7. Fubini Theorem

Definition 59 *Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces. Let $E \subseteq \Omega_1 \times \Omega_2$. For all $\omega_1 \in \Omega_1$, we call ω_1-**section** of E in Ω_2, the set:*

$$E^{\omega_1} \triangleq \{\omega_2 \in \Omega_2 \ : \ (\omega_1, \omega_2) \in E\}$$

EXERCISE 1. Let $(\Omega_1, \mathcal{F}_1)$, $(\Omega_2, \mathcal{F}_2)$ and (S, Σ) be three measurable spaces, and $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to (S, \Sigma)$ be a measurable map. Given $\omega_1 \in \Omega_1$, define:

$$\Gamma^{\omega_1} \triangleq \{E \subseteq \Omega_1 \times \Omega_2 \ , \ E^{\omega_1} \in \mathcal{F}_2\}$$

1. Show that for all $\omega_1 \in \Omega_1$, Γ^{ω_1} is a σ-algebra on $\Omega_1 \times \Omega_2$.
2. Show that for all $\omega_1 \in \Omega_1$, $\mathcal{F}_1 \amalg \mathcal{F}_2 \subseteq \Gamma^{\omega_1}$.
3. Show that for all $\omega_1 \in \Omega_1$ and $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, we have $E^{\omega_1} \in \mathcal{F}_2$.
4. Given $\omega_1 \in \Omega_1$, show that $\omega \to f(\omega_1, \omega)$ is measurable.
5. Show that $\theta : (\Omega_2 \times \Omega_1, \mathcal{F}_2 \otimes \mathcal{F}_1) \to (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ defined by $\theta(\omega_2, \omega_1) = (\omega_1, \omega_2)$ is a measurable map.
6. Given $\omega_2 \in \Omega_2$, show that $\omega \to f(\omega, \omega_2)$ is measurable.

Theorem 29 *Let (S, Σ), $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be three measurable spaces. Let $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to (S, \Sigma)$ be a measurable map. For all $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$, the map $\omega \to f(\omega_1, \omega)$ is measurable w.r. to $\mathcal{F}_2$ and Σ, and $\omega \to f(\omega, \omega_2)$ is measurable w.r. to $\mathcal{F}_1$ and Σ.*

EXERCISE 2. Let $(\Omega_i, \mathcal{F}_i)_{i \in I}$ be a family of measurable spaces with card$I \geq 2$. Let $f : (\Pi_{i \in I} \Omega_i, \otimes_{i \in I} \mathcal{F}_i) \to (E, \mathcal{B}(E))$ be a measurable map, where (E, d) is a metric space. Let $i_1 \in I$. Put $E_1 = \Omega_{i_1}$, $\mathcal{E}_1 = \mathcal{F}_{i_1}$, $E_2 = \Pi_{i \in I \setminus \{i_1\}} \Omega_i$, $\mathcal{E}_2 = \otimes_{i \in I \setminus \{i_1\}} \mathcal{F}_i$.

1. Explain why f can be viewed as a map defined on $E_1 \times E_2$.
2. Show that $f : (E_1 \times E_2, \mathcal{E}_1 \otimes \mathcal{E}_2) \to (E, \mathcal{B}(E))$ is measurable.
3. For all $\omega_{i_1} \in \Omega_{i_1}$, show that the map $\omega \to f(\omega_{i_1}, \omega)$ defined on $\Pi_{i \in I \setminus \{i_1\}} \Omega_i$ is measurable w.r. to $\otimes_{i \in I \setminus \{i_1\}} \mathcal{F}_i$ and $\mathcal{B}(E)$.

Definition 60 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. $(\Omega, \mathcal{F}, \mu)$ is said to be a* **finite measure space***, or we say that μ is a* **finite measure***, if and only if $\mu(\Omega) < +\infty$.*

Definition 61 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. $(\Omega, \mathcal{F}, \mu)$ is said to be a* **σ-finite measure space***, or μ a* **σ-finite measure***, if and only if there exists a sequence $(\Omega_n)_{n \geq 1}$ in $\mathcal{F}$ such that $\Omega_n \uparrow \Omega$ and $\mu(\Omega_n) < +\infty$, for all $n \geq 1$.*

Exercise 3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

1. Show that $(\Omega, \mathcal{F}, \mu)$ is σ-finite if and only if there exists a sequence $(\Omega_n)_{n\geq 1}$ in $\mathcal{F}$ such that $\Omega = \uplus_{n=1}^{+\infty}\Omega_n$, and $\mu(\Omega_n) < +\infty$ for all $n \geq 1$.

2. Show that if $(\Omega, \mathcal{F}, \mu)$ is finite, then μ has values in $\mathbf{R}^+$.

3. Show that if $(\Omega, \mathcal{F}, \mu)$ is finite, then it is σ-finite.

4. Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. Show that the measure space $(\mathbf{R}, \mathcal{B}(\mathbf{R}), dF)$ is σ-finite, where dF is the Stieltjes measure associated with F.

Exercise 4. Let $(\Omega_1, \mathcal{F}_1)$ be a measurable space, and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be a σ-finite measure space. For all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$ and $\omega_1 \in \Omega_1$, define:

$$\Phi_E(\omega_1) \stackrel{\triangle}{=} \int_{\Omega_2} 1_E(\omega_1, x)d\mu_2(x)$$

Let $\mathcal{D}$ be the set of subsets of $\Omega_1 \times \Omega_2$, defined by:

$$\mathcal{D} \stackrel{\triangle}{=} \{E \in \mathcal{F}_1 \otimes \mathcal{F}_2 \ : \ \Phi_E : (\Omega_1, \mathcal{F}_1) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}})) \text{ is measurable}\}$$

1. Explain why for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, the map Φ_E is well defined.

2. Show that $\mathcal{F}_1 \amalg \mathcal{F}_2 \subseteq \mathcal{D}$.

3. Show that if μ_2 is finite, $A, B \in \mathcal{D}$ and $A \subseteq B$, then $B \setminus A \in \mathcal{D}$.

4. Show that if $E_n \in \mathcal{F}_1 \otimes \mathcal{F}_2, n \geq 1$ and $E_n \uparrow E$, then $\Phi_{E_n} \uparrow \Phi_E$.

5. Show that if μ_2 is finite then $\mathcal{D}$ is a Dynkin system on $\Omega_1 \times \Omega_2$.

6. Show that if μ_2 is finite, then the map $\Phi_E : (\Omega_1, \mathcal{F}_1) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$.

7. Let $(\Omega_2^n)_{n\geq 1}$ in $\mathcal{F}_2$ be such that $\Omega_2^n \uparrow \Omega_2$ and $\mu_2(\Omega_2^n) < +\infty$. Define $\mu_2^n = \mu_2^{\Omega_2^n} = \mu_2(\bullet \cap \Omega_2^n)$. For $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, we put:

$$\Phi_E^n(\omega_1) \stackrel{\triangle}{=} \int_{\Omega_2} 1_E(\omega_1, x)d\mu_2^n(x)$$

   Show that $\Phi_E^n : (\Omega_1, \mathcal{F}_1) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, and:

$$\Phi_E^n(\omega_1) = \int_{\Omega_2} 1_{\Omega_2^n}(x)1_E(\omega_1, x)d\mu_2(x)$$

   Deduce that $\Phi_E^n \uparrow \Phi_E$.

8. Show that the map $\Phi_E : (\Omega_1, \mathcal{F}_1) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$.

9. Let s be a simple function on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$. Show that the map $\omega \to \int_{\Omega_2} s(\omega, x) d\mu_2(x)$ is well defined and measurable with respect to $\mathcal{F}_1$ and $\mathcal{B}(\bar{\mathbf{R}})$.

10. Show the following theorem:

Theorem 30 *Let $(\Omega_1, \mathcal{F}_1)$ be a measurable space, and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be a σ-finite measure space. Then for all non-negative and measurable map $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to [0, +\infty]$, the map:*

$$\omega \to \int_{\Omega_2} f(\omega, x) d\mu_2(x)$$

is measurable with respect to $\mathcal{F}_1$ and $\mathcal{B}(\bar{\mathbf{R}})$.

Exercise 5. Let $(\Omega_i, \mathcal{F}_i)_{i \in I}$ be a family of measurable spaces, with card$I \geq 2$. Let $i_0 \in I$, and suppose that μ_0 is a σ-finite measure on $(\Omega_{i_0}, \mathcal{F}_{i_0})$. Show that if $f : (\Pi_{i \in I} \Omega_i, \otimes_{i \in I} \mathcal{F}_i) \to [0, +\infty]$ is a non-negative and measurable map, then:

$$\omega \to \int_{\Omega_{i_0}} f(\omega, x) d\mu_0(x)$$

defined on $\Pi_{i \in I \setminus \{i_0\}} \Omega_i$, is measurable w.r. to $\otimes_{i \in I \setminus \{i_0\}} \mathcal{F}_i$ and $\mathcal{B}(\bar{\mathbf{R}})$.

Exercise 6. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two σ-finite measure spaces. For all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, we define:

$$\mu_1 \otimes \mu_2(E) \stackrel{\triangle}{=} \int_{\Omega_1} \left(\int_{\Omega_2} 1_E(x, y) d\mu_2(y) \right) d\mu_1(x)$$

1. Explain why $\mu_1 \otimes \mu_2 : \mathcal{F}_1 \otimes \mathcal{F}_2 \to [0, +\infty]$ is well defined.

2. Show that $\mu_1 \otimes \mu_2$ is a measure on $\mathcal{F}_1 \otimes \mathcal{F}_2$.

3. Show that if $A \times B \in \mathcal{F}_1 \amalg \mathcal{F}_2$, then:

$$\mu_1 \otimes \mu_2(A \times B) = \mu_1(A)\mu_2(B)$$

Exercise 7. Further to ex. (6), suppose that $\mu : \mathcal{F}_1 \otimes \mathcal{F}_2 \to [0, +\infty]$ is another measure on $\mathcal{F}_1 \otimes \mathcal{F}_2$ with $\mu(A \times B) = \mu_1(A)\mu_2(B)$, for all measurable rectangle $A \times B$. Let $(\Omega_1^n)_{n \geq 1}$ and $(\Omega_2^n)_{n \geq 1}$ be sequences in $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively, such that $\Omega_1^n \uparrow \Omega_1$, $\Omega_2^n \uparrow \Omega_2$, $\mu_1(\Omega_1^n) < +\infty$ and $\mu_2(\Omega_2^n) < +\infty$. Define, for all $n \geq 1$:

$$\mathcal{D}_n \stackrel{\triangle}{=} \{E \in \mathcal{F}_1 \otimes \mathcal{F}_2 : \mu(E \cap (\Omega_1^n \times \Omega_2^n)) = \mu_1 \otimes \mu_2(E \cap (\Omega_1^n \times \Omega_2^n))\}$$

1. Show that for all $n \geq 1$, $\mathcal{F}_1 \amalg \mathcal{F}_2 \subseteq \mathcal{D}_n$.

2. Show that for all $n \geq 1$, $\mathcal{D}_n$ is a Dynkin system on $\Omega_1 \times \Omega_2$.

3. Show that $\mu = \mu_1 \otimes \mu_2$.

4. Show that $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$ is a σ-finite measure space.

5. Show that for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, we have:

$$\mu_1 \otimes \mu_2(E) = \int_{\Omega_2} \left(\int_{\Omega_1} 1_E(x,y) d\mu_1(x) \right) d\mu_2(y)$$

Exercise 8. Let $(\Omega_1, \mathcal{F}_1, \mu_1), \ldots, (\Omega_n, \mathcal{F}_n, \mu_n)$ be n σ-finite measure spaces, $n \geq 2$. Let $i_0 \in \{1, \ldots, n\}$ and put $E_1 = \Omega_{i_0}$, $E_2 = \Pi_{i \neq i_0} \Omega_i$, $\mathcal{E}_1 = \mathcal{F}_{i_0}$ and $\mathcal{E}_2 = \otimes_{i \neq i_0} \mathcal{F}_i$. Put $\nu_1 = \mu_{i_0}$, and suppose that ν_2 is a σ-finite measure on $(E_2, \mathcal{E}_2)$ such that for all measurable rectangle $\Pi_{i \neq i_0} A_i \in \amalg_{i \neq i_0} \mathcal{F}_i$, we have $\nu_2 (\Pi_{i \neq i_0} A_i) = \Pi_{i \neq i_0} \mu_i(A_i)$.

1. Show that $\nu_1 \otimes \nu_2$ is a σ-finite measure on the measure space $(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)$ such that for all measurable rectangles $A_1 \times \ldots \times A_n$, we have:

$$\nu_1 \otimes \nu_2(A_1 \times \ldots \times A_n) = \mu_1(A_1) \ldots \mu_n(A_n)$$

2. Show by induction the existence of a measure μ on $\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$, such that for all measurable rectangles $A_1 \times \ldots \times A_n$, we have:

$$\mu(A_1 \times \ldots \times A_n) = \mu_1(A_1) \ldots \mu_n(A_n)$$

3. Show the uniqueness of such measure, denoted $\mu_1 \otimes \ldots \otimes \mu_n$.

4. Show that $\mu_1 \otimes \ldots \otimes \mu_n$ is σ-finite.

5. Let $i_0 \in \{1, \ldots, n\}$. Show that $\mu_{i_0} \otimes (\otimes_{i \neq i_0} \mu_i) = \mu_1 \otimes \ldots \otimes \mu_n$.

Definition 62 *Let $(\Omega_1, \mathcal{F}_1, \mu_1), \ldots, (\Omega_n, \mathcal{F}_n, \mu_n)$ be n σ-finite measure spaces, with $n \geq 2$. We call* **product measure** *of $\mu_1, \ldots, \mu_n$, the unique measure on $\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$, denoted $\mu_1 \otimes \ldots \otimes \mu_n$, such that for all measurable rectangles $A_1 \times \ldots \times A_n$ in $\mathcal{F}_1 \amalg \ldots \amalg \mathcal{F}_n$, we have:*

$$\mu_1 \otimes \ldots \otimes \mu_n(A_1 \times \ldots \times A_n) = \mu_1(A_1) \ldots \mu_n(A_n)$$

This measure is itself σ-finite.

Exercise 9. Prove that the following definition is legitimate:

Definition 63 *We call* **Lebesgue measure** *in $\mathbf{R}^n$, $n \geq 1$, the unique measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, denoted dx, dx^n or $dx_1 \ldots dx_n$, such that for all $a_i \leq b_i$, $i = 1, \ldots, n$, we have:*

$$dx([a_1, b_1] \times \ldots \times [a_n, b_n]) = \prod_{i=1}^{n} (b_i - a_i)$$

Exercise 10.

1. Show that $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx^n)$ is a σ-finite measure space.
2. For $n, p \geq 1$, show that $dx^{n+p} = dx^n \otimes dx^p$.

Exercise 11. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be σ-finite.

1. Let s be a simple function on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$. Show that:

$$\int_{\Omega_1 \times \Omega_2} s d\mu_1 \otimes \mu_2 = \int_{\Omega_1} \left(\int_{\Omega_2} s d\mu_2 \right) d\mu_1 = \int_{\Omega_2} \left(\int_{\Omega_1} s d\mu_1 \right) d\mu_2$$

2. Show the following:

Theorem 31 (Fubini) *Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two σ-finite measure spaces. Let $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to [0, +\infty]$ be a non-negative and measurable map. Then:*

$$\int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2 = \int_{\Omega_1} \left(\int_{\Omega_2} f d\mu_2 \right) d\mu_1 = \int_{\Omega_2} \left(\int_{\Omega_1} f d\mu_1 \right) d\mu_2$$

Exercise 12. Let $(\Omega_1, \mathcal{F}_1, \mu_1), \ldots, (\Omega_n, \mathcal{F}_n, \mu_n)$ be n σ-finite measure spaces, $n \geq 2$. Let $f : (\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n) \to [0, +\infty]$ be a non-negative, measurable map. Let σ be a permutation of $\mathbf{N}_n$, i.e. a bijection from $\mathbf{N}_n$ to itself.

1. For all $\omega \in \Pi_{i \neq \sigma(1)} \Omega_i$, define:

$$J_1(\omega) \stackrel{\triangle}{=} \int_{\Omega_{\sigma(1)}} f(\omega, x) d\mu_{\sigma(1)}(x)$$

   Explain why $J_1 : (\Pi_{i \neq \sigma(1)} \Omega_i, \otimes_{i \neq \sigma(1)} \mathcal{F}_i) \to [0, +\infty]$ is a well defined, non-negative and measurable map.

2. Suppose $J_k : (\Pi_{i \notin \{\sigma(1), \ldots, \sigma(k)\}} \Omega_i, \otimes_{i \notin \{\sigma(1), \ldots, \sigma(k)\}} \mathcal{F}_i) \to [0, +\infty]$ is a non-negative, measurable map, for $1 \leq k < n - 2$. Define:

$$J_{k+1}(\omega) \stackrel{\triangle}{=} \int_{\Omega_{\sigma(k+1)}} J_k(\omega, x) d\mu_{\sigma(k+1)}(x)$$

   and show that:

$$J_{k+1} : (\Pi_{i \notin \{\sigma(1), \ldots, \sigma(k+1)\}} \Omega_i, \otimes_{i \notin \{\sigma(1), \ldots, \sigma(k+1)\}} \mathcal{F}_i) \to [0, +\infty]$$

   is also well-defined, non-negative and measurable.

3. Propose a rigorous definition for the following notation:

$$\int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$

Exercise 13. Further to ex. (12), Let $(f_p)_{p\geq 1}$ be a sequence of non-negative and measurable maps:

$$f_p : (\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n) \to [0, +\infty]$$

such that $f_p \uparrow f$. Define similarly:

$$\begin{aligned} J_1^p(\omega) &\stackrel{\triangle}{=} \int_{\Omega_{\sigma(1)}} f_p(\omega, x) d\mu_{\sigma(1)}(x) \\ J_{k+1}^p(\omega) &\stackrel{\triangle}{=} \int_{\Omega_{\sigma(k+1)}} J_k^p(\omega, x) d\mu_{\sigma(k+1)}(x) \ , \ 1 \leq k < n-2 \end{aligned}$$

1. Show that $J_1^p \uparrow J_1$.
2. Show that if $J_k^p \uparrow J_k$, then $J_{k+1}^p \uparrow J_{k+1}$, $1 \leq k < n-2$.
3. Show that:
$$\int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f_p d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)} \ \uparrow \ \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$
4. Show that the map $\mu : \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n \to [0, +\infty]$, defined by:
$$\mu(E) = \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} 1_E d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$
is a measure on $\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$.
5. Show that for all $E \in \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$, we have:
$$\mu_1 \otimes \ldots \otimes \mu_n(E) = \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} 1_E d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$
6. Show the following:

Theorem 32 *Let $(\Omega_1, \mathcal{F}_1, \mu_1), \ldots, (\Omega_n, \mathcal{F}_n, \mu_n)$ be n σ-finite measure spaces, with $n \geq 2$. Let $f : (\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n) \to [0, +\infty]$ be a non-negative and measurable map. let σ be a permutation of $\mathbf{N}_n$. Then:*

$$\int_{\Omega_1 \times \ldots \times \Omega_n} f d\mu_1 \otimes \ldots \otimes \mu_n = \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$

Exercise 14. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Define:

$$L^1 \stackrel{\triangle}{=} \{f : \Omega \to \bar{\mathbf{R}} \ , \ \exists g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \ , \ f = g \ \mu\text{-a.s.}\}$$

1. Show that if $f \in L^1$, then $|f| < +\infty$, μ-a.s.
2. Suppose there exists $A \subseteq \Omega$, such that $A \notin \mathcal{F}$ and $A \subseteq N$ for some $N \in \mathcal{F}$ with $\mu(N) = 0$. Show that $1_A \in L^1$ and 1_A is not measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\bar{\mathbf{R}})$.

3. Explain why if $f \in L^1$, the integrals $\int |f| d\mu$ and $\int f d\mu$ may not be well defined.

4. Suppose that $f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is a measurable map with $\int |f| d\mu < +\infty$. Show that $f \in L^1$.

5. Show that if $f \in L^1$ and $f = f_1$ μ-a.s. then $f_1 \in L^1$.

6. Suppose that $f \in L^1$ and $g_1, g_2 \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ are such that $f = g_1$ μ-a.s. and $f = g_2$ μ-a.s.. Show that $\int g_1 d\mu = \int g_2 d\mu$.

7. Propose a definition of the integral $\int f d\mu$ for $f \in L^1$ which extends the integral defined on $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

Exercise 15. Further to ex. (14), Let $(f_n)_{n \geq 1}$ be a sequence in L^1, and $f, h \in L^1$, with $f_n \to f$ μ-a.s. and for all $n \geq 1$, $|f_n| \leq h$ μ-a.s..

1. Show the existence of $N_1 \in \mathcal{F}, \mu(N_1) = 0$, such that for all $\omega \in N_1^c$, $f_n(\omega) \to f(\omega)$, and for all $n \geq 1$, $|f_n(\omega)| \leq h(\omega)$.

2. Show the existence of $g_n, g, h_1 \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ and $N_2 \in \mathcal{F}$, $\mu(N_2) = 0$, such that for all $\omega \in N_2^c$, $g(\omega) = f(\omega)$, $h(\omega) = h_1(\omega)$, and for all $n \geq 1$, $g_n(\omega) = f_n(\omega)$.

3. Show the existence of $N \in \mathcal{F}$, $\mu(N) = 0$, such that for all $\omega \in N^c$, $g_n(\omega) \to g(\omega)$, and for all $n \geq 1$, $|g_n(\omega)| \leq h_1(\omega)$.

4. Show that the Dominated Convergence Theorem can be applied to $g_n 1_{N^c}, g 1_{N^c}$ and $h_1 1_{N^c}$.

5. Recall the definition of $\int |f_n - f| d\mu$ when $f, f_n \in L^1$.

6. Show that $\int |f_n - f| d\mu \to 0$.

Exercise 16. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two σ-finite measure spaces. Let f be an element of $L^1_{\mathbf{R}}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$. Let $\theta : (\Omega_2 \times \Omega_1, \mathcal{F}_2 \otimes \mathcal{F}_1) \to (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ be the map defined by $\theta(\omega_2, \omega_1) = (\omega_1, \omega_2)$ for all $(\omega_2, \omega_1) \in \Omega_2 \times \Omega_1$.

1. Let $A = \{\omega_1 \in \Omega_1 \ : \ \int_{\Omega_2} |f(\omega_1, x)| d\mu_2(x) < +\infty\}$. Show that $A \in \mathcal{F}_1$ and $\mu_1(A^c) = 0$.

2. Show that $f(\omega_1, .) \in L^1_{\mathbf{R}}(\Omega_2, \mathcal{F}_2, \mu_2)$ for all $\omega_1 \in A$.

3. Show that $\bar{I}(\omega_1) = \int_{\Omega_2} f(\omega_1, x) d\mu_2(x)$ is well defined for all $\omega_1 \in A$. Let I be an arbitrary extension of $\bar{I}$, on Ω_1.

4. Define $J = I 1_A$. Show that:

$$J(\omega) = 1_A(\omega) \int_{\Omega_2} f^+(\omega, x) d\mu_2(x) - 1_A(\omega) \int_{\Omega_2} f^-(\omega, x) d\mu_2(x)$$

5. Show that J is $\mathcal{F}_1$-measurable and $\mathbf{R}$-valued.

6. Show that $J \in L^1_{\mathbf{R}}(\Omega_1, \mathcal{F}_1, \mu_1)$ and that $J = I$ μ_1-a.s.

7. Propose a definition for the integral:

$$\int_{\Omega_1} \left(\int_{\Omega_2} f(x,y) d\mu_2(y) \right) d\mu_1(x)$$

8. Show that $\int_{\Omega_1} (1_A \int_{\Omega_2} f^+ d\mu_2) d\mu_1 = \int_{\Omega_1 \times \Omega_2} f^+ d\mu_1 \otimes \mu_2$.

9. Show that:

$$\int_{\Omega_1} \left(\int_{\Omega_2} f(x,y) d\mu_2(y) \right) d\mu_1(x) = \int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2 \qquad (1)$$

10. Show that if $f \in L^1_{\mathbf{C}}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$, then the map $\omega_1 \to \int_{\Omega_2} f(\omega_1, y) d\mu_2(y)$ is μ_1-almost surely equal to an element of $L^1_{\mathbf{C}}(\Omega_1, \mathcal{F}_1, \mu_1)$, and furthermore that (1) is still valid.

11. Show that if $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to [0, +\infty]$ is non-negative and measurable, then $f \circ \theta$ is non-negative and measurable, and:

$$\int_{\Omega_2 \times \Omega_1} f \circ \theta d\mu_2 \otimes \mu_1 = \int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2$$

12. Show that if $f \in L^1_{\mathbf{C}}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$, then $f \circ \theta$ is an element of $L^1_{\mathbf{C}}(\Omega_2 \times \Omega_1, \mathcal{F}_2 \otimes \mathcal{F}_1, \mu_2 \otimes \mu_1)$, and:

$$\int_{\Omega_2 \times \Omega_1} f \circ \theta d\mu_2 \otimes \mu_1 = \int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2$$

13. Show that if $f \in L^1_{\mathbf{C}}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$, then the map $\omega_2 \to \int_{\Omega_1} f(x, \omega_2) d\mu_1(x)$ is μ_2-almost surely equal to an element of $L^1_{\mathbf{C}}(\Omega_2, \mathcal{F}_2, \mu_2)$, and furthermore:

$$\int_{\Omega_2} \left(\int_{\Omega_1} f(x,y) d\mu_1(x) \right) d\mu_2(y) = \int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2$$

Theorem 33 *Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two σ-finite measure spaces. Let $f \in L^1_{\mathbf{C}}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$. Then, the map:*

$$\omega_1 \to \int_{\Omega_2} f(\omega_1, x) d\mu_2(x)$$

is μ_1-almost surely equal to an element of $L^1_{\mathbf{C}}(\Omega_1, \mathcal{F}_1, \mu_1)$ and:

$$\int_{\Omega_1} \left(\int_{\Omega_2} f(x,y) d\mu_2(y) \right) d\mu_1(x) = \int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2$$

Furthermore, the map:

$$\omega_2 \to \int_{\Omega_1} f(x, \omega_2) d\mu_1(x)$$

is μ_2-almost surely equal to an element of $L^1_{\mathbf{C}}(\Omega_2, \mathcal{F}_2, \mu_2)$ and:

$$\int_{\Omega_2} \left(\int_{\Omega_1} f(x, y) d\mu_1(x) \right) d\mu_2(y) = \int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2$$

EXERCISE 17. Let $(\Omega_1, \mathcal{F}_1, \mu_1), \ldots, (\Omega_n, \mathcal{F}_n, \mu_n)$ be n σ-finite measure spaces, $n \geq 2$. Let $f \in L^1_{\mathbf{C}}(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n, \mu_1 \otimes \ldots \otimes \mu_n)$. Let σ be a permutation of $\mathbf{N}_n$.

1. For all $\omega \in \Pi_{i \neq \sigma(1)} \Omega_i$, define:

   $$J_1(\omega) \stackrel{\triangle}{=} \int_{\Omega_{\sigma(1)}} f(\omega, x) d\mu_{\sigma(1)}(x)$$

   Explain why J_1 is well defined and equal to an element of $L^1_{\mathbf{C}}(\Pi_{i \neq \sigma(1)} \Omega_i, \otimes_{i \neq \sigma(1)} \mathcal{F}_i, \otimes_{i \neq \sigma(1)} \mu_i)$, $\otimes_{i \neq \sigma(1)} \mu_i$-almost surely.

2. Suppose $1 \leq k < n-2$ and that $\bar{J}_k$ is well defined and equal to an element of:

   $$L^1_{\mathbf{C}}(\Pi_{i \notin \{\sigma(1), \ldots, \sigma(k)\}} \Omega_i, \otimes_{i \notin \{\sigma(1), \ldots, \sigma(k)\}} \mathcal{F}_i, \otimes_{i \notin \{\sigma(1), \ldots, \sigma(k)\}} \mu_i)$$

   $\otimes_{i \notin \{\sigma(1), \ldots, \sigma(k)\}} \mu_i$-almost surely. Define:

   $$J_{k+1}(\omega) \stackrel{\triangle}{=} \int_{\Omega_{\sigma(k+1)}} \bar{J}_k(\omega, x) d\mu_{\sigma(k+1)}(x)$$

   What can you say about J_{k+1}.

3. Show that:

   $$\int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$

   is a well defined complex number. (Propose a definition for it).

4. Show that:

   $$\int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)} = \int_{\Omega_1 \times \ldots \times \Omega_n} f d\mu_1 \otimes \ldots \otimes \mu_n$$

Solutions to Exercises

Exercise 1.

1. Let $\omega_1 \in \Omega_1$. The ω_1-section of $\Omega_1 \times \Omega_2$ in Ω_2, is equal to $\Omega_2 \in \mathcal{F}_2$. So $\Omega_1 \times \Omega_2 \in \Gamma^{\omega_1}$. Suppose $E \in \Gamma^{\omega_1}$. Then $E^{\omega_1} \in \mathcal{F}_2$. $\mathcal{F}_2$ being closed under complementation, $(E^{\omega_1})^c \in \mathcal{F}_2$. However, given $\omega_2 \in \Omega_2$, $\omega_2 \in (E^{\omega_1})^c$ is equivalent to $(\omega_1, \omega_2) \notin E$, i.e. $(\omega_1, \omega_2) \in E^c$. So $(E^{\omega_1})^c = (E^c)^{\omega_1}$. Hence, we see that $(E^c)^{\omega_1} \in \mathcal{F}_2$. It follows that $E^c \in \Gamma^{\omega_1}$, which is therefore closed under complementation. Let $(E_n)_{n\geq 1}$ be a sequence of elements of Γ^{ω_1}. Let $E = \cup_{n=1}^{+\infty} E_n$. For all $n \geq 1$, $(E_n)^{\omega_1} \in \mathcal{F}_2$. $\mathcal{F}_2$ being closed under countable union, $\cup_{n=1}^{+\infty}(E_n)^{\omega_1} \in \mathcal{F}_2$. However, given $\omega_2 \in \Omega_2$, $\omega_2 \in \cup_{n=1}^{+\infty}(E_n)^{\omega_1}$ is equivalent to the existence of $n \geq 1$, such that $(\omega_1, \omega_2) \in E_n$. Hence, it is equivalent to $(\omega_1, \omega_2) \in \cup_{n=1}^{+\infty} E_n = E$. So $\cup_{n=1}^{+\infty}(E_n)^{\omega_1} = E^{\omega_1}$, and we see that $E^{\omega_1} \in \mathcal{F}_2$. It follows that $E \in \Gamma^{\omega_1}$, which is therefore closed under countable union. We have proved that Γ^{ω_1} is a σ-algebra on $\Omega_1 \times \Omega_2$.

2. Let $\omega_1 \in \Omega_1$, and $E = A \times B \in \mathcal{F}_1 \amalg \mathcal{F}_2$ be a measurable rectangle of $\mathcal{F}_1$ and $\mathcal{F}_2$. Suppose $\omega_1 \in A$. Then $(\omega_1, \omega_2) \in E$, if and only if $\omega_2 \in B$. So $E^{\omega_1} = B \in \mathcal{F}_2$. Suppose $\omega_1 \notin A$. Then for all $\omega_2 \in \Omega_2$, $(\omega_1, \omega_2) \notin E$. So $E^{\omega_1} = \emptyset \in \mathcal{F}_2$. In any case, $E^{\omega_1} \in \mathcal{F}_2$. It follows that $E \in \Gamma^{\omega_1}$. We have proved that $\mathcal{F}_1 \amalg \mathcal{F}_2 \subseteq \Gamma^{\omega_1}$.

3. From $\mathcal{F}_1 \amalg \mathcal{F}_2 \subseteq \Gamma^{\omega_1}$ and the fact that Γ^{ω_1} is a σ-algebra on $\Omega_1 \times \Omega_2$, we conclude that $\mathcal{F}_1 \otimes \mathcal{F}_2 = \sigma(\mathcal{F}_1 \amalg \mathcal{F}_2) \subseteq \Gamma^{\omega_1}$. Hence, for all $\omega_1 \in \Omega_1$ and $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, E is an element of Γ^{ω_1}, or equivalently, $E^{\omega_1} \in \mathcal{F}_2$.

4. Let $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to (S, \Sigma)$ be a measurable map, where (S, Σ) is a measurable space. Let $\omega_1 \in \Omega_1$, and $\phi : \Omega_2 \to S$ be the partial map $\omega \to f(\omega_1, \omega)$. Let $B \in \Sigma$. Then $\{f \in B\}$ is an element of $\mathcal{F}_1 \otimes \mathcal{F}_2$. Using 3. it follows that the ω_1-section $\{f \in B\}^{\omega_1}$ of $\{f \in B\}$ is an element of $\mathcal{F}_2$. However, we have:

$$\begin{aligned} \{f \in B\}^{\omega_1} &= \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in \{f \in B\}\} \\ &= \{\omega_2 \in \Omega_2 : f(\omega_1, \omega_2) \in B\} \\ &= \{\omega_2 \in \Omega_2 : \phi(\omega_2) \in B\} \\ &= \{\phi \in B\} \end{aligned}$$

   Hence we see that $\{\phi \in B\} \in \mathcal{F}_2$. This being true for all $B \in \Sigma$, we conclude that ϕ is measurable. This shows that the map $\omega \to f(\omega_1, \omega)$ is measurable.

5. Let $\theta : (\Omega_2 \times \Omega_1, \mathcal{F}_2 \otimes \mathcal{F}_1) \to (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ be defined by $\theta(\omega_2, \omega_1) = (\omega_1, \omega_2)$. From theorem (28), in order to show that θ is measurable, it is sufficient to prove that each coordinate mapping $\theta_1 : (\omega_2, \omega_1) \to \omega_1$ and $\theta_2 : (\omega_2, \omega_1) \to \omega_2$ is measurable. This is indeed the case, since for all $A_1 \in \mathcal{F}_1$ we have $\theta_1^{-1}(A_1) = \Omega_2 \times A_1 \in \mathcal{F}_2 \otimes \mathcal{F}_1$, and for all $A_2 \in \mathcal{F}_2$ we have $\theta_2^{-1}(A_2) = A_2 \times \Omega_1 \in \mathcal{F}_2 \otimes \mathcal{F}_1$. So θ is measurable.

6. Let $\omega_2 \in \Omega_2$. Let $g : (\Omega_2 \times \Omega_1, \mathcal{F}_2 \otimes \mathcal{F}_1) \to (S, \Sigma)$ be the map defined by $g = f \circ \theta$. Having proved in 5. that θ is measurable, since f is itself measurable, g is a measurable map. Applying 4. to g, it follows that the map $\omega \to g(\omega_2, \omega)$ is measurable with respect to $\mathcal{F}_1$ and Σ. In other words, the map $\omega \to f(\omega, \omega_2)$ is measurable with respect to $\mathcal{F}_1$ and Σ. This completes the proof of theorem (29).

Exercise 1

Exercise 2.

1. There is an obvious bijection Φ between $E_1 \times E_2$ and $\Pi_{i\in I}\Omega_i$, defined by $\Phi(\omega_1, \omega_2)(i_1) = \omega_1$, and $\Phi(\omega_1, \omega_2)(i) = \omega_2(i)$ for $i \neq i_1$. The two sets $E_1 \times E_2$ and $\Pi_{i\in I}\Omega_i$ can therefore identified, and f can be viewed as a map defined on $E_1 \times E_2$.

2. Having identified $E_1 \times E_2$ and $\Pi_{i\in I}\Omega_i$, using exercise (10) of Tutorial 6 for the partition $I = \{i_1\} \uplus (I \setminus \{i_1\})$, we obtain $\otimes_{i\in I}\mathcal{F}_i = \mathcal{E}_1 \otimes \mathcal{E}_2$. So $f : (E_1 \times E_2, \mathcal{E}_1 \otimes \mathcal{E}_2) \to (E, \mathcal{B}(E))$ is measurable.

3. From 2. and theorem (29), given $\omega_1 \in E_1$, the map $\omega \to f(\omega_1, \omega)$ defined on E_2, is measurable with respect to $\mathcal{E}_2$ and $\mathcal{B}(E)$. In other words, given $\omega_{i_1} \in \Omega_{i_1}$, the map $\omega \to f(\omega_{i_1}, \omega)$ defined on $\Pi_{i\in I\setminus\{i_1\}}\Omega_i$, is measurable w.r. to $\otimes_{i\in I\setminus\{i_1\}}\mathcal{F}_i$ and $\mathcal{B}(E)$.

Exercise 2

Exercise 3.

1. Suppose there exists a sequence $(\Omega_n)_{n\geq 1}$ of pairwise disjoint elements of $\mathcal{F}$, such that $\Omega = \uplus_{n=1}^{+\infty}\Omega_n$ and $\mu(\Omega_n) < +\infty$ for all $n \geq 1$. Define $A_n = \uplus_{k=1}^{n}\Omega_k$, for all $n \geq 1$. Then:

$$\mu(A_n) = \sum_{k=1}^{n} \mu(\Omega_k) < +\infty$$

and furthermore, $A_n \uparrow \Omega$. So $(\Omega, \mathcal{F}, \mu)$ is σ-finite. Conversely, suppose $(\Omega, \mathcal{F}, \mu)$ is σ-finite. Let $(A_n)_{n\geq 1}$ be a sequence in $\mathcal{F}$, such that $A_n \uparrow \Omega$ and $\mu(A_n) < +\infty$ for all $n \geq 1$. Define $\Omega_1 = A_1$, and $\Omega_n = A_n \setminus A_{n-1}$ for all $n \geq 2$. Then, $(\Omega_n)_{n\geq 1}$ is a sequence of pairwise disjoint elements of $\mathcal{F}$. Since $\Omega_n \subseteq A_n$ for all $n \geq 1$, we have $\mu(\Omega_n) \leq \mu(A_n) < +\infty$. Given $\omega \in \Omega$, since $\Omega = \cup_{n=1}^{+\infty}A_n$, there exists $n \geq 1$ such that $\omega \in A_n$. Let p be the smallest of such n. Then $\omega \in A_p \setminus A_{p-1}$ if $p \geq 2$, or $\omega \in A_1$. In any case, $\omega \in \Omega_p$. Hence, we see that $\Omega = \cup_{n=1}^{+\infty}\Omega_n$ and finally $\Omega = \uplus_{n=1}^{+\infty}\Omega_n$. We conclude that $(\Omega, \mathcal{F}, \mu)$ is σ-finite, if and only if there exists a sequence $(\Omega_n)_{n\geq 1}$ of pairwise disjoint elements of $\mathcal{F}$, such that $\Omega = \uplus_{n=1}^{+\infty}\Omega_n$ and $\mu(\Omega_n) < +\infty$ for all $n \geq 1$.

2. Suppose $(\Omega, \mathcal{F}, \mu)$ is finite. Then $\mu(\Omega) < +\infty$. For all $A \in \mathcal{F}$, since $A \subseteq \Omega$, $\mu(A) \leq \mu(\Omega) < +\infty$. So μ takes values in $\mathbf{R}^+$.

3. Suppose $(\Omega, \mathcal{F}, \mu)$ is finite. Then $\mu(\Omega) < +\infty$. Define $\Omega_n = \Omega$ for all $n \geq 1$. Then $(\Omega_n)_{n\geq 1}$ is a sequence in $\mathcal{F}$ such that $\Omega_n \uparrow \Omega$ and $\mu(\Omega_n) < +\infty$ for all $n \geq 1$. So $(\Omega, \mathcal{F}, \mu)$ is σ-finite.

4. Take $\Omega_n =]-n, n]$ for all $n \geq 1$. Then, $\Omega_n \subseteq \Omega_{n+1}$ and we have $\mathbf{R} = \cup_{n=1}^{+\infty} \Omega_n$. So $\Omega_n \uparrow \mathbf{R}$. Moreover, by definition of the Stieltjes measure (20), $dF(\Omega_n) = F(n) - F(-n) \in \mathbf{R}^+$. In particular, $dF(\Omega_n) < +\infty$ for all $n \geq 1$. We conclude that $(\mathbf{R}, \mathcal{B}(\mathbf{R}), dF)$ is a σ-finite measure space.

Exercise 3

Exercise 4.

1. Let $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$. The characteristic function 1_E is non-negative and measurable with respect to $\mathcal{F}_1 \otimes \mathcal{F}_2$. From theorem (29), for all $\omega_1 \in \Omega_1$, the partial function $x \to 1_E(\omega_1, x)$ is measurable with respect to $\mathcal{F}_2$. It is also non-negative. It follows that the integral $\int_{\Omega_2} 1_E(\omega_1, x) d\mu_2(x)$ is well-defined, for all $\omega_1 \in \Omega_1$. Hence, we see that Φ_E is a well-defined map on Ω_1.

2. Let $E = A \times B \in \mathcal{F}_1 \amalg \mathcal{F}_2$ be a measurable rectangle of $\mathcal{F}_1$ and $\mathcal{F}_2$. For all $\omega_1 \in \Omega_1$, we have:

$$\Phi_E(\omega_1) = \int_{\Omega_2} 1_A(\omega_1) 1_B(x) d\mu_2(x) = \mu_2(B) 1_A(\omega_1)$$

Since $A \in \mathcal{F}_1$, the map 1_A is $\mathcal{F}_1$-measurable, and consequently $\Phi_E = \mu_2(B) 1_A$ is $\mathcal{F}_1$-measurable. Hence, we see that $E \in \mathcal{D}$. We have proved that $\mathcal{F}_1 \amalg \mathcal{F}_2 \subseteq \mathcal{D}$.

3. Suppose μ_2 is a finite measure. Let $A, B \in \mathcal{D}$ with $A \subseteq B$. For all $\omega_1 \in \Omega_1$, from $1_B = 1_A + 1_{B\setminus A}$, we obtain:

$$\int_{\Omega_2} 1_B(\omega_1, x) d\mu_2(x) = \int_{\Omega_2} 1_A(\omega_1, x) d\mu_2(x) + \int_{\Omega_2} 1_{B\setminus A}(\omega_1, x) d\mu_2(x)$$

i.e. $\Phi_B(\omega_1) = \Phi_A(\omega_1) + \Phi_{B\setminus A}(\omega_1)$. μ_2 being a finite measure, all Φ_E's take values in $\mathbf{R}^+$. Hence, it is legitimate to write:

$$\Phi_{B\setminus A} = \Phi_B - \Phi_A$$

Since $A, B \in \mathcal{D}$, both Φ_A and Φ_B are $\mathcal{F}_1$-measurable. We conclude that $\Phi_{B\setminus A}$ is $\mathcal{F}_1$-measurable, and $B \setminus A \in \mathcal{D}$. We have proved that if $A, B \in \mathcal{D}$ with $A \subseteq B$, then $B \setminus A \in \mathcal{D}$.

4. Let $(E_n)_{n\geq 1}$ be a sequence in $\mathcal{F}_1 \otimes \mathcal{F}_2$ with $E_n \uparrow E$. In particular, $E_n \subseteq E_{n+1}$ for all $n \geq 1$, and therefore $1_{E_n} \leq 1_{E_{n+1}}$. Moreover, $E = \cup_{n=1}^{+\infty} E_n$. Let $\omega \in \Omega_1 \times \Omega_2$. If $\omega \in E$, there exists $N \geq 1$ such that $\omega \in E_N$. For all $n \geq N$, we have $1_{E_n}(\omega) = 1 = 1_E(\omega)$. If $\omega \notin E$, then $1_{E_n}(\omega) = 0 = 1_E(\omega)$, for all $n \geq 1$. In any case, $1_{E_n}(\omega) \to 1_E(\omega)$, and consequently $1_{E_n} \uparrow 1_E$.

Given $\omega_1 \in \Omega_1$, we also have $1_{E_n}(\omega_1, .) \uparrow 1_E(\omega_1, .)$. From the monotone convergence theorem (19), we obtain:

$$\int_{\Omega_2} 1_{E_n}(\omega_1, x) d\mu_2(x) \uparrow \int_{\Omega_2} 1_E(\omega_1, x) d\mu_2(x)$$

i.e. $\Phi_{E_n}(\omega_1) \uparrow \Phi_E(\omega_1)$. We conclude that $\Phi_{E_n} \uparrow \Phi_E$.

5. Suppose that μ_2 is a finite measure. From 2., $\mathcal{F}_1 \amalg \mathcal{F}_2 \subseteq \mathcal{D}$, and in particular $\Omega_1 \times \Omega_2 \in \mathcal{D}$. From 3., whenever $A, B \in \mathcal{D}$ are such that $A \subseteq B$, we have $B \setminus A \in \mathcal{D}$. Let $(E_n)_{n \geq 1}$ be a sequence of elements of $\mathcal{D}$, such that $E_n \uparrow E$. For all $n \geq 1$, Φ_{E_n} is an $\mathcal{F}_1$-measurable map. Moreover from 4., $\Phi_{E_n} \uparrow \Phi_E$. In particular, $\Phi_E = \sup_{n \geq 1} \Phi_{E_n}$ and we conclude that Φ_E is measurable with respect to $\mathcal{F}_1$. So $E \in \mathcal{D}$. We have proved that $\mathcal{D}$ is a Dynkin system on $\Omega_1 \times \Omega_2$.

6. Suppose μ_2 is a finite measure. From 5., $\mathcal{D}$ is a Dynkin system on $\Omega_1 \times \Omega_2$. From 2., we have $\mathcal{F}_1 \amalg \mathcal{F}_2 \subseteq \mathcal{D}$. The set of measurable rectangles $\mathcal{F}_1 \amalg \mathcal{F}_2$ being closed under finite intersection, from the Dynkin system theorem (1), we see that $\mathcal{D}$ also contains the σ-algebra generated by $\mathcal{F}_1 \amalg \mathcal{F}_2$, i.e.

$$\mathcal{F}_1 \otimes \mathcal{F}_2 \stackrel{\triangle}{=} \sigma(\mathcal{F}_1 \amalg \mathcal{F}_2) \subseteq \mathcal{D}$$

We conclude that for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, E is an element of $\mathcal{D}$, or equivalently, the map $\Phi_E : (\Omega_1, \mathcal{F}_1) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

7. For all $n \geq 1$, $\mu_2^n(\Omega_2) = \mu_2(\Omega_2^n) < +\infty$. So μ_2^n is a finite measure. It follows from 6. that for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, the map Φ_E^n defined by:

$$\Phi_E^n(\omega_1) \stackrel{\triangle}{=} \int_{\Omega_2} 1_E(\omega_1, x) d\mu_2^n(x)$$

is measurable with respect to $\mathcal{F}_1$. From definition (45), we have:

$$\Phi_E^n(\omega_1) = \int_{\Omega_2} 1_{\Omega_2^n}(x) 1_E(\omega_1, x) d\mu_2(x)$$

Since $\Omega_2^n \uparrow \Omega_2$, we have $1_{\Omega_2^n} \uparrow 1_{\Omega_2} = 1$ and consequently, $1_{\Omega_2^n}(.)1_E(\omega_1, .) \uparrow 1_E(\omega_1, .)$. From the monotone convergence theorem (19), we obtain:

$$\int_{\Omega_2} 1_{\Omega_2^n}(x) 1_E(\omega_1, x) d\mu_2(x) \uparrow \int_{\Omega_2} 1_E(\omega_1, x) d\mu_2(x)$$

i.e. $\Phi_E^n(\omega_1) \uparrow \Phi_E(\omega_1)$, for all $\omega_1 \in \Omega_1$. So $\Phi_E^n \uparrow \Phi_E$.

8. From 7., each Φ_E^n is $\mathcal{F}_1$-measurable and $\Phi_E = \sup_{n \geq 1} \Phi_E^n$. So Φ_E is $\mathcal{F}_1$-measurable, for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$.

9. Let $s = \sum_{i=1}^n \alpha_i 1_{E_i}$ be a simple function on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$. From theorem (29), the map $x \to s(\omega_1, x)$ is $\mathcal{F}_2$-measurable, for all $\omega_1 \in \Omega_1$.

It is also non-negative. It follows that the integral $\int_{\Omega_2} s(\omega_1, x) d\mu_2(x)$ is well-defined, for all $\omega_1 \in \Omega_1$. Moreover:

$$\int_{\Omega_2} s(\omega_1, x) d\mu_2(x) = \sum_{i=1}^{n} \alpha_i \int_{\Omega_2} 1_{E_i}(\omega_1, x) d\mu_2(x)$$

Since $E_i \in \mathcal{F}_1 \otimes \mathcal{F}_2$, from 8., each $\omega \to \int_{\Omega_2} 1_{E_i}(\omega, x) d\mu_2(x)$ is $\mathcal{F}_1$-measurable. We conclude that $\omega \to \int_{\Omega_2} s(\omega, x) d\mu_2(x)$ is also $\mathcal{F}_1$-measurable.

10. Let $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to [0, +\infty]$ be a non-negative and measurable map. From theorem (18), there exists a sequence $(s_n)_{n \geq 1}$ of simple functions on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ such that $s_n \uparrow f$. In particular for all $\omega \in \Omega_1$, $s_n(\omega, .) \uparrow f(\omega, .)$. From the monotone convergence theorem (19), we obtain:
$$\int_{\Omega_2} s_n(\omega, x) d\mu_2(x) \uparrow \int_{\Omega_2} f(\omega, x) d\mu_2(x)$$
However, from 9., each $\omega \to \int_{\Omega_2} s_n(\omega, x) d\mu_2(x)$ is $\mathcal{F}_1$-measurable. We conclude that $\omega \to \int_{\Omega_2} f(\omega, x) d\mu_2(x)$ is also measurable with respect to $\mathcal{F}_1$ and $\mathcal{B}(\bar{\mathbf{R}})$. This proves theorem (30).

Exercise 4

Exercise 5. Let $f : (\Pi_{i \in I} \Omega_i, \otimes_{i \in I} \mathcal{F}_i) \to [0, +\infty]$ be a non-negative and measurable map. Define $E_1 = \Pi_{i \in I \setminus \{i_0\}} \Omega_i$ and $E_2 = \Omega_{i_0}$. Let $\mathcal{E}_1 = \otimes_{i \in I \setminus \{i_0\}} \mathcal{F}_i$ and $\mathcal{E}_2 = \mathcal{F}_{i_0}$. Using exercise (10) of Tutorial 6, having identified $E_1 \times E_2$ and $\Pi_{i \in I} \Omega_i$, we have:
$$\otimes_{i \in I} \mathcal{F}_i = \left(\otimes_{i \in I \setminus \{i_0\}} \mathcal{F}_i \right) \otimes \mathcal{F}_{i_0}$$
i.e. $\otimes_{i \in I} \mathcal{F}_i = \mathcal{E}_1 \otimes \mathcal{E}_2$. It follows that the map f, viewed as a map defined on $E_1 \times E_2$, is measurable with respect to $\mathcal{E}_1 \otimes \mathcal{E}_2$. μ_0 being a σ-finite measure on $(E_2, \mathcal{E}_2)$, from theorem (30), we see that:

$$\omega \to \int_{\Omega_{i_0}} f(\omega, x) d\mu_0(x)$$

is measurable with respect to $\mathcal{E}_1$ and $\mathcal{B}(\bar{\mathbf{R}})$. In other words, it is measurable with respect to $\otimes_{i \in I \setminus \{i_0\}} \mathcal{F}_i$ and $\mathcal{B}(\bar{\mathbf{R}})$. Exercise 5

Exercise 6.

1. Let $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$. The characteristic function 1_E is measurable with respect to $\mathcal{F}_1 \otimes \mathcal{F}_2$ and non-negative. μ_2 being a σ-finite measure on $(\Omega_2, \mathcal{F}_2)$, applying theorem (30), we see that:

$$x \to \int_{\Omega_2} 1_E(x, y) d\mu_2(y)$$

is measurable with respect to $\mathcal{F}_1$ and $\mathcal{B}(\bar{\mathbf{R}})$. It is also non-negative. Hence, the integral:

$$\mu_1 \otimes \mu_2(E) \stackrel{\triangle}{=} \int_{\Omega_1} \left(\int_{\Omega_2} 1_E(x, y) d\mu_2(y) \right) d\mu_1(x)$$

is well-defined, for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$. So $\mu_1 \otimes \mu_2$ is a well-defined map on $\mathcal{F}_1 \otimes \mathcal{F}_2$, with values in $[0, +\infty]$.

2. Suppose $E = \emptyset$. Then $1_E = 0$ and $\mu_1 \otimes \mu_2(E) = 0$. Let $(E_n)_{n\geq 1}$ be a sequence of pairwise disjoint elements of $\mathcal{F}_1 \otimes \mathcal{F}_2$. Let $E = \uplus_{n=1}^{+\infty} E_n$. Then, $1_E = \sum_{n=1}^{+\infty} 1_{E_n}$. From the monotone convergence theorem (19), for all $x \in \Omega_1$, we have:

$$\int_{\Omega_2} 1_E(x,y)d\mu_2(y) = \sum_{n=1}^{+\infty} \int_{\Omega_2} 1_{E_n}(x,y)d\mu_2(y)$$

Applying the monotone convergence theorem once more:

$$\mu_1 \otimes \mu_2(E) = \sum_{n=1}^{+\infty} \int_{\Omega_1} \left(\int_{\Omega_2} 1_{E_n}(x,y)d\mu_2(y) \right) d\mu_1(x)$$

i.e.

$$\mu_1 \otimes \mu_2(E) = \sum_{n=1}^{+\infty} \mu_1 \otimes \mu_2(E_n)$$

We have proved that $\mu_1 \otimes \mu_2$ is a measure on $\mathcal{F}_1 \otimes \mathcal{F}_2$.

3. Let $E = A \times B \in \mathcal{F}_1 \amalg \mathcal{F}_2$ be a measurable rectangle of $\mathcal{F}_1$ and $\mathcal{F}_2$. For all $x \in \Omega_1$, we have:

$$\int_{\Omega_2} 1_E(x,y)d\mu_2(y) = \int_{\Omega_2} 1_A(x)1_B(y)d\mu_2(y) = \mu_2(B)1_A(x)$$

It follows that:

$$\mu_1 \otimes \mu_2(E) = \int_{\Omega_1} \mu_2(B)1_A(x)d\mu_1(x) = \mu_1(A)\mu_2(B)$$

Exercise 6

Exercise 7.

1. By assumption, if $E = A \times B \in \mathcal{F}_1 \amalg \mathcal{F}_2$ is a measurable rectangle of $\mathcal{F}_1$ and $\mathcal{F}_2$, then $\mu_1 \otimes \mu_2(E) = \mu_1(A)\mu_2(B) = \mu(E)$, i.e. $\mu_1 \otimes \mu_2$ and μ coincide on $\mathcal{F}_1 \amalg \mathcal{F}_2$. Let $E \in \mathcal{F}_1 \amalg \mathcal{F}_2$. Then $E \cap (\Omega_1^n \times \Omega_2^n)$ is still a measurable rectangle, i.e. an element of $\mathcal{F}_1 \amalg \mathcal{F}_2$. Hence $\mu_1 \otimes \mu_2(E \cap (\Omega_1^n \times \Omega_2^n)) = \mu(E \cap (\Omega_1^n \times \Omega_2^n))$. It follows that $E \in \mathcal{D}_n$. So $\mathcal{F}_1 \amalg \mathcal{F}_2 \subseteq \mathcal{D}_n$.

2. $\Omega_1 \times \Omega_2 \in \mathcal{F}_1 \amalg \mathcal{F}_2 \subseteq \mathcal{D}_n$. Let $E, F \in \mathcal{D}_n$ be such that $E \subseteq F$. Then $F = E \uplus (F \setminus E)$, and consequently:

$$\mu(F \cap (\Omega_1^n \times \Omega_2^n)) = \mu(E \cap (\Omega_1^n \times \Omega_2^n)) + \mu((F \setminus E) \cap (\Omega_1^n \times \Omega_2^n)) \quad (2)$$

with a similar expression for $\mu_1 \otimes \mu_2$. Since E and F are elements of $\mathcal{D}_n$, we also have:

$$\mu(F \cap (\Omega_1^n \times \Omega_2^n)) = \mu_1 \otimes \mu_2(F \cap (\Omega_1^n \times \Omega_2^n))$$

and:
$$\mu(E \cap (\Omega_1^n \times \Omega_2^n)) = \mu_1 \otimes \mu_2(E \cap (\Omega_1^n \times \Omega_2^n))$$
All the terms involved being finite, it is legitimate to re-arrange and simplify equation (2) and its counterpart for $\mu_1 \otimes \mu_2$, to obtain:
$$\mu((F \setminus E) \cap (\Omega_1^n \times \Omega_2^n)) = \mu_1 \otimes \mu_2((F \setminus E) \cap (\Omega_1^n \times \Omega_2^n))$$
Hence, we see that $F \setminus E \in \mathcal{D}_n$. Let $(E_p)_{p \geq 1}$ be a sequence of elements of $\mathcal{D}_n$, such that $E_p \uparrow E$. For all $p \geq 1$, we have:
$$\mu(E_p \cap (\Omega_1^n \times \Omega_2^n)) = \mu_1 \otimes \mu_2(E_p \cap (\Omega_1^n \times \Omega_2^n))$$
From theorem (7), taking the limit as $p \to +\infty$, we obtain:
$$\mu(E \cap (\Omega_1^n \times \Omega_2^n)) = \mu_1 \otimes \mu_2(E \cap (\Omega_1^n \times \Omega_2^n))$$
It follows that $E \in \mathcal{D}_n$. We have proved that $\mathcal{D}_n$ is a Dynkin system on $\Omega_1 \times \Omega_2$.

3. From 1., $\mathcal{F}_1 \amalg \mathcal{F}_2 \subseteq \mathcal{D}_n$. From 2., $\mathcal{D}_n$ is in fact a Dynkin system on $\Omega_1 \times \Omega_2$. The set of measurable rectangles $\mathcal{F}_1 \amalg \mathcal{F}_2$ being closed under finite intersection, from the Dynkin system theorem (1), we conclude that $\mathcal{D}_n$ actually contains the σ-algebra generated by $\mathcal{F}_1 \amalg \mathcal{F}_2$, i.e. $\mathcal{F}_1 \otimes \mathcal{F}_2 = \sigma(\mathcal{F}_1 \amalg \mathcal{F}_2) \subseteq \mathcal{D}_n$. Hence, for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, E is an element of $\mathcal{D}_n$, or equivalently:
$$\mu(E \cap (\Omega_1^n \times \Omega_2^n)) = \mu_1 \otimes \mu_2(E \cap (\Omega_1^n \times \Omega_2^n))$$
Since $E \cap (\Omega_1^n \times \Omega_2^n) \uparrow E$, using theorem (7) once more, taking the limit as $n \to +\infty$, we obtain $\mu(E) = \mu_1 \otimes \mu_2(E)$. This being true for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, we have proved that $\mu = \mu_1 \otimes \mu_2$.

4. For all $n \geq 1$, let $E_n = \Omega_1^n \times \Omega_2^n$. Then $E_n \uparrow \Omega_1 \times \Omega_2$, and furthermore, $\mu_1 \otimes \mu_2(E_n) = \mu_1(\Omega_1^n)\mu_2(\Omega_2^n) < +\infty$. We conclude that $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$ is a σ-finite measure space.

5. For all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, define:
$$\nu(E) \stackrel{\triangle}{=} \int_{\Omega_2} \left(\int_{\Omega_1} 1_E(x,y) d\mu_1(x) \right) d\mu_2(y)$$
Note that this is the same definition as that of $\mu_1 \otimes \mu_2(E)$, except that the order of integration has been changed. Similarly to exercise (6), using the monotone convergence theorem (19) twice on infinite series, we see that ν is a measure on $\mathcal{F}_1 \otimes \mathcal{F}_2$. Moreover, for all $E = A \times B \in \mathcal{F}_1 \amalg \mathcal{F}_2$ measurable rectangle of $\mathcal{F}_1$ and $\mathcal{F}_2$, we have:
$$\nu(E) = \int_{\Omega_2} \mu_1(A) 1_B(y) d\mu_2(y) = \mu_1(A)\mu_2(B)$$
So ν is another measure on $\mathcal{F}_1 \otimes \mathcal{F}_2$, coinciding with $\mu_1 \otimes \mu_2$ on the set of measurable rectangles $\mathcal{F}_1 \amalg \mathcal{F}_2$. From 3., we see that $\nu = \mu_1 \otimes \mu_2$. We

have proved that for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$:

$$\mu_1 \otimes \mu_2(E) = \int_{\Omega_2} \left(\int_{\Omega_1} 1_E(x,y) d\mu_1(x) \right) d\mu_2(y)$$

Hence, as far as defining $\mu_1 \otimes \mu_2$ is concerned, the order of integration is irrelevant.

Exercise 7

Exercise 8.

1. $(E_1, \mathcal{E}_1, \nu_1)$ and $(E_2, \mathcal{E}_2, \nu_2)$ being two σ-finite measure spaces, $\nu_1 \otimes \nu_2$ is well-defined as a measure on $(E_1 \times E_2, \mathcal{E}_1 \otimes \mathcal{E}_2)$ (exercise (6)). From exercise (7), such measure is itself σ-finite. Having identified $E_1 \times E_2$ with $\Omega_1 \times \ldots \times \Omega_n$, using exercise (10) of Tutorial 6, we have:

$$\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n = \mathcal{F}_{i_0} \otimes (\otimes_{i \neq i_0} \mathcal{F}_i) = \mathcal{E}_1 \otimes \mathcal{E}_2$$

So $\nu_1 \otimes \nu_2$ is a σ-finite measure on $(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)$. Let $A = A_1 \times \ldots \times A_n$ be a measurable rectangle of $\mathcal{F}_1, \ldots, \mathcal{F}_n$. Identifying A with $A_{i_0} \times (\Pi_{i \neq i_0} A_i)$, we have:

$$\nu_1 \otimes \nu_2(A) = \nu_1(A_{i_0}) \nu_2(\Pi_{i \neq i_0} A_i)$$

Since by assumption, $\nu_2(\Pi_{i \neq i_0} A_i) = \Pi_{i \neq i_0} \mu_i(A_i)$, we conclude:

$$\nu_1 \otimes \nu_2(A) = \mu_1(A_1) \ldots \mu_n(A_n)$$

2. If $n = 2$, there exists a measure μ on $\mathcal{F}_1 \otimes \mathcal{F}_2$, such that for all measurable rectangle $A_1 \times A_2 \in \mathcal{F}_1 \amalg \mathcal{F}_2$, we have:

$$\mu(A_1 \times A_2) = \mu_1(A_1) \mu_2(A_2)$$

In fact, from exercise (7), such measure is unique, σ-finite and equal to $\mu_1 \otimes \mu_2$. Suppose the following induction hypothesis is true for $n \geq 2$: *Given n σ-finite measure spaces $(\Omega_1, \mathcal{F}_1, \mu_1), \ldots, (\Omega_n, \mathcal{F}_n, \mu_n)$, there exists a measure μ on $(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)$, such that for all measurable rectangles $A_1 \times \ldots \times A_n$, we have:*

$$\mu(A_1 \times \ldots \times A_n) = \mu_1(A_1) \ldots \mu_n(A_n)$$

Moreover, such measure μ is σ-finite.
Let us prove this induction hypothesis for $n+1$. Hence, suppose we have $n+1$ σ-finite measure spaces. Take $E_1 = \Omega_1$ and $E_2 = \Omega_2 \times \ldots \times \Omega_{n+1}$. Let $\mathcal{E}_1 = \mathcal{F}_1$ and $\mathcal{E}_2 = \mathcal{F}_2 \otimes \ldots \otimes \mathcal{F}_{n+1}$. Put $\nu_1 = \mu_1$. From our induction hypothesis, there exists a σ-finite measure ν_2 on $(E_2, \mathcal{E}_2)$, such that for all measurable rectangles $A_2 \times \ldots \times A_{n+1}$, we have:

$$\nu_2(A_2 \times \ldots \times A_{n+1}) = \mu_2(A_2) \ldots \mu_{n+1}(A_{n+1})$$

All the conditions of question 1. are met: we conclude that $\nu_1 \otimes \nu_2$ is a σ-finite measure on $(\Omega_1 \times \ldots \times \Omega_{n+1}, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_{n+1})$ such that for all measurable rectangles $A = A_1 \times \ldots \times A_{n+1}$:

$$\nu_1 \otimes \nu_2(A) = \mu_1(A_1) \ldots \mu_{n+1}(A_{n+1})$$

This proves our induction hypothesis for $n+1$.
We have proved that for all $n \geq 2$, and σ-finite measure spaces $(\Omega_1, \mathcal{F}_1, \mu_1)$, $\ldots$, $(\Omega_n, \mathcal{F}_n, \mu_n)$, there exists a σ-finite measure μ on $(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)$, such that for all measurable rectangles $A = A_1 \times \ldots \times A_n$, $\mu(A) = \mu_1(A_1) \ldots \mu_n(A_n)$. Note that this is a little bit stronger (μ is σ-finite !), than what was required by the actual wording of the question. However the σ-finite property was required to carry out the induction argument, based on exercises (6) and (7).

3. Let μ and ν be two measures on $(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)$, such that for all measurable rectangles $A = A_1 \times \ldots \times A_n$:

$$\mu(A) = \nu(A) = \mu_1(A_1) \ldots \mu_n(A_n)$$

For all $i = 1, \ldots, n$, let $(\Omega_i^p)_{p \geq 1}$ be a sequence of elements of $\mathcal{F}_i$, such that $\Omega_i^p \uparrow \Omega_i$, and $\mu_i(\Omega_i^p) < +\infty$ for all $p \geq 1$. Define $E_p = \Omega_1^p \times \ldots \times \Omega_n^p$. Then $E_p \uparrow \Omega_1 \times \ldots \times \Omega_n$, and for all $p \geq 1$, $\mu(E_p) = \nu(E_p) < +\infty$. Define:

$$\mathcal{D}_p \stackrel{\triangle}{=} \{A \in \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n : \mu(A \cap E_p) = \nu(A \cap E_p)\}$$

Then $\mathcal{D}_p$ is a Dynkin system on $\Omega_1 \times \ldots \times \Omega_n$. Moreover, by assumption, $\mathcal{F}_1 \amalg \ldots \amalg \mathcal{F}_n \subseteq \mathcal{D}_p$. The set of measurable rectangles $\mathcal{F}_1 \amalg \ldots \amalg \mathcal{F}_n$ being closed under finite intersection, from the Dynkin system theorem (1), we see that $\mathcal{D}_p$ actually contains the σ-algebra generated by $\mathcal{F}_1 \amalg \ldots \amalg \mathcal{F}_n$, i.e.

$$\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n \stackrel{\triangle}{=} \sigma(\mathcal{F}_1 \amalg \ldots \amalg \mathcal{F}_n) \subseteq \mathcal{D}_p$$

It follows that for all $A \in \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$, we have:

$$\mu(A \cap E_p) = \nu(A \cap E_p)$$

Using theorem (7), taking the limit as $p \to +\infty$, we obtain $\mu(A) = \nu(A)$. This being true for all $A \in \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$, we conclude that $\mu = \nu$. This proves the uniqueness of the measure μ on $(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)$, denoted $\mu_1 \otimes \ldots \otimes \mu_n$, such that $\mu(A) = \mu_1(A_1) \ldots \mu_n(A_n)$, for all measurable rectangles $A = A_1 \times \ldots \times A_n$.

4. The fact that $\mu = \mu_1 \otimes \ldots \otimes \mu_n$ is σ-finite was actually proved as part of the induction argument of 2. However, it is very easy to justify that point directly: if $(\Omega_i^p)_{p \geq 1}$ is a sequence of elements of $\mathcal{F}_i$ such that $\Omega_i^p \uparrow \Omega_i$ and $\mu(\Omega_i^p) < +\infty$ for all $p \geq 1$, defining $E_p = \Omega_1^p \times \ldots \times \Omega_n^p$, we have $E_p \uparrow \Omega_1 \times \ldots \times \Omega_n$, and furthermore:

$$\mu(E_p) = \mu_1(\Omega_1^p) \ldots \mu_n(\Omega_n^p) < +\infty$$

So $\mu_1 \otimes \ldots \otimes \mu_n$ is indeed a σ-finite measure.

5. $\mu_{i_0} \otimes (\otimes_{i \neq i_0} \mu_i)$ is a measure on $(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)$ which coincides with $\mu_1 \otimes \ldots \otimes \mu_n$ on the measurable rectangles. From the uniqueness property proved in 3., the two measures are therefore equal, i.e. $\mu_{i_0} \otimes (\otimes_{i \neq i_0} \mu_i) = \mu_1 \otimes \ldots \otimes \mu_n$.

Exercise 8

Exercise 9. Showing that definition (63) is legitimate amounts to proving the existence and uniqueness of a measure μ on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, such that for all $a_i \leq b_i$, $i \in \mathbf{N}_n$, we have:

$$\mu([a_1, b_1] \times \ldots \times [a_n, b_n]) = \prod_{i=1}^{n}(b_i - a_i) \quad (3)$$

For $i \in \mathbf{N}_n$, let $(\Omega_i, \mathcal{F}_i, \mu_i)$ be the measure space $(\mathbf{R}, \mathcal{B}(\mathbf{R}), dx)$, where dx is the Lebesgue measure on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$. Each $(\Omega_i, \mathcal{F}_i, \mu_i)$ being σ-finite, from definition (62), there exists a measure $\mu = \mu_1 \otimes \ldots \otimes \mu_n$ on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R}))$, such that for all measurable rectangles $A = A_1 \times \ldots \times A_n$, we have:

$$\mu(A) = dx(A_1) \ldots dx(A_n) \quad (4)$$

From exercise (18) of Tutorial 6, we have $\mathcal{B}(\mathbf{R}^n) = \mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R})$. So μ is in fact a measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$. Moreover, taking A_i of the form $A_i = [a_i, b_i]$ for $a_i \leq b_i$, we see from (4) that equation (3) is satisfied. Hence, we have proved the existence of μ. Suppose that ν is another measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ satisfying the property of definition (63). Let $\mathcal{C} = \{[a_1, b_1] \times \ldots \times [a_n, b_n] : a_i \leq b_i, \forall i \in \mathbf{N}_n\}$. Then $\mathcal{C}$ is closed under finite intersection. Given $p \geq 1$, let $E_p = [-p, p]^n$, and define:

$$\mathcal{D}_p \stackrel{\triangle}{=} \{A \in \mathcal{B}(\mathbf{R}^n) : \mu(A \cap E_p) = \nu(A \cap E_p)\}$$

Then $\mathcal{D}_p$ is a Dynkin system on $\mathbf{R}^n$, and we have $\mathcal{C} \subseteq \mathcal{D}_p$. From the Dynkin system theorem (1), we see that $\mathcal{D}_p$ actually contains the σ-algebra generated by $\mathcal{C}$, i.e. $\sigma(\mathcal{C}) \subseteq \mathcal{D}_p$. However, we claim that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbf{R}^n)$. Indeed, from:

$$\mathcal{C} \subseteq \mathcal{B}(\mathbf{R}) \amalg \ldots \amalg \mathcal{B}(\mathbf{R}) \subseteq \mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R}) = \mathcal{B}(\mathbf{R}^n)$$

we obtain $\sigma(\mathcal{C}) \subseteq \mathcal{B}(\mathbf{R}^n)$. Furthermore, if we define:

$$\mathcal{E} \stackrel{\triangle}{=} \{[a, b] : a \leq b, a, b \in \mathbf{R}\}$$

then every open set in $\mathbf{R}$ can be expressed as a countable union of elements of $\mathcal{E}$ (see the proof of theorem (6)), and it is easy to check that $\mathcal{B}(\mathbf{R}) = \sigma(\mathcal{E})$. From theorem (26), we have:

$$\mathcal{B}(\mathbf{R}^n) = \mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R}) = \sigma(\mathcal{E} \amalg \ldots \amalg \mathcal{E})$$

Since any element of $\mathcal{E} \amalg \ldots \amalg \mathcal{E}$ is of the form $A_1 \times \ldots \times A_n$ where each A_i is either equal to $\mathbf{R} = \cup_{p=1}^{+\infty}[-p, p]$, or is an element of $\mathcal{E}$, any element of $\mathcal{E} \amalg \ldots \amalg \mathcal{E}$ can in fact be expressed as a countable union of elements of $\mathcal{C}$. Hence, $\mathcal{E} \amalg \ldots \amalg \mathcal{E} \subseteq \sigma(\mathcal{C})$ and consequently, $\mathcal{B}(\mathbf{R}^n) = \sigma(\mathcal{E} \amalg \ldots \amalg \mathcal{E}) \subseteq \sigma(\mathcal{C})$. We conclude that $\mathcal{B}(\mathbf{R}^n) = \sigma(\mathcal{C})$[1], and finally $\mathcal{B}(\mathbf{R}^n) \subseteq \mathcal{D}_p$. It follows that for all $A \in \mathcal{B}(\mathbf{R}^n)$, we have $\mu(A \cap E_p) = \nu(A \cap E_p)$. Using theorem (7), taking the limit as $p \to +\infty$, we obtain $\mu(A) = \nu(A)$. This being true for all $A \in \mathcal{B}(\mathbf{R}^n)$, we see that $\mu = \nu$. We have proved the uniqueness of μ. Exercise 9

Exercise 10.

[1] We proved something very similar in exercise (7) of Tutorial 6.

1. For all $p \geq 1$, define $E_p = [-p, p]^n$. Then, $E_p \uparrow \mathbf{R}^n$, and furthermore $dx^n(E_p) = (2p)^n < +\infty$, for all $p \geq 1$. So dx^n is a σ-finite measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$.

2. Let $a_i \leq b_i$ for $i \in \mathbf{N}_{n+p}$, and $A = [a_1, b_1] \times \ldots \times [a_{n+p}, b_{n+p}]$. Then, $dx^n \otimes dx^p(A) = dx^{n+p}(A) = \Pi_{i=1}^{n+p}(b_i - a_i)$. From the uniqueness property of definition (63), we conclude that:
$$dx^{n+p} = dx^n \otimes dx^p$$

Exercise 10

Exercise 11.

1. From exercise (6) and exercise (7), for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, we have:
$$\mu_1 \otimes \mu_2(E) = \int_{\Omega_1} \left(\int_{\Omega_2} 1_E(x, y) d\mu_2(y) \right) d\mu_1(x)$$
together with:
$$\mu_1 \otimes \mu_2(E) = \int_{\Omega_2} \left(\int_{\Omega_1} 1_E(x, y) d\mu_1(x) \right) d\mu_2(y)$$
Hence:
$$\int_{\Omega_1 \times \Omega_2} 1_E d\mu_1 \otimes \mu_2 = \int_{\Omega_1} \left(\int_{\Omega_2} 1_E d\mu_2 \right) d\mu_1 = \int_{\Omega_2} \left(\int_{\Omega_1} 1_E d\mu_1 \right) d\mu_2$$
By linearity, it follows that if $s = \sum_{i=1}^n \alpha_i 1_{E_i}$ is a simple function on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$, we have:
$$\int_{\Omega_1 \times \Omega_2} s d\mu_1 \otimes \mu_2 = \int_{\Omega_1} \left(\int_{\Omega_2} s d\mu_2 \right) d\mu_1 = \int_{\Omega_2} \left(\int_{\Omega_1} s d\mu_1 \right) d\mu_2$$

2. Let $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to [0, +\infty]$ be a non-negative and measurable map. From theorem (18), there exists a sequence $(s_n)_{n \geq 1}$ of simple functions on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$, such that $s_n \uparrow f$. In particular, for all $x \in \Omega_1$, $s_n(x, .) \uparrow f(x, .)$. From the monotone convergence theorem (19), for all $x \in \Omega_1$, we have:
$$\int_{\Omega_2} s_n(x, y) d\mu_2(y) \uparrow \int_{\Omega_2} f(x, y) d\mu_2(y)$$
and applying theorem (19) once more, we obtain:
$$\int_{\Omega_1} \left(\int_{\Omega_2} s_n(x, y) d\mu_2(y) \right) d\mu_1(x) \uparrow \int_{\Omega_1} \left(\int_{\Omega_2} f(x, y) d\mu_2(y) \right) d\mu_1(x)$$
and similarly:
$$\int_{\Omega_2} \left(\int_{\Omega_1} s_n(x, y) d\mu_1(x) \right) d\mu_2(y) \uparrow \int_{\Omega_2} \left(\int_{\Omega_1} f(x, y) d\mu_1(x) \right) d\mu_2(y)$$

However, from $s_n \uparrow f$ and the monotone convergence theorem:

$$\int_{\Omega_1\times\Omega_2} s_n d\mu_1\otimes\mu_2 \uparrow \int_{\Omega_1\times\Omega_2} f d\mu_1\otimes\mu_2$$

Using 1., for all $n\geq 1$, we have:

$$\int_{\Omega_1\times\Omega_2} s_n d\mu_1\otimes\mu_2 = \int_{\Omega_1}\left(\int_{\Omega_2} s_n d\mu_2\right)d\mu_1 = \int_{\Omega_2}\left(\int_{\Omega_1} s_n d\mu_1\right)d\mu_2$$

Hence, taking the limit as $n\to+\infty$, we obtain:

$$\int_{\Omega_1\times\Omega_2} f d\mu_1\otimes\mu_2 = \int_{\Omega_1}\left(\int_{\Omega_2} f d\mu_2\right)d\mu_1 = \int_{\Omega_2}\left(\int_{\Omega_1} f d\mu_1\right)d\mu_2$$

This proves theorem (31).

Exercise 11

Exercise 12.

1. Let $f : (\Omega_1\times\ldots\times\Omega_n, \mathcal{F}_1\otimes\ldots\otimes\mathcal{F}_n)\to[0,+\infty]$ be a non-negative and measurable map. Since $\mu_{\sigma(1)}$ is a σ-finite measure, from exercise (5), the map:

$$J_1 : \omega \to \int_{\Omega_{\sigma(1)}} f(\omega,x)d\mu_{\sigma(1)}(x)$$

is well-defined on $\Pi_{i\neq\sigma(1)}\Omega_i$, and measurable w.r. to $\otimes_{i\neq\sigma(1)}\mathcal{F}_i$.

2. If $J_k : (\Pi_{i\not\in\{\sigma(1),\ldots,\sigma(k)\}}\Omega_i, \otimes_{i\not\in\{\sigma(1),\ldots,\sigma(k)\}}\mathcal{F}_i)\to[0,+\infty]$ is non-negative and measurable, for $1\leq k\leq n-2$, from exercise (5):

$$J_{k+1} : \omega \to \int_{\Omega_{\sigma(k+1)}} J_k(\omega,x)d\mu_{\sigma(k+1)}(x)$$

is also well-defined on $\Pi_{i\not\in\{\sigma(1),\ldots,\sigma(k+1)\}}\Omega_i$, and measurable with respect to $\otimes_{i\not\in\{\sigma(1),\ldots,\sigma(k+1)\}}\mathcal{F}_i$.

3. The integral:

$$I = \int_{\Omega_{\sigma(n)}}\ldots\int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)}\ldots d\mu_{\sigma(n)}$$

can be rigorously defined as:

$$I \triangleq \int_{\Omega_{\sigma(n)}} J_{n-1}d\mu_{\sigma(n)}$$

where J_{n-1} is given by 1. and 2.

Exercise 12

Exercise 13.

1. Since $f_p \uparrow f$, for all $\omega \in \Pi_{i\neq\sigma(1)}\Omega_i$, we have $f_p(\omega,.) \uparrow f(\omega,.)$. From the monotone convergence theorem (19), we obtain:

$$\int_{\Omega_{\sigma(1)}} f_p(\omega,x)d\mu_{\sigma(1)}(x) \uparrow \int_{\Omega_{\sigma(1)}} f(\omega,x)d\mu_{\sigma(1)}(x)$$

i.e. $J_1^p \uparrow J_1$.

2. Suppose $J_k^p \uparrow J_k$, $1 \leq k \leq n-2$. For all $\omega \in \Pi_{i\not\in\{\sigma(1),\ldots,\sigma(k+1)\}}\Omega_i$, we have $J_k^p(\omega,.) \uparrow J_k(\omega,.)$. From the monotone convergence theorem (19), we have:

$$\int_{\Omega_{\sigma(k+1)}} J_k^p(\omega,x)d\mu_{\sigma(k+1)}(x) \uparrow \int_{\Omega_{\sigma(k+1)}} J_k(\omega,x)d\mu_{\sigma(k+1)}(x)$$

i.e. $J_{k+1}^p \uparrow J_{k+1}$.

3. From 2., $J_{n-1}^p \uparrow J_{n-1}$. Again from theorem (19):

$$\int_{\Omega_{\sigma(n)}} J_{n-1}^p d\mu_{\sigma(n)} \uparrow \int_{\Omega_{\sigma(n)}} J_{n-1} d\mu_{\sigma(n)}$$

In other words:

$$\int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f_p d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)} \uparrow \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$

4. For all $E \in \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$, we have:

$$\mu(E) \stackrel{\triangle}{=} \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} 1_E d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$

So $\mu(\emptyset) = 0$. If $(E_p)_{p\geq 1}$ is a sequence of pairwise disjoint elements of $\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$, and $E = \uplus_{i=1}^{+\infty} E_i$, defining for $p \geq 1$, $f_p = \sum_{i=1}^p 1_{E_i}$, we have $f_p \uparrow 1_E$. It follows from 3.:

$$\int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f_p d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)} \uparrow \mu(E)$$

By linearity, we obtain $\sum_{i=1}^p \mu(E_i) \uparrow \mu(E)$, or equivalently:

$$\mu(E) = \sum_{i=1}^{+\infty} \mu(E_i)$$

We have proved that μ is indeed a measure on $\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$.

5. Let $E = A_1 \times \ldots \times A_n$ be a measurable rectangle of $(\mathcal{F}_i)_{i\in\mathbf{N}_n}$. Then:

$$\mu(E) = \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} 1_E d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)} = \mu_1(A_1) \ldots \mu_n(A_n)$$

From the uniqueness property of definition (62), it follows that μ coincide with the product measure $\mu_1 \otimes \ldots \otimes \mu_n$. Hence, for all $E \in \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$, we have:

$$\mu_1 \otimes \ldots \otimes \mu_n(E) = \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} 1_E d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$

6. From 5., for all $E \in \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$, we have:

$$\int_{\Omega_1 \times \ldots \times \Omega_n} 1_E d\mu_1 \otimes \ldots \otimes \mu_n = \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} 1_E d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$

If s is a simple function on $(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)$, by linearity, we obtain:

$$\int_{\Omega_1 \times \ldots \times \Omega_n} s d\mu_1 \otimes \ldots \otimes \mu_n = \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} s d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$

Since any $f : (\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n) \to [0, +\infty]$ non-negative and measurable, can be approximated from below by simple functions (theorem (18)), we conclude from the monotone convergence theorem (19) and question 3., that:

$$\int_{\Omega_1 \times \ldots \times \Omega_n} f d\mu_1 \otimes \ldots \otimes \mu_n = \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$

This proves theorem (32).

Exercise 13

Exercise 14.

1. Suppose $f \in L^1$. There exists $g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ such that $f = g$, μ-a.s. Hence, there exists $N \in \mathcal{F}$ with $\mu(N) = 0$, such that $f(\omega) = g(\omega)$ for all $\omega \in N^c$. However, g has values in $\mathbf{R}$. So $|f(\omega)| < +\infty$ for all $\omega \in N^c$. It follows that $|f| < +\infty$ μ-a.s.

2. We assume the existence of $A \subseteq \Omega$, such that $A \notin \mathcal{F}$ and $A \subseteq N$, for some $N \in \mathcal{F}$ with $\mu(N) = 0$. Since $A \notin \mathcal{F}$, 1_A is not measurable. However, for all $\omega \in N^c$, we have $1_A(\omega) = 0$. So $1_A = 0$, μ-a.s. Since $0 \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, we see that $1_A \in L^1$.

3. Suppose $f \in L^1$. As indicated in 2., we have no guarantee that f be a measurable map. Hence, the integrals $\int |f| d\mu$ and $\int f d\mu$ may not be meaningful.

4. Let $f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ be a measurable map, such that $\int |f| d\mu < +\infty$. In particular, we have $\mu(\{|f| = +\infty\}) = 0$ (see exercise (7) of Tutorial 5). Define $g = f 1_{\{|f| < +\infty\}}$. Then, $f(\omega) = g(\omega)$ for all $\omega \in \{|f| < +\infty\}$. So $f = g$ μ-a.s. However, g is measurable, with values in $\mathbf{R}$, and such that:

$$\int |g| d\mu = \int |f| d\mu < +\infty$$

So $g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, and finally $f \in L^1$.

5. Suppose $f \in L^1$ and $f = f_1$ μ-a.s. for some map $f_1 : \Omega \to \bar{\mathbf{R}}$. There exists $g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, such that $f = g$ μ-a.s. There exists $N \in \mathcal{F}$ with $\mu(N) = 0$, such that $f(\omega) = g(\omega)$ for all $\omega \in N^c$. Also, there exists $N_1 \in \mathcal{F}$ with $\mu(N_1) = 0$, such that $f(\omega) = f_1(\omega)$ for all $\omega \in N_1^c$. It follows that $f_1(\omega) = g(\omega)$ for all $\omega \in (N \cup N_1)^c$. Since $\mu(N \cup N_1) \leq \mu(N) + \mu(N_1) = 0$, we see that $f_1 = g$ μ-a.s. We conclude that $f_1 \in L^1$.

6. Let $f \in L^1$. Let $g_1, g_2 \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ with $f = g_1$ μ-a.s. and $f = g_2$ μ-a.s. There exist $N_1, N_2 \in \mathcal{F}$ with $\mu(N_1) = \mu(N_2) = 0$, such that $f(\omega) = g_1(\omega)$ for all $\omega \in N_1^c$, and $f(\omega) = g_2(\omega)$ for all $\omega \in N_2^c$. So $g_1(\omega) = g_2(\omega)$ for all $\omega \in (N_1 \cup N_2)^c$, and $\mu(N_1 \cup N_2) = 0$. So $g_1 = g_2$ μ-a.s. and finally $\int g_1 d\mu = \int g_2 d\mu$.

7. For all $f \in L^1$, we define:

$$\int f d\mu \stackrel{\triangle}{=} \int g d\mu \tag{5}$$

where g is any element of $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ such that $f = g$ μ-a.s. From 6., if $g_1, g_2 \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ are such that $f = g_1$ μ-a.s. and $f = g_2$ μ-a.s., then $\int g_1 d\mu = \int g_2 d\mu$. So $\int f d\mu$ is well-defined. If $f \in L^1 \cap L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, then $\int f d\mu$ as defined in (5) coincide with $\int f d\mu$, in its usual sense.

Exercise 14

Exercise 15.

1. By assumption, $f_n \to f$ μ-a.s. There exists $N \in \mathcal{F}$, $\mu(N) = 0$, such that $f_n(\omega) \to f(\omega)$ for all $\omega \in N^c$. Also, for all $n \geq 1$, $|f_n| \leq h$ μ-a.s. There exists $M_n \in \mathcal{F}$ with $\mu(M_n) = 0$ such that $|f_n(\omega)| \leq h(\omega)$ for all $\omega \in M_n^c$. Let $N_1 = N \cup (\cup_{n \geq 1} M_n)$. Then $N_1 \in \mathcal{F}$, and:

$$\mu(N_1) \leq \mu(N) + \sum_{n=1}^{+\infty} \mu(M_n) = 0$$

So $\mu(N_1) = 0$. Moreover, for all $\omega \in N_1^c$, we have $f_n(\omega) \to f(\omega)$ and for all $n \geq 1$, $|f_n(\omega)| \leq h(\omega)$.

2. Since $f \in L^1$, there exists $g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ such that $f = g$ μ-a.s. There exists $N \in \mathcal{F}$ with $\mu(N) = 0$, such that $f(\omega) = g(\omega)$ for all $\omega \in N^c$. Similarly, there exists $h_1 \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, and a set $M_1' \in \mathcal{F}$ with $\mu(M_1') = 0$, such that $h(\omega) = h_1(\omega)$ for all $\omega \in (M_1')^c$. For all $n \geq 1$, there exist $g_n \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ and $M_n \in \mathcal{F}$ with $\mu(M_n) = 0$ such that $g_n(\omega) = f_n(\omega)$ for all $\omega \in M_n^c$. Let $N_2 = N \cup M_1' \cup (\cup_{n \geq 1} M_n)$. Then $N_2 \in \mathcal{F}$, $\mu(N_2) = 0$, and for all $\omega \in N_2^c$, we have $g(\omega) = f(\omega)$, $h_1(\omega) = h(\omega)$ and $g_n(\omega) = f_n(\omega)$ for all $n \geq 1$.

3. Let $N = N_1 \cup N_2$ where N_1 and N_2 are given by 1. and 2. respectively. Then $N \in \mathcal{F}$, $\mu(N) = 0$, and for all $\omega \in N^c$, we have $g_n(\omega) \to g(\omega)$ and $|g_n(\omega)| \leq h_1(\omega)$ for all $n \geq 1$.

4. $(g_n 1_{N^c})_{n\geq 1}$ is a sequence of $\mathbf{C}$-valued (in fact $\mathbf{R}$-valued) measurable maps, such that $g_n 1_{N^c}(\omega) \to g 1_{N^c}(\omega)$ for all $\omega \in \Omega$. Moreover, $h_1 1_{N^c}$ is an element of $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ such that for all $n \geq 1$, $|g_n 1_{N^c}| \leq h_1 1_{N^c}$. Hence, we can apply the dominated convergence theorem (23).

5. When $f, f_n \in L^1$, we have $|f_n - f| \in L^1$, and $\int |f_n - f| d\mu$ is defined as $\int k d\mu$ where k is any element of $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ such that $|f_n - f| = k$ μ-a.s. In fact, $|g_n - g| \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ and $|f_n - f| = |g_n - g|$ μ-a.s. So $\int |f_n - f| d\mu = \int |g_n - g| d\mu$.

6. From 4., and the dominated convergence theorem (23), we have $\lim \int 1_{N^c} |g_n - g_n| d\mu = 0$ and consequently, $\int |g_n - g| d\mu \to 0$. It follows from 5. that $\int |f_n - f| d\mu \to 0$.

Exercise 15

Exercise 16.

1. We define $A = \{\omega_1 \in \Omega_1 : \int_{\Omega_2} |f(\omega_1, x)| d\mu_2(x) < +\infty\}$. From theorem (30), the map $\phi : \omega_1 \to \int_{\Omega_2} |f(\omega_1, x)| d\mu_2(x)$ is measurable with respect to $\mathcal{F}_1$ and $\mathcal{B}(\bar{\mathbf{R}})$. It follows that:

$$A = \phi^{-1}([-\infty, +\infty[) \in \mathcal{F}_1$$

From theorem (31), we have:

$$\int_{\Omega_1} \left(\int_{\Omega_2} |f(\omega_1, x)| d\mu_2(x) \right) d\mu_1(\omega_1) = \int_{\Omega_1 \times \Omega_2} |f| d\mu_1 \otimes \mu_2 < +\infty$$

Using exercise (7) (11.) of Tutorial 5, we have $\mu_1(A^c) = 0$.

2. For all $\omega_1 \in A$, we have $\int_{\Omega_2} |f(\omega_1, x)| d\mu_2(x) < +\infty$. From theorem (29), the map $f(\omega_1, .)$ is measurable with respect to $\mathcal{F}_2$, for all $\omega_1 \in \mathcal{F}_1$. f being $\mathbf{R}$-valued, we conclude that for all $\omega_1 \in A$, $f(\omega_1, .) \in L^1_{\mathbf{R}}(\Omega_2, \mathcal{F}_2, \mu_2)$.

3. For all $\omega_1 \in A$, the map $f(\omega_1, .)$ lies in $L^1_{\mathbf{R}}(\Omega_2, \mathcal{F}_2, \mu_2)$. Hence, $\bar{I}(\omega_1) = \int_{\Omega_2} f(\omega_1, x) d\mu_2(x)$ is well-defined for all $\omega_1 \in A$.

4. If $\omega \in A$, then $J(\omega) = I(\omega) = \bar{I}(\omega) = \int_{\Omega_2} f(\omega, x) d\mu_2(x)$. Hence:

$$J(\omega) = 1_A(\omega) \int_{\Omega_2} f^+(\omega, x) d\mu_2(x) - 1_A(\omega) \int_{\Omega_2} f^-(\omega, x) d\mu_2(x)$$

This equation still holds if $\omega \notin A$.

5. $\int_{\Omega_2} f^+(\omega, x) d\mu_2(x) < +\infty$ and $\int_{\Omega_2} f^-(\omega, x) d\mu_2(x) < +\infty$, for all $\omega \in A$. If $\omega \notin A$, then $J(\omega) = 0$. It follows that $J(\omega) \in \mathbf{R}$, for all $\omega \in \Omega_1$. From theorem (30), $\omega \to \int_{\Omega_2} f^+(\omega, x) d\mu_2(x)$ and $\omega \to \int_{\Omega_2} f^-(\omega, x) d\mu_2(x)$

are $\mathcal{F}_1$-measurable maps. Furthermore, $A \in \mathcal{F}_1$. So 1_A is also an $\mathcal{F}_1$-measurable map. From 4. we conclude that J is itself $\mathcal{F}_1$-measurable.

6. For all $\omega \in \Omega_1$, using 4., we have:

$$|J(\omega)| \leq \int_{\Omega_2} f^+ d\mu_2 + \int_{\Omega_2} f^- d\mu_2 = \int_{\Omega_2} |f(\omega, x)| d\mu_2(x)$$

and therefore:

$$\int_{\Omega_1} |J(\omega)| d\mu_1(\omega) \leq \int_{\Omega_1} \left(\int_{\Omega_2} |f(\omega, x)| d\mu_2(x) \right) d\mu_1(\omega) < +\infty$$

Since J is $\mathbf{R}$-valued and $\mathcal{F}_1$-measurable, $J \in L^1_{\mathbf{R}}(\Omega_1, \mathcal{F}_1, \mu)$. Furthermore, for all $\omega \in A$, we have $J(\omega) = I(\omega)$. Since $\mu_1(A^c) = 0$, we conclude that $J = I$ μ_1-a.s.

7. The map $x \to \int_{\Omega_2} f(x, y) d\mu_2(y)$ is defined for all $x \in A$, but may not be defined for all $x \in \Omega_1$. Hence, strictly speaking, the integral $\int_{\Omega_1} (\int_{\Omega_2} f d\mu_2) d\mu_1$ may not be meaningful. However, whichever way we choose to extend $x \to \int_{\Omega_2} f(x, y) d\mu_2(y)$ (the map I), we have $J = I$, $\mu_1 - a.s.$ where $J \in L^1_{\mathbf{R}}(\Omega_1, \mathcal{F}_1, \mu_1)$. Following the previous exercise, we see that $I \in L^1$, and the integral $\int_{\Omega_1} I(x) d\mu_1(x)$ can in fact be defined as:

$$\int_{\Omega_1} \left(\int_{\Omega_2} f(x, y) d\mu_2(y) \right) d\mu_1(x) \stackrel{\triangle}{=} \int_{\Omega_1} J(x) d\mu_1(x)$$

8. Since $\mu_1(A^c) = 0$, we have:

$$\int_{\Omega_1} \left(1_A \int_{\Omega_2} f^+ d\mu_2 \right) d\mu_1 = \int_{\Omega_1} \left(\int_{\Omega_2} f^+ d\mu_2 \right) d\mu_1$$

Using theorem (31), we conclude that:

$$\int_{\Omega_1} \left(1_A \int_{\Omega_2} f^+ d\mu_2 \right) d\mu_1 = \int_{\Omega_1 \times \Omega_2} f^+ d\mu_1 \otimes \mu_2$$

9. Using 4., 8. and its counterpart for f^-, we obtain:

$$\int_{\Omega_1} J(x) d\mu_1(x) = \int_{\Omega_1 \times \Omega_2} f^+ d\mu_1 \otimes \mu_2 - \int_{\Omega_1 \times \Omega_2} f^- d\mu_1 \otimes \mu_2$$

In other words:

$$\int_{\Omega_1} \left(\int_{\Omega_2} f(x, y) d\mu_2(y) \right) d\mu_1(x) = \int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2$$

10. Suppose that $f \in L^1_{\mathbf{C}}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$, i.e. we no longer assume that f is $\mathbf{R}$-valued. Then $f = u + iv$ where both u and v are elements of $L^1_{\mathbf{R}}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$. Applying 6. the map

$\omega_1 \to \int_{\Omega_2} u(\omega_1, x)d\mu_2(x)$ and the map $\omega_1 \to \int_{\Omega_2} v(\omega_1, x)d\mu_2(x)$ are μ_1-almost surely equal to elements of $L^1_{\mathbf{R}}(\Omega_1, \mathcal{F}_1, \mu_1)$ (say J_u and J_v respectively). Furthermore, from (1) we have:

$$\int_{\Omega_1}\left(\int_{\Omega_2} u(x,y)d\mu_2(y)\right)d\mu_1(x) = \int_{\Omega_1\times\Omega_2} ud\mu_1\otimes\mu_2$$

and:

$$\int_{\Omega_1}\left(\int_{\Omega_2} v(x,y)d\mu_2(y)\right)d\mu_1(x) = \int_{\Omega_1\times\Omega_2} vd\mu_1\otimes\mu_2$$

It follows that $\omega_1 \to \int_{\Omega_2} f(\omega_1, x)d\mu_2(x)$ is μ_1-almost surely equal to $J_u + iJ_v \in L^1_{\mathbf{C}}(\Omega_1, \mathcal{F}_1, \mu_1)$, and:

$$\begin{aligned}
\int_{\Omega_1}\left(\int_{\Omega_2} f(x,y)d\mu_2(y)\right)d\mu_1(x) &\stackrel{\triangle}{=} \int_{\Omega_1}(J_u + iJ_v)d\mu_1 \\
&= \int_{\Omega_1} J_u d\mu_1 + i\int_{\Omega_1} J_v d\mu_1 \\
&= \int_{\Omega_1}\left(\int_{\Omega_2} u(x,y)d\mu_2(y)\right)d\mu_1(x) \\
&+ i\int_{\Omega_1}\left(\int_{\Omega_2} v(x,y)d\mu_2(y)\right)d\mu_1(x) \\
&= \int_{\Omega_1\times\Omega_2} ud\mu_1\otimes\mu_2 \\
&+ i\int_{\Omega_1\times\Omega_2} vd\mu_1\otimes\mu_2 \\
&= \int_{\Omega_1\times\Omega_2} fd\mu_1\otimes\mu_2
\end{aligned}$$

This proves equation (1).

11. From 5. of exercise (1), the map θ is measurable. It follows that $f\circ\theta : (\Omega_2\times\Omega_1, \mathcal{F}_2\otimes\mathcal{F}_1) \to [0, +\infty]$ is indeed non-negative and measurable. Furthermore, from theorem (31), we have:

$$\begin{aligned}
\int_{\Omega_2\times\Omega_1} f\circ\theta d\mu_2\otimes\mu_1 &= \int_{\Omega_2}\left(\int_{\Omega_1} f\circ\theta(\omega_2,\omega_1)d\mu_1(\omega_1)\right)d\mu_2(\omega_2) \\
&= \int_{\Omega_2}\left(\int_{\Omega_1} f(\omega_1,\omega_2)d\mu_1(\omega_1)\right)d\mu_2(\omega_2) \\
\text{Theorem (31)} \ \to &= \int_{\Omega_1\times\Omega_2} fd\mu_1\otimes\mu_2
\end{aligned}$$

12. From 5. of exercise (1), the map θ is measurable. So $f\circ\theta$ is itself measurable. Applying 11. to $|f|$ we obtain:

$$\int_{\Omega_2\times\Omega_1} |f\circ\theta|d\mu_2\otimes\mu_1 \quad = \quad \int_{\Omega_2\times\Omega_1} |f|\circ\theta d\mu_2\otimes\mu_1$$

$$= \int_{\Omega_1\times\Omega_2} |f| d\mu_1 \otimes \mu_2 < +\infty$$

So $f \circ \theta \in L^1_{\mathbf{C}}(\Omega_2 \times \Omega_1, \mathcal{F}_2 \otimes \mathcal{F}_1, \mu_2 \otimes \mu_1)$. If $u = Re(f)$ and $v = Im(f)$, using 11. once more, we obtain:

$$\begin{aligned}
\int_{\Omega_2\times\Omega_1} f\circ\theta d\mu_2\otimes\mu_1 &= \int_{\Omega_2\times\Omega_1} u^+\circ\theta d\mu_2\otimes\mu_1 \\
&- \int_{\Omega_2\times\Omega_1} u^-\circ\theta d\mu_2\otimes\mu_1 \\
&+ i\int_{\Omega_2\times\Omega_1} v^+\circ\theta d\mu_2\otimes\mu_1 \\
&- i\int_{\Omega_2\times\Omega_1} v^-\circ\theta d\mu_2\otimes\mu_1 \\
&= \int_{\Omega_1\times\Omega_2} u^+ d\mu_1\otimes\mu_2 - \int_{\Omega_1\times\Omega_2} u^- d\mu_1\otimes\mu_2 \\
&+ i\int_{\Omega_1\times\Omega_2} v^+ d\mu_1\otimes\mu_2 - i\int_{\Omega_1\times\Omega_2} v^- d\mu_1\otimes\mu_2 \\
&= \int_{\Omega_1\times\Omega_2} f d\mu_1\otimes\mu_2
\end{aligned}$$

13. Let $f \in L^1_{\mathbf{C}}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$. From 12. $g = f \circ \theta$ is an element of $L^1_{\mathbf{C}}(\Omega_2 \times \Omega_1, \mathcal{F}_2 \otimes \mathcal{F}_1, \mu_2 \otimes \mu_1)$. Applying 10. to g, it follows that the map $\omega_2 \to \int_{\Omega_1} g(\omega_2, x) d\mu_1(x)$ is μ_2-almost surely equal to an element of $L^1_{\mathbf{C}}(\Omega_2, \mathcal{F}_2, \mu_2)$. In other words, the map $\omega_2 \to \int_{\Omega_1} f(x, \omega_2) d\mu_1(x)$ is μ_2-almost surely equal to an element of $L^1_{\mathbf{C}}(\Omega_2, \mathcal{F}_2, \mu_2)$. Furthermore, we have:

$$\begin{aligned}
\int_{\Omega_2}\left(\int_{\Omega_1} f(x,y)d\mu_1(x)\right)d\mu_2(y) &= \int_{\Omega_2}\left(\int_{\Omega_1} g(y,x)d\mu_1(x)\right)d\mu_2(y) \\
\text{From 10. } \rightarrow &= \int_{\Omega_2\times\Omega_1} g d\mu_2\otimes\mu_1 \\
\text{From 12. } \rightarrow &= \int_{\Omega_1\times\Omega_2} f d\mu_1\otimes\mu_2
\end{aligned}$$

This completes the proof of theorem (33).

Exercise 16

Exercise 17.

1. Let $f \in L^1_{\mathbf{C}}(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n, \mu_1 \otimes \ldots \otimes \mu_n)$. Define $E_1 = \Pi_{i\neq\sigma(1)}\Omega_i$, $E_2 = \Omega_{\sigma(1)}$, $\mathcal{E}_1 = \otimes_{i\neq\sigma(1)}\mathcal{F}_i$ and $\mathcal{E}_2 = \mathcal{F}_{\sigma(1)}$. Let $\nu_1 = \otimes_{i\neq\sigma(1)}\mu_i$ and $\nu_2 = \mu_{\sigma(1)}$. Then:

$$f \in L^1_{\mathbf{C}}(E_1 \times E_2, \mathcal{E}_1 \otimes \mathcal{E}_2, \nu_1 \otimes \nu_2)$$

From theorem (33), the map $\omega \to \int_{E_2} f(\omega, x)d\nu_2(x)$ (defined ν_1-almost surely and arbitrarily extended on E_1), is ν_1-almost surely equal to an element of $L^1_{\mathbf{C}}(E_1, \mathcal{E}_1, \nu_1)$. In other words:

$$J_1(\omega) \stackrel{\triangle}{=} \int_{\Omega_{\sigma(1)}} f(\omega, x)d\mu_{\sigma(1)}(x)$$

is almost surely[2] equal to an element of $L^1_{\mathbf{C}}(\Pi_{i\neq\sigma(1)}\Omega_i)$[3].

2. J_{k+1} is a.s. equal to an element of $L^1_{\mathbf{C}}(\Pi_{i\not\in\{\sigma(1),\ldots,\sigma(k+1)\}}\Omega_i)$.

3. From 1., $J_1(\omega) = \int_{\Omega_{\sigma(1)}} f(\omega, x)d\mu_{\sigma(1)}(x)$ is almost surely equal to an element of $L^1_{\mathbf{C}}(\Pi_{i\neq\sigma(1)}\Omega_i)$, say $\bar{J}_1$. Similarly, from 2., $J_2(\omega) = \int_{\Omega_{\sigma(2)}} \bar{J}_1(\omega, x)d\mu_{\sigma(2)}(x)$ is almost surely equal to an element of $L^1_{\mathbf{C}}(\Pi_{i\not\in\{\sigma(1),\sigma(2)\}}\Omega_i)$, say $\bar{J}_2$. By induction, we obtain a map J_{n-1} defined on $\Omega_{\sigma(n)}$, and $\mu_{\sigma(n)}$-almost surely equal to an element of $L^1_{\mathbf{C}}(\Omega_{\sigma(n)})$, say $\bar{J}_{n-1}$. We define:

$$\int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)} \stackrel{\triangle}{=} \int_{\Omega_{\sigma(n)}} \bar{J}_{n-1} d\mu_{\sigma(n)}$$

This multiple integral is a well-defined complex number. It is easy to check by induction that which ever choice is made of $\bar{J}_1, \ldots, \bar{J}_{n-2}$, the map $\bar{J}_{n-1}$ is unique up to $\mu_{\sigma(n)}$-almost sure equality. Hence, this multiple integral is uniquely defined.

4. From theorem (33), we have:

$$\int_{\Pi_{i\neq\sigma(1)}\Omega_i} \bar{J}_1(\omega)d \otimes_{i\neq\sigma(1)} \mu_i = \int_{\Omega_1\times\ldots\times\Omega_n} f d\mu_1 \otimes \ldots \otimes \mu_n$$

Following an induction argument, we obtain:

$$\int_{\Omega_{\sigma(n)}} \bar{J}_{n-1} d\mu_{\sigma(n)} = \int_{\Omega_1\times\ldots\times\Omega_n} f d\mu_1 \otimes \ldots \otimes \mu_n$$

i.e.

$$\int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)} = \int_{\Omega_1\times\ldots\times\Omega_n} f d\mu_1 \otimes \ldots \otimes \mu_n$$

This solution is not as detailed as it could have been. . .

Exercise 17

[2]A case of sloppy terminology: we are trying to make the whole thing readable.
[3]A case of sloppy notations.