19. Fourier Transform

Exercise 1. Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a, b] \to \mathbf{C}$ be a map such that $f'(t)$ exists for all $t \in [a, b]$. We assume that:

$$\int_a^b |f'(t)|dt < +\infty$$

1. Show that $f' : ([a, b], \mathcal{B}([a, b])) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is measurable.
2. Show that:
$$f(b) - f(a) = \int_a^b f'(t)dt$$

Exercise 2. We define the maps $\psi : \mathbf{R}^2 \to \mathbf{C}$ and $\phi : \mathbf{R} \to \mathbf{C}$:

$$\forall (u, x) \in \mathbf{R}^2 \ , \ \psi(u, x) \stackrel{\triangle}{=} e^{iux - x^2/2}$$

$$\forall u \in \mathbf{R} \ , \ \phi(u) \stackrel{\triangle}{=} \int_{-\infty}^{+\infty} \psi(u, x)dx$$

1. Show that for all $u \in \mathbf{R}$, the map $x \to \psi(u, x)$ is measurable.
2. Show that for all $u \in \mathbf{R}$, we have:
$$\int_{-\infty}^{+\infty} |\psi(u, x)|dx = \sqrt{2\pi} < +\infty$$
and conclude that ϕ is well defined.
3. Let $u \in \mathbf{R}$ and $(u_n)_{n\geq 1}$ be a sequence in $\mathbf{R}$ converging to u. Show that $\phi(u_n) \to \phi(u)$ and conclude that ϕ is continuous.
4. Show that:
$$\int_0^{+\infty} xe^{-x^2/2}dx = 1$$
5. Show that for all $u \in \mathbf{R}$, we have:
$$\int_{-\infty}^{+\infty} \left| \frac{\partial \psi}{\partial u}(u, x) \right| dx = 2 < +\infty$$
6. Let $a, b \in \mathbf{R}$, $a < b$. Show that:
$$e^{ib} - e^{ia} = \int_a^b ie^{ix}dx$$
7. Let $a, b \in \mathbf{R}$, $a < b$. Show that:
$$|e^{ib} - e^{ia}| \leq |b - a|$$

8. Let $a, b \in \mathbf{R},\ a \neq b$. Show that for all $x \in \mathbf{R}$:

$$\left|\frac{\psi(b,x)-\psi(a,x)}{b-a}\right| \leq |x|e^{-x^2/2}$$

9. Let $u \in \mathbf{R}$ and $(u_n)_{n\geq 1}$ be a sequence in $\mathbf{R}$ converging to u, with $u_n \neq u$ for all n. Show that:

$$\lim_{n\to+\infty} \frac{\phi(u_n)-\phi(u)}{u_n-u} = \int_{-\infty}^{+\infty} \frac{\partial\psi}{\partial u}(u,x)dx$$

10. Show that ϕ is differentiable with:

$$\forall u \in \mathbf{R}\ ,\ \phi'(u) = \int_{-\infty}^{+\infty} \frac{\partial\psi}{\partial u}(u,x)dx$$

11. Show that ϕ is of class C^1.

12. Show that for all $(u,x) \in \mathbf{R}^2$, we have:

$$\frac{\partial\psi}{\partial u}(u,x) = -u\psi(u,x) - i\frac{\partial\psi}{\partial x}(u,x)$$

13. Show that for all $u \in \mathbf{R}$:

$$\int_{-\infty}^{+\infty} \left|\frac{\partial\psi}{\partial x}(u,x)\right| dx < +\infty$$

14. Let $a, b \in \mathbf{R},\ a < b$. Show that for all $u \in \mathbf{R}$:

$$\psi(u,b) - \psi(u,a) = \int_a^b \frac{\partial\psi}{\partial x}(u,x)dx$$

15. Show that for all $u \in \mathbf{R}$:

$$\int_{-\infty}^{+\infty} \frac{\partial\psi}{\partial x}(u,x)dx = 0$$

16. Show that for all $u \in \mathbf{R}$:

$$\phi'(u) = -u\phi(u)$$

Exercise 3. Let $\mathcal{S}$ be the set of functions defined by:

$$\mathcal{S} \stackrel{\triangle}{=} \{h:\ h \in C^1(\mathbf{R},\mathbf{R})\ ,\ \forall u \in \mathbf{R}\ ,\ h'(u) = -uh(u)\}$$

1. Let ϕ be as in ex. (2). Show that $Re(\phi)$ and $Im(\phi)$ lie in $\mathcal{S}$.

2. Given $h \in \mathcal{S}$, we define $g : \mathbf{R} \to \mathbf{R}$, by:

$$\forall u \in \mathbf{R}\ ,\ g(u) \stackrel{\triangle}{=} h(u)e^{u^2/2}$$

Show that g is of class C^1 with $g' = 0$.

3. Let $a, b \in \mathbf{R}$, $a < b$. Show the existence of $c \in]a, b[$, such that:
$$g(b) - g(a) = g'(c)(b-a)$$
4. Conclude that for all $h \in \mathcal{S}$, we have:
$$\forall u \in \mathbf{R}\ ,\ h(u) = h(0)e^{-u^2/2}$$
5. Prove the following:

Theorem 124 *For all $u \in \mathbf{R}$, we have:*
$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{iux - x^2/2}dx = e^{-u^2/2}$$

Definition 135 *Let $\mu_1, \ldots, \mu_p$ be complex measures on $\mathbf{R}^n$, where $n, p \geq 1$. We call* **convolution** *of $\mu_1, \ldots, \mu_p$, denoted $\mu_1 \star \ldots \star \mu_p$, the image measure of the product measure $\mu_1 \otimes \ldots \otimes \mu_p$ by the measurable map $S : (\mathbf{R}^n)^p \to \mathbf{R}^n$ defined by:*
$$S(x_1, \ldots, x_p) \stackrel{\triangle}{=} x_1 + \ldots + x_p$$
In other words, $\mu_1 \star \ldots \star \mu_p$ is the complex measure on $\mathbf{R}^n$, defined by:
$$\mu_1 \star \ldots \star \mu_p \stackrel{\triangle}{=} S(\mu_1 \otimes \ldots \otimes \mu_p)$$

Recall that the product $\mu_1 \otimes \ldots \otimes \mu_p$ is defined in theorem (66).

Exercise 4. Let μ, ν be complex measures on $\mathbf{R}^n$.

1. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$:
$$\mu \star \nu(B) = \int_{\mathbf{R}^n \times \mathbf{R}^n} 1_B(x+y) d\mu \otimes \nu(x, y)$$
2. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$:
$$\mu \star \nu(B) = \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} 1_B(x+y) d\mu(x)\right) d\nu(y)$$
3. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$:
$$\mu \star \nu(B) = \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} 1_B(x+y) d\nu(x)\right) d\mu(y)$$
4. Show that $\mu \star \nu = \nu \star \mu$.
5. Let $f : \mathbf{R}^n \to \mathbf{C}$ be bounded and measurable. Show that:
$$\int_{\mathbf{R}^n} f d\mu \star \nu = \int_{\mathbf{R}^n \times \mathbf{R}^n} f(x+y) d\mu \otimes \nu(x, y)$$

EXERCISE 5. Let μ, ν be complex measures on $\mathbf{R}^n$. Given $B \subseteq \mathbf{R}^n$ and $x \in \mathbf{R}^n$, we define $B - x = \{y \in \mathbf{R}^n \ , \ y + x \in B\}$.

1. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$, $B - x \in \mathcal{B}(\mathbf{R}^n)$.
2. Show $x \to \mu(B - x)$ is measurable and bounded, for $B \in \mathcal{B}(\mathbf{R}^n)$.
3. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$:
$$\mu \star \nu(B) = \int_{\mathbf{R}^n} \mu(B - x) d\nu(x)$$
4. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$:
$$\mu \star \nu(B) = \int_{\mathbf{R}^n} \nu(B - x) d\mu(x)$$

EXERCISE 6. Let μ_1, μ_2, μ_3 be complex measures on $\mathbf{R}^n$.

1. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$:
$$\mu_1 \star (\mu_2 \star \mu_3)(B) = \int_{\mathbf{R}^n \times \mathbf{R}^n} 1_B(x + y) d\mu_1 \otimes (\mu_2 \star \mu_3)(x, y)$$
2. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$:
$$\int_{\mathbf{R}^n} 1_B(x + y) d\mu_2 \star \mu_3(y) = \int_{\mathbf{R}^n \times \mathbf{R}^n} 1_B(x + y + z) d\mu_2 \otimes \mu_3(y, z)$$
3. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$:
$$\mu_1 \star (\mu_2 \star \mu_3)(B) = \int_{\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n} 1_B(x + y + z) d\mu_1 \otimes \mu_2 \otimes \mu_3(x, y, z)$$
4. Show that $\mu_1 \star (\mu_2 \star \mu_3) = \mu_1 \star \mu_2 \star \mu_3 = (\mu_1 \star \mu_2) \star \mu_3$

Definition 136 *Let $n \geq 1$ and $a \in \mathbf{R}^n$. We define $\delta_a : \mathcal{B}(\mathbf{R}^n) \to \mathbf{R}^+$:*
$$\forall B \in \mathcal{B}(\mathbf{R}^n) \ , \ \delta_a(B) \stackrel{\triangle}{=} 1_B(a)$$
δ_a is called the **Dirac probability measure** *on $\mathbf{R}^n$, centered in a.*

EXERCISE 7. Let $n \geq 1$ and $a \in \mathbf{R}^n$.

1. Show that δ_a is indeed a probability measure on $\mathbf{R}^n$.
2. Show for all $f : \mathbf{R}^n \to [0, +\infty]$ non-negative and measurable:
$$\int_{\mathbf{R}^n} f d\delta_a = f(a)$$

3. Show if $f:\mathbf{R}^n\to\mathbf{C}$ is measurable, $f\in L^1_C(\mathbf{R}^n,\mathcal{B}(\mathbf{R}^n),\delta_a)$ and:

$$\int_{\mathbf{R}^n} f d\delta_a = f(a)$$

4. Show that for any complex measure μ on $\mathbf{R}^n$:

$$\mu\star\delta_0=\delta_0\star\mu=\mu$$

5. Let $\tau_a(x)=a+x$ define the translation of vector a in $\mathbf{R}^n$. Show that for any complex measure μ on $\mathbf{R}^n$:

$$\mu\star\delta_a=\delta_a\star\mu=\tau_a(\mu)$$

Exercise 8. Let $f,g:(\Omega,\mathcal{F})\to(\mathbf{C},\mathcal{B}(\mathbf{C}))$ be two measurable maps, where $(\Omega,\mathcal{F})$ is a measurable space. Let $u=Re(f)$, $v=Im(f)$, $u'=Re(g)$ and $v'=Im(g)$.

1. Show that $u,v,u',v':(\Omega,\mathcal{F})\to(\mathbf{R},\mathcal{B}(\mathbf{R}))$ are all measurable.
2. Show that $u+u'$, $v+v'$, $uu'-vv'$ and $uv'+u'v$ are measurable.
3. Show that $f+g,fg:(\Omega,\mathcal{F})\to(\mathbf{C},\mathcal{B}(\mathbf{C}))$ are measurable.
4. Show that $\mathbf{C}=\mathbf{R}^2$ has a countable base.
5. Show that $\mathcal{B}(\mathbf{C}\times\mathbf{C})=\mathcal{B}(\mathbf{C})\otimes\mathcal{B}(\mathbf{C})$.
6. Show that $(z,z')\to z+z'$ and $(z,z')\to zz'$ are continuous.
7. Show that $\omega\to(f(\omega),g(\omega))$ is measurable w.r. to $\mathcal{B}(\mathbf{C})\otimes\mathcal{B}(\mathbf{C})$.
8. Conclude once more that $f+g$ and fg are measurable.

Exercise 9. Let $n\geq 1$ and μ,ν be complex measures on $\mathbf{R}^n$. We assume that $\nu << dx$, i.e. that ν is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}^n$.

1. Show there is $f\in L^1_C(\mathbf{R}^n,\mathcal{B}(\mathbf{R}^n),dx)$, such that $\nu=\int f dx$.
2. Show that for all $B\in\mathcal{B}(\mathbf{R}^n)$, we have:

$$\mu\star\nu(B)=\int_{\mathbf{R}^n}\nu(B-x)d\mu(x)$$

3. Show that for all $B\in\mathcal{B}(\mathbf{R}^n)$ and $x\in\mathbf{R}^n$:

$$\nu(B-x)=\int_{\mathbf{R}^n}1_B(y)f(y-x)dy$$

4. Show that for all $B\in\mathcal{B}(\mathbf{R}^n)$ the map:

$$(x,y)\to 1_B(y)f(y-x)$$

lies in $L^1_{\mathbf{C}}(\mathbf{R}^n\times\mathbf{R}^n,\mathcal{B}(\mathbf{R}^n)\otimes\mathcal{B}(\mathbf{R}^n),|\mu|\otimes dy)$.

5. Let $h \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), |\mu|)$ with $|h| = 1$, $\mu = \int hd|\mu|$. Show:
$$(x, y) \to 1_B(y)f(y - x)h(x)$$
also lies in $L^1_{\mathbf{C}}(\mathbf{R}^n \times \mathbf{R}^n, \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^n), |\mu| \otimes dy)$.

6. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$, we have:
$$\mu \star \nu(B) = \int_B \left(\int_{\mathbf{R}^n} f(y - x)d\mu(x) \right) dy$$

7. Let g be the map defined by $g(y) \stackrel{\triangle}{=} \int_{\mathbf{R}^n} f(y - x)d\mu(x)$. Recall why g is dy-almost surely well-defined, and dy-almost surely equal to an element of $L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dy)$.

8. Show that $\mu \star \nu = \int gdx$ and $\mu \star \nu << dx$.

Theorem 125 *Let μ, ν be two complex measures on $\mathbf{R}^n$, $n \geq 1$. If $\nu << dx$, i.e. ν is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}^n$, with density $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, then the convolution $\mu \star \nu = \nu \star \mu$ is itself absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}^n$, with density:*
$$g(y) = \int_{\mathbf{R}^n} f(y - x)d\mu(x) \ , \ dy - a.s.$$
In other words, $\mu \star \nu = \nu \star \mu = \int gdx$.

Exercise 10. Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ where $(\Omega, \mathcal{F}, \mu)$ is a measure space. Let ν be the complex measure on $(\Omega, \mathcal{F})$ defined by $\nu = \int fd\mu$. Let $g : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ be a measurable map.

1. Show that $g \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu) \Leftrightarrow gf \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

2. Show that for all $g \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu)$:
$$\int gd\nu = \int gfd\mu$$

Exercise 11. Further to theorem (125), show that if $\mu = \int hdx$ for some $h \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, then:
$$g(y) = \int_{\mathbf{R}^n} f(y - x)h(x)dx \ , \ dy - a.s.$$

Definition 137 *Let μ be a complex measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, $n \geq 1$. We call* **Fourier transform** *of μ, the map $\mathcal{F}\mu : \mathbf{R}^n \to \mathbf{C}$ defined by:*
$$\forall u \in \mathbf{R}^n \ , \ \mathcal{F}\mu(u) \stackrel{\triangle}{=} \int_{\mathbf{R}^n} e^{i\langle u,x\rangle} d\mu(x)$$
where $\langle \cdot, \cdot \rangle$ is the usual inner-product in $\mathbf{R}^n$.

Exercise 12. Further to definition (137):

1. Show that $\mathcal{F}\mu$ is well-defined.
2. Show that $\mathcal{F}\mu \in C^b_{\mathbf{C}}(\mathbf{R}^n)$, i.e $\mathcal{F}\mu$ is continuous and bounded.
3. Show that for all $a, u \in \mathbf{R}^n$, we have $\mathcal{F}\delta_a(u) = e^{i\langle u,a\rangle}$.
4. Let μ be the probability measure on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ defined by:
$$\forall B \in \mathcal{B}(\mathbf{R}) \ , \ \mu(B) \stackrel{\triangle}{=} \frac{1}{\sqrt{2\pi}} \int_B e^{-x^2/2} dx$$
Show that $\mathcal{F}\mu(u) = e^{-u^2/2}$, for all $u \in \mathbf{R}$.

Exercise 13. Let $\mu_1, \ldots, \mu_p$ be complex measures on $\mathbf{R}^n$, $p \geq 2$.

1. Show that for all $u \in \mathbf{R}^n$, we have:
$$\mathcal{F}(\mu_1 \star \ldots \star \mu_p)(u) = \int_{(\mathbf{R}^n)^p} e^{i\langle u, x_1 + \ldots + x_p\rangle} d\mu_1 \otimes \ldots \otimes \mu_p(x)$$
2. Show that if $p \geq 3$ then $\mu_1 \star \ldots \star \mu_p = (\mu_1 \star \ldots \star \mu_{p-1}) \star \mu_p$.
3. Show that $\mathcal{F}(\mu_1 \star \ldots \star \mu_p) = \Pi_{j=1}^p \mathcal{F}\mu_j$.

Exercise 14. Let $n \geq 1$, $\sigma > 0$ and $g_\sigma : \mathbf{R}^n \to \mathbf{R}^+$ defined by:
$$\forall x \in \mathbf{R}^n \ , \ g_\sigma(x) \stackrel{\triangle}{=} \frac{1}{(2\pi)^{\frac{n}{2}} \sigma^n} e^{-\|x\|^2/2\sigma^2}$$

1. Show that $\int_{\mathbf{R}^n} g_\sigma(x) dx = 1$.
2. Show that for all $u \in \mathbf{R}^n$, we have:
$$\int_{\mathbf{R}^n} g_\sigma(x) e^{i\langle u,x\rangle} dx = e^{-\sigma^2\|u\|^2/2}$$
3. Show that $P_\sigma = \int g_\sigma dx$ is a probability on $\mathbf{R}^n$, and:
$$\forall u \in \mathbf{R}^n \ , \ \mathcal{F}P_\sigma(u) = e^{-\sigma^2\|u\|^2/2}$$
4. Show that for all $x \in \mathbf{R}^n$, we have:
$$g_\sigma(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{i\langle x,u\rangle - \sigma^2\|u\|^2/2} du$$

Exercise 15. Further to ex. (14), let μ be a complex measure on $\mathbf{R}^n$.

1. Show that $\mu \star P_\sigma = \int \phi_\sigma dx$ where:
$$\phi_\sigma(x) = \int_{\mathbf{R}^n} g_\sigma(x - y) d\mu(y) \ , \ dx - a.s.$$

2. Show that we also have:

$$\phi_\sigma(x) = \int_{\mathbf{R}^n} g_\sigma(y-x)d\mu(y) \ , \ dx - a.s.$$

3. Show that:

$$\phi_\sigma(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^n} e^{i\langle y-x,u\rangle - \sigma^2\|u\|^2/2}du\right)d\mu(y) \ , \ dx - a.s.$$

4. Show that:

$$\phi_\sigma(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{-i\langle x,u\rangle - \sigma^2\|u\|^2/2}(\mathcal{F}\mu)(u)du$$

5. Show that if μ,ν are two complex measures on $\mathbf{R}^n$ such that $\mathcal{F}\mu = \mathcal{F}\nu$, then for all $\sigma > 0$, we have $\mu \star P_\sigma = \nu \star P_\sigma$.

Definition 138 *Let $(\Omega, \mathcal{T})$ be a topological space. Let $(\mu_k)_{k\geq 1}$ be a sequence of complex measures on $(\Omega, \mathcal{B}(\Omega))$. We say that the sequence $(\mu_k)_{k\geq 1}$* **narrowly converges**, *or* **weakly converges** *to a complex measure μ on $(\Omega, \mathcal{B}(\Omega))$, and we write $\mu_k \to \mu$, if and only if:*

$$\forall f \in C^b_{\mathbf{R}}(\Omega) \ , \ \lim_{k\to+\infty}\int f d\mu_k = \int f d\mu$$

Exercise 16. Further to definition (138):

1. Show that $\mu_k \to \mu$ narrowly, is equivalent to:

$$\forall f \in C^b_{\mathbf{C}}(\Omega) \ , \ \lim_{k\to+\infty}\int f d\mu_k = \int f d\mu$$

2. Show that if $(\Omega, \mathcal{T})$ is metrizable and ν is a complex measure on $(\Omega, \mathcal{B}(\Omega))$ such that $\mu_k \to \mu$ and $\mu_k \to \nu$ narrowly, then $\mu = \nu$.

Theorem 126 *On a metrizable topological space, the narrow or weak limit when it exists, of any sequence of complex measures, is unique.*

Exercise 17.

1. Show that on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$, we have $\delta_{1/n} \to \delta_0$ narrowly.

2. Show there is $B \in \mathcal{B}(\mathbf{R})$, such that $\delta_{1/n}(B) \not\to \delta_0(B)$.

Exercise 18. Let $n \geq 1$. Given $\sigma > 0$, let P_σ be the probability measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ defined as in ex. (14). Let $(\sigma_k)_{k\geq 1}$ be a sequence in $\mathbf{R}^+$ such that $\sigma_k > 0$ and $\sigma_k \to 0$.

1. Show that for all $f \in C^b_{\mathbf{R}}(\mathbf{R}^n)$, we have:
$$\int_{\mathbf{R}^n} f(x) g_{\sigma_k}(x) dx = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbf{R}^n} f(\sigma_k x) e^{-\|x\|^2/2} dx$$

2. Show that for all $f \in C^b_{\mathbf{R}}(\mathbf{R}^n)$, we have:
$$\lim_{k\to+\infty} \int_{\mathbf{R}^n} f(x) g_{\sigma_k}(x) dx = f(0)$$

3. Show that $P_{\sigma_k} \to \delta_0$ narrowly.

Exercise 19. Let μ, ν be two complex measures on $\mathbf{R}^n$. Let $(\nu_k)_{k\geq 1}$ be a sequence of complex measures on $\mathbf{R}^n$, which narrowly converges to ν. Let $f \in C^b_{\mathbf{R}}(\mathbf{R}^n)$, and $\phi : \mathbf{R}^n \to \mathbf{R}$ be defined by:
$$\forall y \in \mathbf{R}^n \ , \ \phi(y) \stackrel{\triangle}{=} \int_{\mathbf{R}^n} f(x+y) d\mu(x)$$

1. Show that:
$$\int_{\mathbf{R}^n} f d\mu \star \nu_k = \int_{\mathbf{R}^n \times \mathbf{R}^n} f(x+y) d\mu \otimes \nu_k(x,y)$$

2. Show that:
$$\int_{\mathbf{R}^n} f d\mu \star \nu_k = \int_{\mathbf{R}^n} \phi d\nu_k$$

3. Show that $\phi \in C^b_{\mathbf{C}}(\mathbf{R}^n)$.

4. Show that:
$$\lim_{k\to+\infty} \int_{\mathbf{R}^n} \phi d\nu_k = \int_{\mathbf{R}^n} \phi d\nu$$

5. Show that:
$$\lim_{k\to+\infty} \int_{\mathbf{R}^n} f d\mu \star \nu_k = \int_{\mathbf{R}^n} f d\mu \star \nu$$

6. Show that $\mu \star \nu_k \to \mu \star \nu$ narrowly.

Theorem 127 *Let μ, ν be two complex measures on $\mathbf{R}^n$, $n \geq 1$. Let $(\nu_k)_{k\geq 1}$ be a sequence of complex measures on $\mathbf{R}^n$. Then:*
$$\nu_k \to \nu \text{ narrowly} \ \Rightarrow \ \mu \star \nu_k \to \mu \star \nu \text{ narrowly}$$

Exercise 20. Let μ, ν be two complex measures on $\mathbf{R}^n$, such that $\mathcal{F}\mu = \mathcal{F}\nu$. For all $\sigma > 0$, let P_σ be the probability measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ as defined in ex. (14). Let $(\sigma_k)_{k\geq 1}$ be a sequence in $\mathbf{R}^+$ such that $\sigma_k > 0$ and $\sigma_k \to 0$.

1. Show that $\mu \star P_{\sigma_k} = \nu \star P_{\sigma_k}$, for all $k \geq 1$.

2. Show that $\mu \star P_{\sigma_k} \to \mu \star \delta_0$ narrowly.

3. Show that $(\mu \star P_{\sigma_k})_{k\geq 1}$ narrowly converges to both μ and ν.

4. Prove the following:

Theorem 128 *Let μ, ν be two complex measures on $\mathbf{R}^n$. Then:*

$$\mathcal{F}\mu = \mathcal{F}\nu \quad \Rightarrow \quad \mu = \nu$$

i.e. the Fourier transform is an injective mapping on $M^1(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$.

Definition 139 *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Given $n \geq 1$, and a measurable map $X : (\Omega, \mathcal{F}) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, the mapping ϕ_X defined as:*

$$\forall u \in \mathbf{R}^n \ , \ \phi_X(u) \stackrel{\triangle}{=} E[e^{i\langle u,X\rangle}]$$

is called the **characteristic function**[1] *of the random variable X.*

Exercise 21. Further to definition (139):

1. Show that ϕ_X is well-defined, bounded and continuous.

2. Show that we have:

$$\forall u \in \mathbf{R}^n \ , \ \phi_X(u) = \int_{\mathbf{R}^n} e^{i\langle u,x\rangle} dX(P)(x)$$

3. Show ϕ_X is the Fourier transform of the image measure $X(P)$.

4. Show the following:

Theorem 129 *Let $X, Y : (\Omega, \mathcal{F}) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, $n \geq 1$, be two random variables on a probability space $(\Omega, \mathcal{F}, P)$. If X and Y have the same characteristic functions, i.e.*

$$\forall u \in \mathbf{R}^n \ , \ E[e^{i\langle u,X\rangle}] = E[e^{i\langle u,Y\rangle}]$$

then X and Y have the same distributions, i.e.

$$\forall B \in \mathcal{B}(\mathbf{R}^n) \ , \ P(\{X \in B\}) = P(\{Y \in B\})$$

Definition 140 *Let $n \geq 1$. Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$, we define the modulus of α, denoted $|\alpha|$, by $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Given $x \in \mathbf{R}^n$ and $\alpha \in \mathbf{N}^n$, we put:*

$$x^\alpha \stackrel{\triangle}{=} x_1^{\alpha_1} \ldots x_n^{\alpha_n}$$

where it is understood that $x_j^{\alpha_j} = 1$ whenever $\alpha_j = 0$. Given a map $f : U \to \mathbf{C}$, where U is an open subset of $\mathbf{R}^n$, we denote $\partial^\alpha f$ the $|\alpha|$-th partial derivative, when it exists:

$$\partial^\alpha f \stackrel{\triangle}{=} \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

Note that $\partial^\alpha f = f$, whenever $|\alpha| = 0$. Given $k \geq 0$, we say that f is of **class** *C^k, if and only if for all $\alpha \in \mathbf{N}^n$ with $|\alpha| \leq k$, $\partial^\alpha f$ exists and is continuous on U.*

[1] Do not confuse with the *characteristic function* 1_A of a set A, definition (39).

Exercise 22. Explain why def. (140) is consistent with def. (130).

Exercise 23. Let μ be a complex measure on $\mathbf{R}^n$, and $\alpha \in \mathbf{N}^n$, with:

$$\int_{\mathbf{R}^n} |x^\alpha| d|\mu|(x) < +\infty \tag{1}$$

Let $x^\alpha\mu$ the complex measure on $\mathbf{R}^n$ defined by $x^\alpha\mu = \int x^\alpha d\mu$.

1. Explain why the above integral (1) is well-defined.
2. Show that $x^\alpha\mu$ is a well-defined complex measure on $\mathbf{R}^n$.
3. Show that the total variation of $x^\alpha\mu$ is given by:

$$\forall B \in \mathcal{B}(\mathbf{R}^n)\ ,\ |x^\alpha\mu|(B) = \int_B |x^\alpha| d|\mu|(x)$$

4. Show that the Fourier transform of $x^\alpha\mu$ is given by:

$$\forall u \in \mathbf{R}^n\ ,\ \mathcal{F}(x^\alpha\mu)(u) = \int_{\mathbf{R}^n} x^\alpha e^{i\langle u,x\rangle} d\mu(x)$$

Exercise 24. Let μ be a complex measure on $\mathbf{R}^n$. Let $\beta \in \mathbf{N}^n$ with $|\beta| = 1$, and:

$$\int_{\mathbf{R}^n} |x^\beta| d|\mu|(x) < +\infty$$

Let $x^\beta\mu$ be the complex measure on $\mathbf{R}^n$ defined as in ex. (23).

1. Show that there is $j \in \mathbf{N}_n$ with $x^\beta = x_j$ for all $x \in \mathbf{R}^n$.
2. Show that for all $u \in \mathbf{R}^n$, $\frac{\partial\mathcal{F}\mu}{\partial u_j}(u)$ exists and that we have:

$$\frac{\partial\mathcal{F}\mu}{\partial u_j}(u) = i\int_{\mathbf{R}^n} x_j e^{i\langle u,x\rangle} d\mu(x)$$

3. Conclude that $\partial^\beta\mathcal{F}\mu$ exists and that we have:

$$\partial^\beta\mathcal{F}\mu = i\mathcal{F}(x^\beta\mu)$$

4. Explain why $\partial^\beta\mathcal{F}\mu$ is continuous.

Exercise 25. Let μ be a complex measure on $\mathbf{R}^n$. Let $k \geq 0$ be an integer. We assume that for all $\alpha \in \mathbf{N}^n$, we have:

$$|\alpha| \leq k\ \Rightarrow\ \int_{\mathbf{R}^n} |x^\alpha| d|\mu|(x) < +\infty \tag{2}$$

In particular, if $|\alpha| \leq k$, the measure $x^\alpha\mu$ of ex. (23) is well-defined. We claim that for all $\alpha \in \mathbf{N}^n$ with $|\alpha| \leq k$, $\partial^\alpha\mathcal{F}\mu$ exists, and:

$$\partial^\alpha\mathcal{F}\mu = i^{|\alpha|}\mathcal{F}(x^\alpha\mu)$$

1. Show that if $k = 0$, then the property is obviously true. We assume the property is true for some $k \geq 0$, and that the above integrability condition (2) holds for $k+1$.

2. Let $\alpha' \in \mathbf{N}^n$ be such that $|\alpha'| \leq k+1$. Explain why if $|\alpha'| \leq k$, then $\partial^{\alpha'}\mathcal{F}\mu$ exists, with:
$$\partial^{\alpha'}\mathcal{F}\mu = i^{|\alpha'|}\mathcal{F}(x^{\alpha'}\mu)$$

3. We assume that $|\alpha'| = k+1$. Show the existence of $\alpha, \beta \in \mathbf{N}^n$ such that $\alpha + \beta = \alpha'$, $|\alpha| = k$ and $|\beta| = 1$.

4. Explain why $\partial^\alpha \mathcal{F}\mu$ exists, and:
$$\partial^\alpha \mathcal{F}\mu = i^{|\alpha|}\mathcal{F}(x^\alpha \mu)$$

5. Show that:
$$\int_{\mathbf{R}^n} |x^\beta| d|x^\alpha \mu|(x) < +\infty$$

6. Show that $\partial^\beta \mathcal{F}(x^\alpha \mu)$ exists, with:
$$\partial^\beta \mathcal{F}(x^\alpha \mu) = i\mathcal{F}(x^\beta(x^\alpha \mu))$$

7. Show that $\partial^\beta(\partial^\alpha \mathcal{F}\mu)$ exists, with:
$$\partial^\beta(\partial^\alpha \mathcal{F}\mu) = i^{|\alpha|+1}\mathcal{F}(x^\beta(x^\alpha \mu))$$

8. Show that $x^\beta(x^\alpha\mu) = x^{\alpha'}\mu$.

9. Conclude that the property is true for $k+1$.

10. Show the following:

Theorem 130 *Let μ be a complex measure on $\mathbf{R}^n$, $n \geq 1$. Let $k \geq 0$ be an integer such that for all $\alpha \in \mathbf{N}^n$ with $|\alpha| \leq k$, we have:*
$$\int_{\mathbf{R}^n} |x^\alpha| d|\mu|(x) < +\infty$$
Then, the Fourier transform $\mathcal{F}\mu$ is of class C^k on $\mathbf{R}^n$, and for all $\alpha \in \mathbf{N}^n$ with $|\alpha| \leq k$, we have:
$$\forall u \in \mathbf{R}^n\ ,\ \partial^\alpha \mathcal{F}\mu(u) = i^{|\alpha|}\int_{\mathbf{R}^n} x^\alpha e^{i\langle u,x\rangle} d\mu(x)$$
In particular:
$$\int_{\mathbf{R}^n} x^\alpha d\mu(x) = i^{-|\alpha|}\partial^\alpha \mathcal{F}\mu(0)$$