$$P(t) = A + 2tB + t^2 C$$

where $A = \int |f|^2 d\mu$, $B = \int |f||g|d\mu$ and $C = \int |g|^2 d\mu$. All three numbers A, B and C are elements of $\mathbf{R}^+$.[14] If $C = 0$, then $g = 0$ μ-a.s. and consequently $B = 0$. In particular, the inequality $B^2 \leq AC$ holds. If $C \neq 0$, from $P(-B/C) \geq 0$ we obtain $B^2 \leq AC$, and consequently:

$$\int |fg|d\mu \leq \left(\int |f|^2 d\mu\right)^{\frac{1}{2}} \left(\int |g|^2 d\mu\right)^{\frac{1}{2}}$$

6. Let $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be non-negative and measurable. Suppose both integrals $\int f^2 d\mu$ and $\int g^2 d\mu$ are finite. Then f and g are μ-almost surely finite, and therefore μ-almost surely equal to $f1_{\{f<+\infty\}}$ and $g1_{\{g<+\infty\}}$

[14] B can be shown to be finite from $|fg| \leq (|f|^2 + |g|^2)/2$.

respectively. It follows that f and g are μ-almost surely equal to elements of $L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Applying 5. to $f1_{\{f<+\infty\}}$ and $g1_{\{g<+\infty\}}$, we obtain:

$$\int fgd\mu \leq \left(\int f^2 d\mu\right)^{\frac{1}{2}} \left(\int g^2 d\mu\right)^{\frac{1}{2}}$$

If $\int f^2 d\mu = +\infty$ or $\int g^2 d\mu = +\infty$, such inequality still holds. We have effectively proved theorem (42), without using holder inequality (41).

Exercise 16

Exercise 17.

1. Let $x, y \in \mathcal{H}$. Using (ii) of definition (81), we have:

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle$$

Furthermore, using (i) and (ii):

$$\langle x, x+y\rangle = \overline{\langle x+y, x\rangle} = \overline{\langle x, x\rangle} + \overline{\langle y, x\rangle} = \|x\|^2 + \langle x, y\rangle$$

and also:

$$\langle y, x+y\rangle = \overline{\langle x+y, y\rangle} = \|y\|^2 + \overline{\langle x, y\rangle}$$

We conclude that:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle}$$

2. From the Cauchy-Schwarz inequality of theorem (50):

$$|\overline{\langle x, y\rangle}| = |\langle x, y\rangle| \leq \|x\|.\|y\|$$

Consequently, using 1., we have:

$$\|x+y\|^2 \leq \|x\|^2 + \|y\|^2 + 2\|x\|.\|y\| = (\|x\| + \|y\|)^2$$

We conclude that for all $x, y \in \mathcal{H}$, we have:

$$\|x+y\| \leq \|x\| + \|y\|$$

3. Let $d = d_{\langle\cdot,\cdot\rangle}$ be the map defined by $d(x,y) = \|x-y\|$. Note that from (iv) of definition (81):

$$d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle}$$

is well-defined, and non-negative. So d is indeed a map from $\mathcal{H} \times \mathcal{H}$, with values in $\mathbf{R}^+$. Let $x, y, z \in \mathcal{H}$. $d(x,y) = 0$ is equivalent to $\langle x-y, x-y\rangle = 0$, which from (v) of definition (81), is itself equivalent to $x = y$. So (i) of definition (28) is satisfied by d. Furthermore, we have:

$$\|-x\|^2 = \langle -x, -x\rangle = -\overline{\langle -x, x\rangle} = \|x\|^2$$

and consequently, $d(x,y) = \|x-y\| = \|y-x\| = d(y,x)$. So (ii) of definition (28) is satisfied by d. Finally, using 2.:

$$\|x-y\| = \|x-z+z-y\| \leq \|x-z\| + \|z-y\|$$

and we see that $d(x,y) \leq d(x,z) + d(z,y)$. So (iii) of definition (28) is also satisfied by d. Having checked conditions (i), (ii) and (iii) of definition (28), we conclude that d is indeed a metric on $\mathcal{H}$.

Exercise 17

Exercise 18.

1. $\mathcal{M}$ being a linear subspace of the $\mathbf{K}$-vector space $\mathcal{H}$, is itself a $\mathbf{K}$-vector space. $[\cdot,\cdot]$ being the restriction of $\langle\cdot,\cdot\rangle$ to $\mathcal{M} \times \mathcal{M}$, is indeed a map $[\cdot,\cdot] : \mathcal{M} \times \mathcal{M} \to K$. For all $x, y \in \mathcal{M}$, we have:

$$[x,y] = \langle x,y\rangle = \overline{\langle y,x\rangle} = \overline{[y,x]}$$

So (i) of definition (81) is satisfied by $[\cdot,\cdot]$. Similarly, it is clear that all properties $(ii)-(v)$ of definition (81) are also satisfied by $[\cdot,\cdot]$. We conclude that $[\cdot,\cdot]$ is indeed an inner-product on the $\mathbf{K}$-vector space $\mathcal{M}$.

2. Recall that from definition (83), the metric $d_{[\cdot,\cdot]}$ is defined by:

$$d_{[\cdot,\cdot]}(x,y) = \sqrt{[x-y, x-y]}$$

$[\cdot,\cdot]$ being the restriction of $\langle\cdot,\cdot\rangle$ to $\mathcal{M} \times \mathcal{M}$, we have:

$$d_{[\cdot,\cdot]}(x,y) = \sqrt{\langle x-y, x-y\rangle} = d_{\langle\cdot,\cdot\rangle}(x,y)$$

We conclude that the metric $d_{[\cdot,\cdot]}$ is nothing but the restriction of the metric $d_{\langle\cdot,\cdot\rangle}$ to $\mathcal{M} \times \mathcal{M}$, i.e. $d_{[\cdot,\cdot]} = (d_{\langle\cdot,\cdot\rangle})_{|\mathcal{M}\times\mathcal{M}}$.

3. From theorem (12), the topology induced on $\mathcal{M}$ by the norm topology $\mathcal{T}_{\langle\cdot,\cdot\rangle}$ (the latter being the metric topology associated with $d_{\langle\cdot,\cdot\rangle}$, by definition (82)), is nothing but the metric topology associated with $(d_{\langle\cdot,\cdot\rangle})_{\mathcal{M}\times\mathcal{M}} = d_{[\cdot,\cdot]}$ (which by definition (82), is the norm topology on $\mathcal{M}$, i.e. $\mathcal{T}_{[\cdot,\cdot]}$). So $(\mathcal{T}_{\langle\cdot,\cdot\rangle})_{|\mathcal{M}} = \mathcal{T}_{[\cdot,\cdot]}$.

Exercise 18

Exercise 19.

1. Since $(x_n)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{M}$, with respect to the metric $d_{[\cdot,\cdot]}$, from definition (79), for all $\epsilon > 0$, there exists an integer $n_0 \geq 1$, such that:

$$n, m \geq n_0 \ \Rightarrow\ d_{[\cdot,\cdot]}(x_n, x_m) \leq \epsilon$$

However, since $d_{[\cdot,\cdot]}$ is the restriction of $d_{\langle\cdot,\cdot\rangle}$ to $\mathcal{M}\times\mathcal{M}$, we have $d_{[\cdot,\cdot]}(x,y) = d_{\langle\cdot,\cdot\rangle}(x,y)$ for all $x, y \in \mathcal{M}$. It follows that $(x_n)_{n\geq 1}$ is also a Cauchy sequence in $\mathcal{H}$, with respect to the metric $d_{\langle\cdot,\cdot\rangle}$.

2. $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ being a Hilbert space, from definition (83), $\mathcal{H}$ is a also a complete metric space. From definition (80), $(x_n)_{n\geq 1}$ being a Cauchy sequence in $\mathcal{H}$, there exists $x \in \mathcal{H}$ such that $x_n \to x$.

3. $\mathcal{M}$ is a closed subset of $\mathcal{H}$, and $(x_n)_{n\geq 1}$ is a sequence of elements of $\mathcal{M}$ converging to $x \in \mathcal{H}$. From exercise (13) [4.], we conclude that $x \in \mathcal{M}$.

4. As seen in the previous exercise, the norm topology $\mathcal{T}_{[\cdot,\cdot]}$ on $\mathcal{M}$ is induced by the norm topology $\mathcal{T}_{\langle\cdot,\cdot\rangle}$ on $\mathcal{H}$. Since $(x_n)_{n\geq 1}$ is a sequence in $\mathcal{M}$ and $x \in \mathcal{M}$, the convergence $x_n \to x$ relative to the topology $\mathcal{T}_{[\cdot,\cdot]}$, is equivalent to the convergence $x_n \to x$ relative to the topology $\mathcal{T}_{\langle\cdot,\cdot\rangle}$.

5. Given a Cauchy sequence $(x_n)_{n\geq 1}$ in $\mathcal{M}$, we have found an element $x \in \mathcal{M}$, such that $x_n \to x$. From definition (80), this shows that $(\mathcal{M}, d_{[\cdot,\cdot]})$ is a complete metric space. It follows that $\mathcal{M}$ is a $\mathbf{K}$-vector space, that $[\cdot,\cdot]$ is an inner-product on $\mathcal{M}$, under which $\mathcal{M}$ is complete. From definition (83), we conclude that $(\mathcal{M}, [\cdot,\cdot]) = (\mathcal{M}, \langle\cdot,\cdot\rangle_{|\mathcal{M}\times\mathcal{M}})$ is a Hilbert space over $\mathbf{K}$. The purpose of this exercise is to show that any closed linear subspace of a Hilbert space, is itself a Hilbert space, together with its restricted inner-product.

Exercise 19

Exercise 20.

1. Let $z, z', z'' \in \mathbf{C}^n$ and $\alpha \in \mathbf{C}$. We have:

$$\langle z, z'\rangle = \sum_{i=1}^n z_i \bar{z}_i{}' = \overline{\sum_{i=1}^n \bar{z}_i z_i'} = \overline{\langle z', z\rangle}$$

$$\langle z + z', z''\rangle = \sum_{i=1}^n (z_i + z_i')\bar{z}_i{}'' = \langle z, z''\rangle + \langle z', z''\rangle$$

$$\langle \alpha z, z'\rangle = \sum_{i=1}^n (\alpha z_i)\bar{z}_i{}' = \alpha\langle z, z'\rangle$$

$$\langle z, z\rangle = \sum_{i=1}^n z_i \bar{z}_i = \sum_{i=1}^n |z_i|^2 \geq 0$$

and finally, $\langle z, z\rangle = 0$ is equivalent to $z_i = 0$ for all $i \in \mathbf{N}_n$, itself equivalent to $z = 0$. Hence, we see that all five conditions $(i) - (v)$ of definition (81) are satisfied by $\langle\cdot,\cdot\rangle$. So $\langle\cdot,\cdot\rangle$ is indeed an inner-product on $\mathbf{C}^n$.

2. The metric $d_{\langle\cdot,\cdot\rangle}$ is defined by:

$$d_{\langle\cdot,\cdot\rangle}(z, z') = \sqrt{\langle z - z', z - z'\rangle} = \sqrt{\sum_{i=1}^n |z_i - z_i'|^2}$$

It therefore coincides with the usual metric on $\mathbf{C}^n$.

3. From theorem (49), $\mathbf{C}^n$ is a complete metric space, with respect to its usual metric. The latter being the same as the metric $d_{\langle\cdot,\cdot\rangle}$, we conclude from definition (83) that $(\mathbf{C}^n, \langle\cdot,\cdot\rangle)$ is a Hilbert space over $\mathbf{C}$.

4. For all $i \in \mathbf{N}_n$, let $\phi_i : \mathbf{C}^n \to \mathbf{R}$ be defined by $\phi_i(z) = Im(z_i)$. For all $z, z' \in \mathbf{C}^n$, we have:
$$|\phi_i(z) - \phi_i(z')| = |Im(z_i - z_i')| \leq \|z - z'\| = d_{\mathbf{C}^n}(z, z')$$
So each ϕ_i is a continuous map. The set $\{0\}$ being a closed subset of $\mathbf{R}$, the inverse image $\phi_i^{-1}(\{0\})$ is a closed subset of $\mathbf{C}^n$. It follows that $\mathbf{R}^n = \cap_{i=1}^n \phi_i^{-1}(\{0\})$ as an intersection of closed subsets of $\mathbf{C}^n$, is itself a closed subset of $\mathbf{C}^n$.

5. Given $x \in \mathbf{R}^n$ and $\alpha \in \mathbf{C}$, $\alpha.x$ is not in general an element of $\mathbf{R}^n$. So $\mathbf{R}^n$ is not a linear subspace of $\mathbf{C}^n$. It is of course an $\mathbf{R}$-vector space...

6. Since $\mathbf{R}^n$ is not a linear subspace of $\mathbf{C}^n$, we cannot rely on exercise (19) to argue that $(\mathbf{R}^n, \langle\cdot,\cdot\rangle)$ is a Hilbert space. In fact, we want to show that $\mathbf{R}^n$ is a Hilbert space over $\mathbf{R}$, (not $\mathbf{C}$), so exercise (19) is no good to us... However, the restriction of $\langle\cdot,\cdot\rangle$ to $\mathbf{R}^n \times \mathbf{R}^n$ also satisfies conditions $(i)-(v)$ of definition (81), and is therefore an inner-product on $\mathbf{R}^n$, which furthermore induces the usual metric on $\mathbf{R}^n$. Since from theorem (49), $\mathbf{R}^n$ is complete with respect to its usual metric, we conclude from definition (83) that it is a Hilbert space over $\mathbf{R}$.

Exercise 20

Exercise 21.

1. Since $\mathcal{C} \neq \emptyset$, there exists $y \in \mathcal{C}$. From $\delta_{\min} \leq \|y - x_0\|$, we obtain $\delta_{\min} < +\infty$. In particular, $\delta_{\min} < \delta_{\min} + 1/n$ for all $n \geq 1$. $\delta_{\min}$ being the greatest of all lower-bound of $\|x - x_0\|$ for $x \in \mathcal{C}$, it follows that $\delta_{\min} + 1/n$ cannot be such lower-bound. There exists $x_n \in \mathcal{C}$, such that $\|x_n - x_0\| < \delta_{\min} + 1/n$. This being true for all $n \geq 1$, we have found a sequence $(x_n)_{n\geq 1}$ in $\mathcal{C}$, such that $\delta_{\min} \leq \|x_n - x_0\| < \delta_{\min} + 1/n$, for all $n \geq 1$. In particular, $\|x_n - x_0\| \to \delta_{\min}$.

2. For all $x, y \in \mathcal{H}$:
$$\|x - y\|^2 = \langle x - y, x - y\rangle = \|x\|^2 + \|y\|^2 - \langle x, y\rangle - \overline{\langle x, y\rangle}$$
$$\|x + y\|^2 = \langle x + y, x + y\rangle = \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle}$$
and therefore:
$$\|x - y\|^2 + \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2$$
or equivalently:
$$\|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 - 4\left\|\frac{x + y}{2}\right\|^2 \tag{6}$$

3. Let $n, m \geq 1$. x_n and x_m are both elements of $\mathcal{C}$. Since we have $1/2 \in [0, 1]$, from definition (85), $\mathcal{C}$ being convex, $(x_n + x_m)/2$ is also an element of $\mathcal{C}$. Since $\delta_{\min}$ is a lower-bound of $\|x - x_0\|$ for $x \in \mathcal{C}$, we conclude that:
$$\delta_{\min} \leq \left\|\frac{x_n + x_m}{2} - x_0\right\| \tag{7}$$

4. Let $n, m \geq 1$. Applying (6) to $x = x_n - x_0$ and $y = x_m - x_0$:

$$\|x_n - x_m\|^2 = 2\|x_n - x_0\|^2 + 2\|x_m - x_0\|^2 - 4\left\|\frac{x_n + x_m}{2} - x_0\right\|^2$$

and therefore, from (7):

$$\|x_n - x_m\|^2 \leq 2\|x_n - x_0\|^2 + 2\|x_m - x_0\|^2 - 4\delta_{\min}^2 \qquad (8)$$

5. Let $\epsilon > 0$. Since $(x_n)_{n\geq 1}$ is such that $\|x_n - x_0\| \to \delta_{\min}$, in particular, there exists $N \geq 1$ such that:

$$n \geq N \;\Rightarrow\; 2\|x_n - x_0\|^2 \leq 2\delta_{\min}^2 + \epsilon^2/2$$

Using (8), we have:

$$n, m \geq N \;\Rightarrow\; \|x_n - x_m\|^2 \leq \epsilon^2$$

It follows from definition (79) that $(x_n)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{H}$. Since $\mathcal{H}$ is a Hilbert space, it is also a complete metric space. So $(x_n)_{n\geq 1}$ has a limit in $\mathcal{H}$. There exists $x^* \in \mathcal{H}$, such that $x_n \to x^*$[15].

6. From 5., we have $x_n \to x^*$, while $(x_n)_{n\geq 1}$ is a sequence of elements of $\mathcal{C}$. Since by assumption, $\mathcal{C}$ is a closed subset of $\mathcal{H}$, using exercise (13) [4.], we conclude that $x^* \in \mathcal{C}$.

7. Let $x, y \in \mathcal{H}$. From exercise (17), we have:

$$\|x\| \leq \|x - y\| + \|y\|$$
$$\|y\| \leq \|x - y\| + \|x\|$$

where we have used the fact that $\|x - y\| = \|y - x\|$. Hence:

$$-\|x - y\| \leq \|x\| - \|y\| \leq \|x - y\|$$

or equivalently $|\,\|x\| - \|y\|\,| \leq \|x - y\|$.

8. For all $n \geq 1$, from 7., we have:

$$|\,\|x_n - x_0\| - \|x^* - x_0\|\,| \leq \|x^* - x_n\|$$

Since $x_n \to x^*$, $\|x^* - x_n\| \to 0$, and so $\|x_n - x_0\| \to \|x^* - x_0\|$.

9. By construction, $(x_n)_{n\geq 1}$ is such that $\|x_n - x_0\| \to \delta_{\min}$. However, from 8., $\|x_n - x_0\| \to \|x^* - x_0\|$. So $\|x^* - x_0\| = \delta_{\min}$. Since $x^* \in \mathcal{C}$, we have found $x^* \in \mathcal{C}$, such that:

$$\|x^* - x_0\| = \inf\{\|x - x_0\| : x \in \mathcal{C}\}$$

---

[15] Convergence relative to the norm topology, so $x_n \stackrel{\mathcal{T}_{\langle\cdot,\cdot\rangle}}{\to} x^*$.

10. Suppose y^* is another element of $\mathcal{C}$, such that:

$$\|y^* - x_0\| = \inf\{\|x - x_0\| : x \in \mathcal{C}\}$$

Applying (6) to $x = x^* - x_0$ and $y = y^* - x_0$, we obtain:

$$\|x^* - y^*\|^2 = 2\|x^* - x_0\|^2 + 2\|y^* - x_0\|^2 - 4\left\|\frac{x^* + y^*}{2} - x_0\right\|^2$$

Since $\mathcal{C}$ is convex and x^*, y^* are elements of $\mathcal{C}$, $(x^* + y^*)/2$ is also an element of $\mathcal{C}$. It follows that:

$$\delta_{\min} \leq \left\|\frac{x^* + y^*}{2} - x_0\right\|$$

and finally $\|x^* - y^*\|^2 \leq 2\|x^* - x_0\|^2 + 2\|y^* - x_0\|^2 - 4\delta_{\min}^2$.

11. Since $\delta_{\min} = \|x^* - x_0\| = \|y^* - x_0\|$, we see from 10. that $\|x^* - y^*\| = 0$, and finally $x^* = y^*$. This proves theorem (52).

Exercise 21

Exercise 22.

1. For all $y \in \mathcal{G}$, $\langle 0, y\rangle = 0.\langle 0, y\rangle = 0$. So $0 \in \mathcal{G}^\perp$ and in particular $\mathcal{G}^\perp \neq \emptyset$. Let $x_1, x_2 \in \mathcal{G}^\perp$ and $\alpha \in \mathbf{K}$. For all $y \in \mathcal{G}$, we have $\langle x_1, y\rangle = 0$ and $\langle x_2, y\rangle = 0$. Hence:

$$\langle x_1 + \alpha x_2, y\rangle = \langle x_1, y\rangle + \alpha\langle x_2, y\rangle = 0$$

This being true for all $y \in \mathcal{G}$, $x_1 + \alpha x_2 \in \mathcal{G}^\perp$. We conclude that $\mathcal{G}^\perp$ is a linear sub-space of $\mathcal{H}$. Note that no assumption was made, as to whether $\mathcal{G}$ is itself a linear sub-space or not.

2. Given $y \in \mathcal{H}$, let $\phi_y : \mathcal{H} \to \mathbf{K}$ be defined by $\phi_y(x) = \langle x, y\rangle$. From the Cauchy-Schwarz inequality of theorem (50), if $x_1, x_2 \in \mathcal{H}$, we have $|\phi_y(x_1) - \phi_y(x_2)| = |\langle x_1 - x_2, y\rangle| \leq \|y\|.\|x_1 - x_2\|$ or equivalently $d_{\mathbf{K}}(\phi_y(x_1), \phi_y(x_2)) \leq \|y\|.d_{\langle\cdot,\cdot\rangle}(x_1, x_2)$, where $d_{\mathbf{K}}$ is the usual metric on $\mathbf{K}$. It follows that $\phi_y : \mathcal{H} \to \mathbf{K}$ is a continuous map, with respect to the norm topology on $\mathcal{H}$, and the usual topology on $\mathbf{K}$.

3. Suppose $x \in \mathcal{G}^\perp$. For all $y \in \mathcal{G}$, we have $\langle x, y\rangle = 0 = \phi_y(x)$. So $x \in \cap_{y\in\mathcal{G}}\phi_y^{-1}(\{0\})$. Conversely, if $x \in \cap_{y\in\mathcal{G}}\phi_y^{-1}(\{0\})$, then for all $y \in \mathcal{G}$, we have $\phi_y(x) = 0 = \langle x, y\rangle$, and therefore $x \in \mathcal{G}^\perp$. This proves that $\mathcal{G}^\perp = \cap_{y\in\mathcal{G}}\phi_y^{-1}(\{0\})$.

4. The set $\{0\}$ is a closed subset of $\mathbf{K}$. Since $\phi_y : \mathcal{H} \to \mathbf{K}$ is a continuous map for all $y \in \mathcal{H}$, the inverse image $\phi_y^{-1}(\{0\})$ is a closed subset of $\mathcal{H}$. From 3., $\mathcal{G}^\perp$ being an arbitrary intersection of closed subsets of $\mathcal{H}$, we conclude that $\mathcal{G}^\perp$ is itself a closed subset of $\mathcal{H}$.

5. $\emptyset^{\perp} \subseteq \mathcal{H}$ and $\{0\}^{\perp} \subseteq \mathcal{H}$ are obviously true. Furthermore, a statement such that $[\forall y \in \emptyset, \langle x, y\rangle = 0]$ is also true for any $x \in \mathcal{H}$. So $\mathcal{H} \subseteq \emptyset^{\perp}$. Moreover, for all $x \in \mathcal{H}$, $\langle x, 0\rangle = 0$, i.e. $x \in \{0\}^{\perp}$. So $\mathcal{H} \subseteq \{0\}^{\perp}$. We have proved that $\mathcal{H} = \emptyset^{\perp} = \{0\}^{\perp}$.

6. For all $y \in \mathcal{H}$, $\langle 0, y\rangle = 0$. So $\{0\} \subseteq \mathcal{H}^{\perp}$. Conversely, if $x \in \mathcal{H}^{\perp}$, then $\langle x, x\rangle = 0$ and therefore $x = 0$. So $\mathcal{H}^{\perp} \subseteq \{0\}$.

Exercise 22

Exercise 23.

1. $\mathcal{M}$ being a linear sub-space of $\mathcal{H}$, it has at least one element, namely 0. So $\mathcal{M} \neq \emptyset$. Furthermore, for all $x, y \in \mathcal{M}$ and $\alpha, \beta \in \mathbf{K}$, we have $\alpha x + \beta y \in \mathcal{M}$. In particular, for all $t \in [0, 1]$, $tx + (1-t)y \in \mathcal{M}$. From definition (85), it follows that $\mathcal{M}$ is also a convex subset of $\mathcal{H}$. Having assumed $\mathcal{M}$ to be closed, it is therefore a non-empty, closed and convex subset of $\mathcal{H}$. Applying theorem (52), there exists $x^* \in \mathcal{M}$ such that:
$$\|x^* - x_0\| = \inf\{\|x - x_0\| : x \in \mathcal{M}\}$$

2. Let $y^* = x_0 - x^*$. Since $x^* \in \mathcal{M}$, for all $y \in \mathcal{M}$ and $\alpha \in \mathbf{K}$, $x^* + \alpha y$ is also an element of $\mathcal{M}$. It follows that:
$$\|x^* - x_0\| \leq \|x^* + \alpha y - x_0\|$$
or equivalently:
$$\|y^*\|^2 \leq \|y^* - \alpha y\|^2 \tag{9}$$

3. Let $y \in \mathcal{M}$ and $\alpha \in \mathbf{K}$. We have:
$$\|y^* - \alpha y\|^2 = \|y^*\|^2 - \alpha\langle y, y^*\rangle - \overline{\alpha\langle y, y^*\rangle} + |\alpha|^2\|y\|^2$$
Hence, using (9), we obtain:
$$0 \leq -\alpha\langle y, y^*\rangle - \overline{\alpha\langle y, y^*\rangle} + |\alpha|^2\|y\|^2 \tag{10}$$

4. Given $y \in \mathcal{M} \setminus \{0\}$, take $\alpha = \overline{\langle y, y^*\rangle}/\|y\|^2$ in (10). We obtain:
$$0 \leq -\frac{|\langle y, y^*\rangle|^2}{\|y\|^2}$$

5. It follows from 4. that $|\langle y, y^*\rangle|^2 \leq 0$ for all $y \in \mathcal{M} \setminus \{0\}$. So $\langle y^*, y\rangle = \langle y, y^*\rangle = 0$, for all $y \in \mathcal{M} \setminus \{0\}$. Since $\langle y^*, 0\rangle = 0$, we in fact have $\langle y^*, y\rangle = 0$ for all $y \in \mathcal{M}$, and we see that $y^* \in \mathcal{M}^{\perp}$. So $x^* \in \mathcal{M}$, $y^* \in \mathcal{M}^{\perp}$, and since $y^* = x_0 - x^*$, we conclude that $x_0 = x^* + y^*$.

6. $\mathcal{M}$ and $\mathcal{M}^{\perp}$ being linear sub-spaces of $\mathcal{H}$, 0 is an element of both $\mathcal{M}$ and $\mathcal{M}^{\perp}$. So $\{0\} \subseteq \mathcal{M} \cap \mathcal{M}^{\perp}$. Conversely, suppose $x \in \mathcal{M} \cap \mathcal{M}^{\perp}$. From $x \in \mathcal{M}^{\perp}$, we have $\langle x, y\rangle = 0$ for all $y \in \mathcal{M}$. From $x \in \mathcal{M}$, we see in particular that $\langle x, x\rangle = 0$. From (v) of definition (81), we conclude that $x = 0$. So $\mathcal{M} \cap \mathcal{M}^{\perp} = \{0\}$.

7. Suppose there exist $\bar{x} \in \mathcal{M}$ and $\bar{y} \in \mathcal{M}^{\perp}$, such that $x_0 = \bar{x} + \bar{y}$. Then $x^* + y^* = \bar{x} + \bar{y}$ and consequently $x^* - \bar{x} = \bar{y} - y^*$, while $x^* - \bar{x} \in \mathcal{M}$ and $\bar{y} - y^* \in \mathcal{M}^{\perp}$. Since $\mathcal{M} \cap \mathcal{M}^{\perp} = \{0\}$, we conclude that $x^* = \bar{x}$ and $y^* = \bar{y}$. So $x^* \in \mathcal{M}$ and $y^* \in \mathcal{M}^{\perp}$ such that $x_0 = x^* + y^*$ are unique. This proves theorem (53).

Exercise 23

Exercise 24.

1. Let $\lambda : \mathcal{H} \to \mathbf{K}$ be a linear functional, which is continuous at $x_0 \in \mathcal{H}$[16]. Given an open set V in $\mathbf{K}$ containing $\lambda(x_0)$, there exists an open set U in $\mathcal{H}$ containing x_0, such that $f(U) \subseteq V$. Since the two topologies on $\mathcal{H}$ and $\mathbf{K}$ are metric, this is easily shown to be equivalent to the property that for all $\epsilon > 0$, there exists $\delta > 0$, such that:

$$\forall x \in \mathcal{H} \ , \ \|x - x_0\| < \delta \ \Rightarrow \ |\lambda(x) - \lambda(x_0)| < \epsilon$$

In particular, taking $\epsilon = 1$ and some $\eta > 0$ strictly smaller than the associated δ, we have:

$$\forall x \in \mathcal{H} \ , \ \|x - x_0\| \leq \eta \ \Rightarrow \ |\lambda(x) - \lambda(x_0)| \leq 1$$

Hence, given $x \in \mathcal{H}$, $x \neq 0$, we have:

$$|\lambda(\eta x/\|x\|)| = |\lambda(x_0 + \eta x/\|x\|) - \lambda(x_0)| \leq 1$$

2. If λ is continuous at some $x_0 \in \mathcal{H}$, from 1., there exists $\eta > 0$ such that $|\lambda(\eta x/\|x\|)| \leq 1$ for all $x \in \mathcal{H} \setminus \{0\}$. So $|\lambda(x)| \leq \|x\|/\eta$ for all $x \in \mathcal{H} \setminus \{0\}$, which is obviously still valid if $x = 0$. We have found $M = 1/\eta \in \mathbf{R}^+$, such that:

$$\forall x \in \mathcal{H} \ , \ |\lambda(x)| \leq M\|x\| \tag{11}$$

3. Suppose $\lambda : \mathcal{H} \to \mathbf{K}$ is a linear functional, such that (11) holds for some $M \in \mathbf{R}^+$. Then for all $x_1, x_2 \in \mathcal{H}$, we have:

$$|\lambda(x_1) - \lambda(x_2)| = |\lambda(x_1 - x_2)| \leq M\|x_1 - x_2\|$$

So λ is continuous (everywhere).

Exercise 24

Exercise 25.

1. Let $x_0 \in \mathcal{H}$ such that $\lambda(x_0) \neq 0$. Then $x_0 \notin \mathcal{M} = \lambda^{-1}(\{0\})$.

2. $\mathcal{M} = \lambda^{-1}(\{0\})$ is a linear sub-space of $\mathcal{H}$. Indeed, it is not empty ($\lambda(0) = 0$), and if $\lambda(x_1) = \lambda(x_2) = 0$ and $\alpha \in \mathbf{K}$, then:

$$\lambda(x_1 + \alpha x_2) = \lambda(x_1) + \alpha\lambda(x_2) = 0$$

[16] Continuity at a given point is defined in what follows.

Furthermore, λ being a bounded linear functional, is continuous, and $\mathcal{M} = \lambda^{-1}(\{0\})$ is therefore a closed subset of $\mathcal{H}$. So $\mathcal{M}$ is a closed linear subspace of $\mathcal{H}$. From theorem (53), there exists $x^* \in \mathcal{M}$, $y^* \in \mathcal{M}^\perp$, such that $x_0 = x^* + y^*$.

3. Since $x^* \in \mathcal{M}$, $\lambda(y^*) = \lambda(x_0)$ and therefore $\lambda(y^*) \neq 0$. In particular, $y^* \neq 0$. Taking $z = y^*/\|y^*\|$, we have found $z \in \mathcal{M}^\perp$, such that $\|z\| = 1$.

4. Let $\alpha \in \mathbf{K} \setminus \{0\}$. We have $\langle z, \alpha z\rangle/\bar{\alpha} = \langle z, (\alpha z)/\alpha\rangle = \langle z, z\rangle = 1$. It follows that $\lambda(x)\langle z, \alpha z\rangle/\bar{\alpha} = \lambda(x)$ for all $x \in \mathcal{H}$.

5. In order to have $\lambda(x) = \langle x, \alpha z\rangle$ for all $x \in \mathcal{H}$, we need:

$$0 = \lambda(x) - \langle x, \alpha z\rangle = \lambda(x)\langle z, \alpha z\rangle/\bar{\alpha} - \langle x, \alpha z\rangle = \langle \lambda(x)z/\bar{\alpha} - x, \alpha z\rangle$$

Since $z \in \mathcal{M}^\perp$, it is sufficient to choose $\alpha \in \mathbf{K} \setminus \{0\}$, with:

$$\forall x \in \mathcal{H} \ , \ \frac{\lambda(x)z}{\bar{\alpha}} - x \in \mathcal{M} \tag{12}$$

6. Since $\mathcal{M} = \lambda^{-1}(\{0\})$, property (12) is equivalent to:

$$0 = \lambda\left(\frac{\lambda(x)z}{\bar{\alpha}} - x\right) = \lambda(x)\lambda(z)/\bar{\alpha} - \lambda(x)$$

for all $x \in \mathcal{H}$, which is satisfied for $\alpha = \overline{\lambda(z)}$, provided $\lambda(z) \neq 0$. But if $\lambda(z) = 0$, then $z \in \mathcal{M}$. So $z \in \mathcal{M} \cap \mathcal{M}^\perp$ and $\langle z, z\rangle = 0$, contradicting the fact that $\|z\| = 1$. Hence, if we take $\alpha = \overline{\lambda(z)}$, then condition (12) is satisfied, and therefore $\lambda(x) = \langle x, \alpha z\rangle$ for all $x \in \mathcal{H}$. Taking $y = \alpha z = \overline{\lambda(z)}z$, we have found $y \in \mathcal{H}$, with:

$$\forall x \in \mathcal{H} \ , \ \lambda(x) = \langle x, y\rangle \tag{13}$$

In case one has any doubt about (13), one can quickly check:

$$\begin{aligned} \lambda(x) - \langle x, \overline{\lambda(z)}z\rangle &= \lambda(x)\langle z, z\rangle - \lambda(z)\langle x, z\rangle \\ &= \langle \lambda(x)z - \lambda(z)x, z\rangle \\ &= 0 \end{aligned}$$

the last equality arising from $\lambda(x)z - \lambda(z)x \in \mathcal{M}$ and $z \in \mathcal{M}^\perp$.

7. Suppose $\bar{y} \in \mathcal{H}$ is such that $\lambda(x) = \langle x, \bar{y}\rangle$ for all $x \in \mathcal{H}$. Then $\langle x, y - \bar{y}\rangle = 0$ for all $x \in \mathcal{H}$, and in particular $\|y - \bar{y}\|^2 = 0$, i.e. $\bar{y} = y$. So $y \in \mathcal{H}$ satisfying (13) is unique. This proves theorem (54) [17] .

Exercise 25

Exercise 26.

[17] The case $\lambda = 0$ is easy to handle.

1. Suppose $f = g$ μ-a.s. For all $h \in [f]$, we have $h = f$ μ-a.s. and therefore $h = g$ μ-a.s., i.e. $h \in [g]$. So $[f] \subseteq [g]$, and similarly $[g] \subseteq [f]$. Conversely, if $[f] = [g]$, then in particular $f \in [g]$ and therefore $f = g$ μ-a.s. We have proved that $f = g$ μ-a.s. is equivalent to $[f] = [g]$.

2. Suppose $[f] = [f']$ and $[g] = [g']$. Then $f = f'$ μ-a.s. and $g = g'$ μ-a.s. So $f + g = f' + g'$ μ-a.s. and $[f + g] = [f' + g']$.

3. $\oplus$ is defined as $[f] \oplus [g] = [f + g]$. This definition may not be legitimate, as $[f] \oplus [g]$ is defined in terms of particular representatives f and g of the equivalence classes $[f]$ and $[g]$. Since such representative are normally far from being unique, this may lead to different values of $[f + g]$, as f and g range over all possible choices. However, as shown in 2., $[f + g]$ is in fact independent of the particular choice of $f \in [f]$ and $g \in [g]$. So $[f] \oplus [g]$ is unambiguously defined, i.e. the operator $\oplus$ is well-defined.

4. Let $\alpha \in \mathbf{K}$. If $[f] = [f']$, then $f = f'$ μ-a.s. and $\alpha f = \alpha f'$ μ-a.s. So $[\alpha f] = [\alpha f']$. It follows that $[\alpha f]$ is independent of the particular choice of $f \in [f]$. So $\alpha \otimes [f]$ is unambiguously defined, i.e. the operator $\otimes$ is well-defined.

5. For all $[f], [g], [h] \in \mathcal{H}$ and $\alpha, \beta \in \mathbf{K}$, we have:

$$
\begin{array}{rl}
(i) & [0] \oplus [f] = [0 + f] = [f] \\
(ii) & [-f] \oplus [f] = [-f + f] = [0] \\
(iii) & [f] \oplus ([g] \oplus [h]) = [f + g + h] = ([f] \oplus [g]) \oplus [h] \\
(iv) & [f] \oplus [g] = [f + g] = [g] \oplus [f] \\
(v) & 1 \otimes [f] = [1.f] = [f] \\
(vi) & \alpha \otimes (\beta \otimes [f]) = [\alpha\beta f] = (\alpha\beta) \otimes [f] \\
(vii) & (\alpha + \beta) \otimes [f] = [\alpha f + \beta f] = (\alpha \otimes [f]) \oplus (\beta \otimes [f]) \\
(viii) & \alpha \otimes ([f] \oplus [g]) = [\alpha f + \alpha g] = (\alpha \otimes [f]) \oplus (\alpha \otimes [g])
\end{array}
$$

Exercise 26

Exercise 27.

1. Suppose $[f] = [f']$ and $[g] = [g']$. Then $f = f'$ μ-a.s. and $g = g'$ μ-a.s. So $f\bar{g} = f'\bar{g}'$ μ-a.s. and therefore:

$$\int f\bar{g}d\mu = \int f'\bar{g}'d\mu \tag{14}$$

It follows that (14) is independent of the of choice of $f \in [f]$ and $g \in [g]$. We conclude that $\langle [f], [g] \rangle_{\mathcal{H}}$ is unambiguously defined, i.e. $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is well-defined.

2. Let $[f], [g] \in \mathcal{H}$, $\alpha \in \mathbf{K}$ and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathcal{H}}$. We have:

$$(i) \qquad \langle [f], [g] \rangle = \int f\bar{g}d\mu = \overline{\langle [g], [f] \rangle}$$

$$(ii) \qquad \langle [f] \oplus [g], [h] \rangle = \int (f+g)\bar{h}d\mu = \langle [f], [h] \rangle + \langle [g], [h] \rangle$$

$$(iii) \qquad \langle \alpha \otimes [f], [g] \rangle = \int (\alpha f)\bar{g}d\mu = \alpha \langle [f], [g] \rangle$$

$$(iv) \qquad \langle [f], [f] \rangle = \int |f|^2 d\mu \in \mathbf{R}^+$$

   and finally, $\langle [f], [f] \rangle = 0$ is equivalent to $\int |f|^2 d\mu = 0$, which is in turn equivalent to $f = 0$ μ-a.s., i.e. $[f] = [0]$. From definition (81), we conclude that $\langle \cdot, \cdot \rangle$ is an inner-product on $\mathcal{H}$.

3. $\mathcal{H}$ is a $\mathbf{K}$-vector space, and $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is an inner-product on $\mathcal{H}$. From definition (83), to show that $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ is a Hilbert space over $\mathbf{K}$, we need to prove that $\mathcal{H}$ is in fact complete with respect to the metric induced by the inner-product. Let $([f_n])_{n \geq 1}$ be a Cauchy sequence in $\mathcal{H}$. For all $\epsilon > 0$, there exists $n_0 \geq 1$ with:

$$n, m \geq n_0 \;\Rightarrow\; \|[f_n] - [f_m]\|_{\mathcal{H}} \leq \epsilon^{18}$$

   However, for all $f \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$, we have:

$$\|[f]\|_{\mathcal{H}} = (\langle [f], [f] \rangle_{\mathcal{H}})^{\frac{1}{2}} = \left(\int |f|^2 d\mu \right)^{\frac{1}{2}} = \|f\|_2$$

   It follows that $(f_n)_{n \geq 1}$ is a Cauchy sequence in $L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$. From theorem (46), there exists $f \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$, such that $f_n \to f$ in L^2. In other words, for all $\epsilon > 0$, there exists $n_0 \geq 1$, such that:

$$n \geq n_0 \;\Rightarrow\; \|f_n - f\|_2 \leq \epsilon$$

   Since $\|f_n - f\|_2 = \|[f_n] - [f]\|_{\mathcal{H}}$, we conclude that $[f_n] \to [f]$ with respect to the norm topology on $\mathcal{H}$. Having found a limit for the Cauchy sequence $([f_n])_{n \geq 1}$, we have proved that $\mathcal{H}$ is complete, and $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ is finally a Hilbert space over $\mathbf{K}$.

4. $\langle f, g \rangle = \int f\bar{g}d\mu$ is not an inner-product on $L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$, as property (v) of definition (81) fails to be satisfied. If $\langle f, f \rangle = 0$, then we know for sure that $f = 0$ μ-a.s. There is no reason why f should be 0 everywhere. This is the very reason why in this exercise, we go through so much trouble considering the quotient set $\mathcal{H} = (L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu))_{|\mathcal{R}}$, where $\mathcal{R}$ is the μ-a.s. equivalence relation on $L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$.

Exercise 27

Exercise 28.

[18] $[f_n] - [f_m]$ is a light notation to indicate $[f_n] \oplus [-f_m]$.

1. Since $L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$ is not a Hilbert space, we cannot use exercise (24) in its literal form. However, most of what we did then, can be reproduced here. Let $\lambda : L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \to \mathbf{K}$ be a continuous linear functional. The open ball $B(0,1) = \{z \in \mathbf{K} : |z| < 1\}$ being open in $\mathbf{K}$, the inverse image $\lambda^{-1}(B(0,1))$ is an open subset of $L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$. Since $0 \in \lambda^{-1}(B(0,1))$, there exists $\delta > 0$, such that $B(0,\delta) \subseteq \lambda^{-1}(B(0,1))$, where $B(0,\delta)$ is the open ball in $L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$. Taking an arbitrary $\eta > 0$, strictly smaller than δ, for all $f \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$, we have:
$$\|f\|_2 \leq \eta \ \Rightarrow \ |\lambda(f)| \leq 1$$
It follows that $|\lambda(\eta f/\|f\|_2)| \leq 1$ for all $f \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$, $f \neq 0$, and finally:
$$\forall f \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \ , \ |\lambda(f)| \leq \frac{1}{\eta}\|f\|_2 \tag{15}$$

2. If $[f] = [g]$, then $f - g = 0$ μ-a.s. and $\|f - g\|_2 = 0$. It follows from (15) that $\lambda(f) = \lambda(g)$.

3. $\Lambda : \mathcal{H} \to \mathbf{K}$ is defined by $\Lambda([f]) = \lambda(f)$. Since $\lambda(f)$ is independent of the particular choice of $f \in [f]$, $\Lambda([f])$ is unambiguously defined, i.e. Λ is well-defined. For all $[f], [g] \in \mathcal{H}$ and $\alpha \in \mathbf{K}$:
$$\Lambda([f] \oplus (\alpha \otimes [g])) = \Lambda([f + \alpha g]) = \lambda(f) + \alpha\lambda(g) = \Lambda([f]) + \alpha\Lambda([g])$$
So Λ is a linear functional on $\mathcal{H}$. Furthermore, since we have $\|[f]\|_{\mathcal{H}} = \|f\|_2$ for all $f \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$, we obtain immediately from (15) that:
$$\forall [f] \in \mathcal{H} \ , \ |\Lambda([f])| \leq \frac{1}{\eta}\|[f]\|_{\mathcal{H}}$$
and we conclude from definition (88) that Λ is a well-defined bounded linear functional on $\mathcal{H}$.

4. Let $\lambda : L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \to \mathbf{K}$ be a continuous linear functional. Then from 3., $\Lambda : \mathcal{H} \to \mathbf{K}$ defined by $\Lambda([f]) = \lambda(f)$ is a bounded linear functional on the Hilbert space $\mathcal{H}$. Applying theorem (54), there exists $[g] \in \mathcal{H}$, such that:
$$\forall [f] \in \mathcal{H} \ , \ \Lambda([f]) = \langle [f], [g] \rangle_{\mathcal{H}}$$
It follows that:
$$\forall f \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \ , \ \lambda(f) = \int f\bar{g}d\mu$$
This proves theorem (55).

Exercise 28