

1. Dynkin systems

Definition 1 A **Dynkin system** on a set Ω is a subset \mathcal{D} of the power set $\mathcal{P}(\Omega)$, with the following properties:

- (i) $\Omega \in \mathcal{D}$
- (ii) $A, B \in \mathcal{D}, A \subseteq B \Rightarrow B \setminus A \in \mathcal{D}$
- (iii) $A_n \in \mathcal{D}, A_n \subseteq A_{n+1}, n \geq 1 \Rightarrow \bigcup_{n=1}^{+\infty} A_n \in \mathcal{D}$

Definition 2 A σ -**algebra** on a set Ω is a subset \mathcal{F} of the power set $\mathcal{P}(\Omega)$ with the following properties:

- (i) $\Omega \in \mathcal{F}$
- (ii) $A \in \mathcal{F} \Rightarrow A^c \triangleq \Omega \setminus A \in \mathcal{F}$
- (iii) $A_n \in \mathcal{F}, n \geq 1 \Rightarrow \bigcup_{n=1}^{+\infty} A_n \in \mathcal{F}$

EXERCISE 1. Let \mathcal{F} be a σ -algebra on Ω . Show that $\emptyset \in \mathcal{F}$, that if $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$ and also $A \cap B \in \mathcal{F}$. Recall that $B \setminus A = B \cap A^c$ and conclude that \mathcal{F} is also a Dynkin system on Ω .

EXERCISE 2. Let $(\mathcal{D}_i)_{i \in I}$ be an arbitrary family of Dynkin systems on Ω , with $I \neq \emptyset$. Show that $\mathcal{D} \triangleq \bigcap_{i \in I} \mathcal{D}_i$ is also a Dynkin system on Ω .

EXERCISE 3. Let $(\mathcal{F}_i)_{i \in I}$ be an arbitrary family of σ -algebras on Ω , with $I \neq \emptyset$. Show that $\mathcal{F} \triangleq \bigcap_{i \in I} \mathcal{F}_i$ is also a σ -algebra on Ω .

EXERCISE 4. Let \mathcal{A} be a subset of the power set $\mathcal{P}(\Omega)$. Define:

$$D(\mathcal{A}) \triangleq \{\mathcal{D} \text{ Dynkin system on } \Omega : \mathcal{A} \subseteq \mathcal{D}\}$$

Show that $\mathcal{P}(\Omega)$ is a Dynkin system on Ω , and that $D(\mathcal{A})$ is not empty. Define:

$$\mathcal{D}(\mathcal{A}) \triangleq \bigcap_{\mathcal{D} \in D(\mathcal{A})} \mathcal{D}$$

Show that $\mathcal{D}(\mathcal{A})$ is a Dynkin system on Ω such that $\mathcal{A} \subseteq \mathcal{D}(\mathcal{A})$, and that it is the smallest Dynkin system on Ω with such property, (i.e. if \mathcal{D} is a Dynkin system on Ω with $\mathcal{A} \subseteq \mathcal{D}$, then $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{D}$).

Definition 3 Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. We call **Dynkin system generated by \mathcal{A}** , the Dynkin system on Ω , denoted $\mathcal{D}(\mathcal{A})$, equal to the intersection of all Dynkin systems on Ω , which contain \mathcal{A} .

EXERCISE 5. Do exactly as before, but replacing Dynkin systems by σ -algebras.

Definition 4 Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. We call σ -algebra generated by \mathcal{A} , the σ -algebra on Ω , denoted $\sigma(\mathcal{A})$, equal to the intersection of all σ -algebras on Ω , which contain \mathcal{A} .

Definition 5 A subset \mathcal{A} of the power set $\mathcal{P}(\Omega)$ is called a π -system on Ω , if and only if it is closed under finite intersection, i.e. if it has the property:

$$A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$$

EXERCISE 6. Let \mathcal{A} be a π -system on Ω . For all $A \in \mathcal{D}(\mathcal{A})$, we define:

$$\Gamma(A) \triangleq \{B \in \mathcal{D}(\mathcal{A}) : A \cap B \in \mathcal{D}(\mathcal{A})\}$$

1. If $A \in \mathcal{A}$, show that $\mathcal{A} \subseteq \Gamma(A)$
2. Show that for all $A \in \mathcal{D}(\mathcal{A})$, $\Gamma(A)$ is a Dynkin system on Ω .
3. Show that if $A \in \mathcal{A}$, then $\mathcal{D}(\mathcal{A}) \subseteq \Gamma(A)$.
4. Show that if $B \in \mathcal{D}(\mathcal{A})$, then $\mathcal{A} \subseteq \Gamma(B)$.
5. Show that for all $B \in \mathcal{D}(\mathcal{A})$, $\mathcal{D}(\mathcal{A}) \subseteq \Gamma(B)$.
6. Conclude that $\mathcal{D}(\mathcal{A})$ is also a π -system on Ω .

EXERCISE 7. Let \mathcal{D} be a Dynkin system on Ω which is also a π -system.

1. Show that if $A, B \in \mathcal{D}$ then $A \cup B \in \mathcal{D}$.
2. Let $A_n \in \mathcal{D}, n \geq 1$. Consider $B_n \triangleq \cup_{i=1}^n A_i$. Show that $\cup_{n=1}^{+\infty} A_n = \cup_{n=1}^{+\infty} B_n$.
3. Show that \mathcal{D} is a σ -algebra on Ω .

EXERCISE 8. Let \mathcal{A} be a π -system on Ω . Explain why $\mathcal{D}(\mathcal{A})$ is a σ -algebra on Ω , and $\sigma(\mathcal{A})$ is a Dynkin system on Ω . Conclude that $\mathcal{D}(\mathcal{A}) = \sigma(\mathcal{A})$. Prove the theorem:

Theorem 1 (Dynkin system) Let \mathcal{C} be a collection of subsets of Ω which is closed under pairwise intersection. If \mathcal{D} is a Dynkin system containing \mathcal{C} then \mathcal{D} also contains the σ -algebra $\sigma(\mathcal{C})$ generated by \mathcal{C} .

Solutions to Exercises

Exercise 1.

1. From (i) of definition (2), $\Omega \in \mathcal{F}$. Hence, from (ii), $\emptyset = \Omega^c \in \mathcal{F}$.
2. If $A, B \in \mathcal{F}$, we can construct a sequence $(A_n)_{n \geq 1}$ of elements of \mathcal{F} by setting $A_1 = A$, $A_2 = B$ and $A_k = \emptyset$ for all $k \geq 3$. Then, using (iii) of definition (2):

$$A \cup B = \bigcup_{n=1}^{+\infty} A_n \in \mathcal{F}$$

3. From (ii), A^c and B^c are also elements of \mathcal{F} . So $A^c \cup B^c \in \mathcal{F}$. Finally, again from (ii):

$$A \cap B = (A^c \cup B^c)^c \in \mathcal{F}$$

4. B and A^c are both elements of \mathcal{F} , so:

$$B \setminus A \triangleq B \cap A^c \in \mathcal{F}$$

5. \mathcal{F} being a σ -algebra, conditions (i) and (iii) of definition (1) are immediately satisfied. But we have just proved that if $A, B \in \mathcal{F}$, then $B \setminus A \in \mathcal{F}$. Hence, condition (ii) of definition (1) is also satisfied, and \mathcal{F} is therefore a Dynkin system.

Exercise 1

Exercise 2.

1. Each \mathcal{D}_i is a Dynkin system. From (i) of definition (1), $\Omega \in \mathcal{D}_i$. This being true for all $i \in I$, $\Omega \in \bigcap_{i \in I} \mathcal{D}_i = \mathcal{D}$. Hence (i) of definition (1) is satisfied for \mathcal{D} .
2. Let $A, B \in \mathcal{D}$ with $A \subseteq B$. Then for all $i \in I$, $A, B \in \mathcal{D}_i$, with $A \subseteq B$. Since each \mathcal{D}_i is a Dynkin system, from (ii) of definition (1) we see that $B \setminus A \in \mathcal{D}_i$. This being true for all $i \in I$, $B \setminus A \in \bigcap_{i \in I} \mathcal{D}_i = \mathcal{D}$. Hence, (ii) of definition (1) is satisfied for \mathcal{D} .
3. Let $(A_n)_{n \geq 1}$ be a sequence of elements of \mathcal{D} with $A_n \subseteq A_{n+1}$. Then, for all $i \in I$, $(A_n)_{n \geq 1}$ is a sequence of elements of \mathcal{D}_i with $A_n \subseteq A_{n+1}$. Since each \mathcal{D}_i is a Dynkin system, from (iii) of definition (1) we see that $\bigcup_{n=1}^{+\infty} A_n \in \mathcal{D}_i$. This being true for all $i \in I$, $\bigcup_{n=1}^{+\infty} A_n \in \bigcap_{i \in I} \mathcal{D}_i = \mathcal{D}$. Hence, (iii) of definition (1) is satisfied for \mathcal{D} .
4. Having checked (i), (ii), (iii) of definition (1), we conclude that \mathcal{D} is indeed a Dynkin system on Ω .

Exercise 2

Exercise 3.

1. Each \mathcal{F}_i is a σ -algebra. From (i) of definition (2), $\Omega \in \mathcal{F}_i$. This being true for all $i \in I$, $\Omega \in \cap_{i \in I} \mathcal{F}_i = \mathcal{F}$. Hence (i) of definition (2) is satisfied for \mathcal{F} .
2. Let $A \in \mathcal{F}$. Then for all $i \in I$, $A \in \mathcal{F}_i$. Since each \mathcal{F}_i is a σ -algebra, from (ii) of definition (2) we see that $A^c \in \mathcal{F}_i$. This being true for all $i \in I$, $A^c \in \cap_{i \in I} \mathcal{F}_i = \mathcal{F}$. Hence, (ii) of definition (2) is satisfied for \mathcal{F} .
3. Let $(A_n)_{n \geq 1}$ be a sequence of elements of \mathcal{F} . Then, for all $i \in I$, $(A_n)_{n \geq 1}$ is a sequence of elements of \mathcal{F}_i . Since each \mathcal{F}_i is a σ -algebra, from (iii) of definition (2) we see that $\cup_{n=1}^{+\infty} A_n \in \mathcal{F}_i$. This being true for all $i \in I$, $\cup_{n=1}^{+\infty} A_n \in \cap_{i \in I} \mathcal{F}_i = \mathcal{F}$. Hence, (iii) of definition (2) is satisfied for \mathcal{F} .
4. Having checked (i), (ii), (iii) of definition (2), we conclude that \mathcal{F} is indeed a σ -algebra on Ω .

Exercise 3

Exercise 4.

1. Ω is obviously a subset of Ω , so $\Omega \in \mathcal{P}(\Omega)$, and (i) of definition (1) is satisfied for $\mathcal{P}(\Omega)$. If $A, B \in \mathcal{P}(\Omega)$, whether or not $A \subseteq B$, $B \setminus A$ is still a subset of Ω , i.e. $B \setminus A \in \mathcal{P}(\Omega)$. So (ii) of definition (1) is also satisfied for $\mathcal{P}(\Omega)$. If $(A_n)_{n \geq 1}$ is a sequence of subsets of Ω , whether or not this sequence is increasing (i.e. $A_n \subseteq A_{n+1}$), $\cup_{n=1}^{+\infty} A_n$ is still a subset of Ω , i.e. belongs to $\mathcal{P}(\Omega)$. So (iii) of definition (1) is satisfied for $\mathcal{P}(\Omega)$, and finally, $\mathcal{P}(\Omega)$ is a Dynkin system on Ω .
2. By assumption, $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. Since $\mathcal{P}(\Omega)$ is also a Dynkin system on Ω , we see that $\mathcal{P}(\Omega) \in D(\mathcal{A})$. In particular, $D(\mathcal{A})$ is not empty.
3. Take $I = D(\mathcal{A})$, and for all $i \in I$, define $\mathcal{D}_i = i$. Then $(\mathcal{D}_i)_{i \in I}$ is a family of Dynkin systems on Ω (where $I \neq \emptyset$) and since:

$$\mathcal{D}(\mathcal{A}) \triangleq \bigcap_{\mathcal{D} \in D(\mathcal{A})} \mathcal{D} = \bigcap_{i \in I} \mathcal{D}_i$$

using exercise (2), we conclude that $\mathcal{D}(\mathcal{A})$ is a Dynkin system on Ω .

4. Let $A \in \mathcal{A}$. For all $\mathcal{D} \in D(\mathcal{A})$, we have $\mathcal{A} \subseteq \mathcal{D}$. Hence, for all $\mathcal{D} \in D(\mathcal{A})$, $A \in \mathcal{D}$. So:

$$A \in \bigcap_{\mathcal{D} \in D(\mathcal{A})} \mathcal{D} \triangleq \mathcal{D}(\mathcal{A})$$

It follows that $\mathcal{A} \subseteq \mathcal{D}(\mathcal{A})$.

5. Suppose \mathcal{D} is another Dynkin system on Ω such that $\mathcal{A} \subseteq \mathcal{D}$. Then $\mathcal{D} \in D(\mathcal{A})$, from which we conclude that:

$$\mathcal{D}(\mathcal{A}) \triangleq \bigcap_{\mathcal{D}' \in D(\mathcal{A})} \mathcal{D}' \subseteq \mathcal{D}$$

Exercise 4

Exercise 5.

1. We define similarly: $F(\mathcal{A}) \triangleq \{\mathcal{F} \text{ } \sigma\text{-algebra on } \Omega : \mathcal{A} \subseteq \mathcal{F}\}$ and:

$$\sigma(\mathcal{A}) \triangleq \bigcap_{\mathcal{F} \in F(\mathcal{A})} \mathcal{F}$$

2. $\Omega \in \mathcal{P}(\Omega)$, and (i) of definition (2) is satisfied for $\mathcal{P}(\Omega)$. If $A \in \mathcal{P}(\Omega)$, then $A^c \in \mathcal{P}(\Omega)$ and (ii) of definition (2) is also satisfied for $\mathcal{P}(\Omega)$. If $(A_n)_{n \geq 1}$ is a sequence of subsets of Ω , $\cup_{n=1}^{+\infty} A_n$ is still a subset of Ω , and (iii) of definition (2) is satisfied for $\mathcal{P}(\Omega)$. Finally, $\mathcal{P}(\Omega)$ is a σ -algebra on Ω .
3. By assumption, $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. Since $\mathcal{P}(\Omega)$ is also a σ -algebra on Ω , we see that $\mathcal{P}(\Omega) \in F(\mathcal{A})$. In particular, $F(\mathcal{A})$ is not empty.
4. Take $I = F(\mathcal{A})$, and for all $i \in I$, define $\mathcal{F}_i = i$. Then $(\mathcal{F}_i)_{i \in I}$ is a family of σ -algebra on Ω (where $I \neq \emptyset$) and since:

$$\sigma(\mathcal{A}) \triangleq \bigcap_{\mathcal{F} \in F(\mathcal{A})} \mathcal{F} = \bigcap_{i \in I} \mathcal{F}_i$$

using exercise (3), we conclude that $\sigma(\mathcal{A})$ is a σ -algebra on Ω .

5. Let $A \in \mathcal{A}$. For all $\mathcal{F} \in F(\mathcal{A})$, we have $\mathcal{A} \subseteq \mathcal{F}$. Hence, for all $\mathcal{F} \in F(\mathcal{A})$, $A \in \mathcal{F}$. So:

$$A \in \bigcap_{\mathcal{F} \in F(\mathcal{A})} \mathcal{F} \triangleq \sigma(\mathcal{A})$$

It follows that $\mathcal{A} \subseteq \sigma(\mathcal{A})$.

6. Suppose \mathcal{F} is another σ -algebra on Ω such that $\mathcal{A} \subseteq \mathcal{F}$. Then $\mathcal{F} \in F(\mathcal{A})$, from which we conclude that:

$$\sigma(\mathcal{A}) \triangleq \bigcap_{\mathcal{F}' \in F(\mathcal{A})} \mathcal{F}' \subseteq \mathcal{F}$$

Exercise 5

Exercise 6.

1. Suppose $A \in \mathcal{A}$, and let $B \in \mathcal{A}$. $\mathcal{D}(\mathcal{A})$ being the Dynkin system generated by \mathcal{A} , $\mathcal{A} \subseteq \mathcal{D}(\mathcal{A})$ (see exercise (4)). In particular, $B \in \mathcal{D}(\mathcal{A})$. Since both A, B lie in \mathcal{A} , \mathcal{A} being a π -system, we have $A \cap B \in \mathcal{A} \subseteq \mathcal{D}(\mathcal{A})$. Hence, we see that $B \in \Gamma(A)$. We have proved that for all $A \in \mathcal{A}$, $\mathcal{A} \subseteq \Gamma(A)$.
2. Let $A \in \mathcal{D}(\mathcal{A})$. $\mathcal{D}(\mathcal{A})$ being a Dynkin system, $\Omega \in \mathcal{D}(\mathcal{A})$. Moreover, $A \cap \Omega = A \in \mathcal{D}(\mathcal{A})$. So $\Omega \in \Gamma(A)$, and condition (i) of definition (1) is satisfied for $\Gamma(A)$. Let $B, C \in \Gamma(A)$ with $B \subseteq C$. Both B, C belong to

$\mathcal{D}(\mathcal{A})$, with $B \subseteq C$. $\mathcal{D}(\mathcal{A})$ being a Dynkin system on Ω , $C \setminus B \in \mathcal{D}(\mathcal{A})$. Also:

$$A \cap (C \setminus B) = (A \cap C) \setminus (A \cap B)$$

Since $B, C \in \Gamma(\mathcal{A})$, both $A \cap B$ and $A \cap C$ belong to $\mathcal{D}(\mathcal{A})$, with $A \cap B \subseteq A \cap C$. It follows that $(A \cap C) \setminus (A \cap B) \in \mathcal{D}(\mathcal{A})$, i.e. $A \cap (C \setminus B) \in \mathcal{D}(\mathcal{A})$. Hence, we see that $C \setminus B \in \Gamma(\mathcal{A})$, and (ii) of definition (1) is satisfied for $\Gamma(\mathcal{A})$. Let $(B_n)_{n \geq 1}$ be a sequence of elements of $\Gamma(\mathcal{A})$, with $B_n \subseteq B_{n+1}$. Let $B = \cup_{n=1}^{+\infty} B_n$. Then $(B_n)_{n \geq 1}$ is a sequence of elements of $\mathcal{D}(\mathcal{A})$ with $B_n \subseteq B_{n+1}$. $\mathcal{D}(\mathcal{A})$ being a Dynkin system, we see that $B \in \mathcal{D}(\mathcal{A})$. Moreover, $(A \cap B_n)_{n \geq 1}$ is a sequence of elements of $\mathcal{D}(\mathcal{A})$, with $A \cap B_n \subseteq A \cap B_{n+1}$. It follows that $\cup_{n=1}^{+\infty} A \cap B_n = A \cap B \in \mathcal{D}(\mathcal{A})$. Hence we see that $B \in \Gamma(\mathcal{A})$, and condition (iii) of definition (1) is satisfied for $\Gamma(\mathcal{A})$. We have proved that for all $A \in \mathcal{D}(\mathcal{A})$, $\Gamma(\mathcal{A})$ is a Dynkin system on Ω .

3. If $A \in \mathcal{A}$, we saw in 1. that $\mathcal{A} \subseteq \Gamma(\mathcal{A})$. But $\Gamma(\mathcal{A})$ being a Dynkin system on Ω , using exercise (4), we conclude that $\mathcal{D}(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$.
4. From the previous point, it follows that if $A \in \mathcal{A}$ and $B \in \mathcal{D}(\mathcal{A})$, then $A \cap B \in \mathcal{D}(\mathcal{A})$. In other words, if $B \in \mathcal{D}(\mathcal{A})$ and $A \in \mathcal{A}$, then $A \in \Gamma(B)$. We have proved that for all $B \in \mathcal{D}(\mathcal{A})$, we have $\mathcal{A} \subseteq \Gamma(B)$.
5. From 2., we know that $\Gamma(B)$ is a Dynkin system on Ω . Using exercise (4), it follows from the previous point that $\mathcal{D}(\mathcal{A}) \subseteq \Gamma(B)$, for all $B \in \mathcal{D}(\mathcal{A})$.
6. Another way of writing the previous property, is that for all $B \in \mathcal{D}(\mathcal{A})$, and $A \in \mathcal{D}(\mathcal{A})$, we have $A \in \Gamma(B)$, i.e. $A \cap B \in \mathcal{D}(\mathcal{A})$. Hence, we see that $\mathcal{D}(\mathcal{A})$ is *closed under finite intersection*, i.e. it is a π -system on Ω . The purpose of this exercise is to show that whenever \mathcal{A} is a π -system on Ω , its *generated Dynkin system* $\mathcal{D}(\mathcal{A})$ is also a π -system on Ω .

Exercise 6

Exercise 7.

1. Let $A \in \mathcal{D}$. Since \mathcal{D} is a Dynkin system on Ω , $\Omega \in \mathcal{D}$. We obviously have $A \subseteq \Omega$. It follows that $\Omega \setminus A \in \mathcal{D}$, i.e. $A^c \in \mathcal{D}$. Hence we see that \mathcal{D} is *closed under complementation*. Now, if $A, B \in \mathcal{D}$, then $A^c, B^c \in \mathcal{D}$. Since \mathcal{D} is also assumed to be a π -system on Ω , we have $A^c \cap B^c \in \mathcal{D}$, and finally:

$$A \cup B = (A^c \cap B^c)^c \in \mathcal{D}$$

2. Let $(A_n)_{n \geq 1}$ be a sequence of elements of \mathcal{D} . Having defined $B_n = \cup_{i=1}^n A_i$ for all $n \geq 1$, we put $A = \cup_{n=1}^{+\infty} A_n$ and $B = \cup_{n=1}^{+\infty} B_n$. Let $x \in A$. There exists $n \geq 1$ such that $x \in A_n \subseteq B_n$. So $x \in B$, and $A \subseteq B$. Let $x \in B$. There exists $n \geq 1$ such that $x \in B_n = \cup_{i=1}^n A_i$. Hence, there exists $i \in \{1, \dots, n\}$ such that $x \in A_i$. So $x \in A$ and $B \subseteq A$. We have proved that $A = B$.

3. \mathcal{D} being a Dynkin system, $\Omega \in \mathcal{D}$ and condition (i) of definition (2) is satisfied for \mathcal{D} . If $A \in \mathcal{D}$, we saw in 1. that $A^c \in \mathcal{D}$. So condition (ii) of definition (2) is also satisfied for \mathcal{D} . Let $(A_n)_{n \geq 1}$ be a sequence of elements of \mathcal{D} . Having defined $B_n = \cup_{i=1}^n A_i$ for all $n \geq 1$, we saw in 1. that \mathcal{D} was *closed under finite union*, i.e. B_n is an element of \mathcal{D} for all $n \geq 1$. Moreover, $B_n \subseteq B_{n+1}$ for all $n \geq 1$. \mathcal{D} being a Dynkin system, $\cup_{n=1}^{+\infty} B_n$ is an element of \mathcal{D} . But from 2., $\cup_{n=1}^{+\infty} B_n = \cup_{n=1}^{+\infty} A_n$. Hence, we see that $\cup_{n=1}^{+\infty} A_n$ is also an element of \mathcal{D} , and condition (iii) of definition (2) is satisfied for \mathcal{D} . We have proved that \mathcal{D} is indeed a σ -algebra on Ω . The purpose of this exercise is to show that whenever a Dynkin system \mathcal{D} is also a π -system, then it is in fact a σ -algebra on Ω .

Exercise 7

Exercise 8.

1. From exercise (6), we know that since \mathcal{A} is a π -system on Ω , its generated Dynkin system $\mathcal{D}(\mathcal{A})$ is also a π -system on Ω . However, from exercise (7), we know that any Dynkin system which is also a π -system, is in fact a σ -algebra on Ω . Hence, $\mathcal{D}(\mathcal{A})$ is a σ -algebra on Ω .
2. The σ -algebra $\sigma(\mathcal{A})$ generated by \mathcal{A} being a σ -algebra, it is also a Dynkin system on Ω (see exercise (1)).
3. From $\mathcal{A} \subseteq \sigma(\mathcal{A})$ and the fact that $\sigma(\mathcal{A})$ is a Dynkin system on Ω , we conclude that $\mathcal{D}(\mathcal{A}) \subseteq \sigma(\mathcal{A})$ (see exercise (4)). From $\mathcal{A} \subseteq \mathcal{D}(\mathcal{A})$ and the fact that $\mathcal{D}(\mathcal{A})$ is also a σ -algebra on Ω , we conclude that $\sigma(\mathcal{A}) \subseteq \mathcal{D}(\mathcal{A})$ (see exercise (5)). Finally, $\sigma(\mathcal{A}) = \mathcal{D}(\mathcal{A})$. The purpose of this exercise is to show that for any π -system \mathcal{A} on Ω , its generated σ -algebra $\sigma(\mathcal{A})$ and Dynkin system $\mathcal{D}(\mathcal{A})$ coincide.
4. If \mathcal{C} is a π -system and \mathcal{D} is a Dynkin system with $\mathcal{C} \subseteq \mathcal{D}$, then $\mathcal{D}(\mathcal{C}) \subseteq \mathcal{D}$ (see exercise (4)). But we have just seen that \mathcal{C} being a π -system, $\mathcal{D}(\mathcal{C}) = \sigma(\mathcal{C})$. Hence $\sigma(\mathcal{C}) \subseteq \mathcal{D}$, which proves theorem (1).

Exercise 8