17. Image Measure

In the following, $\mathbf{K}$ denotes $\mathbf{R}$ or $\mathbf{C}$. We denote $\mathcal{M}_n(\mathbf{K})$, $n \geq 1$, the set of all $n \times n$-matrices with $\mathbf{K}$-valued entries. We recall that for all $M = (m_{ij}) \in \mathcal{M}_n(\mathbf{K})$, M is identified with the linear map $M : \mathbf{K}^n \to \mathbf{K}^n$ uniquely determined by:

$$\forall j = 1, \ldots, n \ , \ Me_j \stackrel{\triangle}{=} \sum_{i=1}^{n} m_{ij} e_i$$

where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbf{K}^n$, i.e. $e_i \stackrel{\triangle}{=} (0, ., \overbrace{1}^{i}, ., 0)$.

Exercise 1. For all $\alpha \in \mathbf{K}$, let $H_\alpha \in \mathcal{M}_n(\mathbf{K})$ be defined by:

$$H_\alpha \stackrel{\triangle}{=} \begin{pmatrix} \alpha & & & \\ & 1 & 0 & \\ & 0 & \ddots & \\ & & & 1 \end{pmatrix}$$

i.e. by $H_\alpha e_1 = \alpha e_1$, $H_\alpha e_j = e_j$, for all $j \geq 2$. Note that H_α is obtained from the identity matrix, by multiplying the top left entry by α. For $k, l \in \{1, \ldots, n\}$, we define the matrix $\Sigma_{kl} \in \mathcal{M}_n(\mathbf{K})$ by $\Sigma_{kl} e_k = e_l$, $\Sigma_{kl} e_l = e_k$ and $\Sigma_{kl} e_j = e_j$, for all $j \in \{1, \ldots, n\} \setminus \{k, l\}$. Note that Σ_{kl} is obtained from the identity matrix, by interchanging column k and column l. If $n \geq 2$, we define the matrix $U \in \mathcal{M}_n(\mathbf{K})$ by:

$$U \stackrel{\triangle}{=} \begin{pmatrix} 1 & 0 & & \\ 1 & 1 & 0 & \\ & 0 & \ddots & \\ & & & 1 \end{pmatrix}$$

i.e. by $Ue_1 = e_1 + e_2$, $Ue_j = e_j$ for all $j \geq 2$. Note that the matrix U is obtained from the identity matrix, by adding column 2 to column 1. If $n = 1$, we put $U = 1$. We define $\mathcal{N}_n(\mathbf{K}) = \{H_\alpha : \ \alpha \in \mathbf{K}\} \cup \{\Sigma_{kl} : \ k, l = 1, \ldots, n\} \cup \{U\}$, and $\mathcal{M}'_n(\mathbf{K})$ to be the set of all finite products of elements of $\mathcal{N}_n(\mathbf{K})$:

$$\mathcal{M}'_n(\mathbf{K}) \stackrel{\triangle}{=} \{M \in \mathcal{M}_n(\mathbf{K}) : M = Q_1. \ldots .Q_p \ , \ p \geq 1 \ , \ Q_j \in \mathcal{N}_n(\mathbf{K}) \ , \ \forall j\}$$

We shall prove that $\mathcal{M}_n(\mathbf{K}) = \mathcal{M}'_n(\mathbf{K})$.

1. Show that if $\alpha \in \mathbf{K} \setminus \{0\}$, H_α is non-singular with $H_\alpha^{-1} = H_{1/\alpha}$
2. Show that if $k, l = 1, \ldots, n$, Σ_{kl} is non-singular with $\Sigma_{kl}^{-1} = \Sigma_{kl}$.
3. Show that U is non-singular, and that for $n \geq 2$:

$$U^{-1} = \begin{pmatrix} 1 & 0 & & \\ -1 & 1 & 0 & \\ & 0 & \ddots & \\ & & & 1 \end{pmatrix}$$

4. Let $M = (m_{ij}) \in \mathcal{M}_n(\mathbf{K})$. Let $R_1, \ldots, R_n$ be the rows of M:

$$M \triangleq \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$$

Show that for all $\alpha \in \mathbf{K}$:

$$H_\alpha.M = \begin{pmatrix} \alpha R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$$

Conclude that *multiplying M by H_α from the left, amounts to multiplying the first row of M by α.*

5. Show that *multiplying M by H_α from the right, amounts to multiplying the first column of M by α.*

6. Show that *multiplying M by Σ_{kl} from the left, amounts to interchanging the rows R_l and R_k.*

7. Show that *multiplying M by Σ_{kl} from the right, amounts to interchanging the columns C_l and C_k.*

8. Show that *multiplying M by U^{-1} from the left ($n \geq 2$), amounts to subtracting R_1 from R_2*, i.e.:

$$U^{-1}.\begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 - R_1 \\ \vdots \\ R_n \end{pmatrix}$$

9. Show that *multiplying M by U^{-1} from the right (for $n \geq 2$), amounts to subtracting C_2 from C_1.*

10. Define $U' = \Sigma_{12}.U^{-1}.\Sigma_{12}$, $(n \geq 2)$. Show that multiplying M by U' from the right, amounts to subtracting C_1 from C_2.

11. Show that if $n = 1$, then indeed we have $\mathcal{M}_1(\mathbf{K}) = \mathcal{M}'_1(\mathbf{K})$.

Exercise 2. Further to exercise (1), we now assume that $n \geq 2$, and make the induction hypothesis that $\mathcal{M}_{n-1}(\mathbf{K}) = \mathcal{M}'_{n-1}(\mathbf{K})$.

1. Let $O_n \in \mathcal{M}_n(\mathbf{K})$ be the matrix with all entries equal to zero. Show the existence of $Q'_1, \ldots, Q'_p \in \mathcal{N}_{n-1}(\mathbf{K})$, $p \geq 1$, such that:

$$O_{n-1} = Q'_1. \ldots .Q'_p$$

2. For $k = 1, \ldots, p$, we define $Q_k \in \mathcal{M}_n(\mathbf{K})$, by:

$$Q_k \triangleq \begin{pmatrix} & & & 0 \\ & Q'_k & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}$$

Show that $Q_k \in \mathcal{N}_n(\mathbf{K})$, and that we have:

$$\Sigma_{1n}.Q_1.\ldots.Q_p.\Sigma_{1n} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & O_{n-1} & \\ 0 & & & \end{pmatrix}$$

3. Conclude that $O_n \in \mathcal{M}'_n(\mathbf{K})$.

4. We now consider $M = (m_{ij}) \in \mathcal{M}_n(\mathbf{K})$, $M \neq O_n$. We want to show that $M \in \mathcal{M}'_n(\mathbf{K})$. Show that for some $k, l \in \{1, \ldots, n\}$:

$$H^{-1}_{m_{kl}}.\Sigma_{1k}.M.\Sigma_{1l} = \begin{pmatrix} 1 & * & \ldots & * \\ * & & & \\ \vdots & & * & \\ * & & & \end{pmatrix}$$

5. Show that if $H^{-1}_{m_{kl}}.\Sigma_{1k}.M.\Sigma_{1l} \in \mathcal{M}'_n(\mathbf{K})$, then $M \in \mathcal{M}'_n(\mathbf{K})$. Conclude that without loss of generality, in order to prove that M lies in $\mathcal{M}'_n(\mathbf{K})$ we can assume that $m_{11} = 1$.

6. Let $i = 2, \ldots, n$. Show that if $m_{i1} \neq 0$, we have:

$$H^{-1}_{m_{i1}}.\Sigma_{2i}.U^{-1}.\Sigma_{2i}.H^{-1}_{1/m_{i1}}.M = \begin{pmatrix} 1 & * & \ldots & * \\ * & & & \\ 0 & \leftarrow i & * & \\ * & & & \end{pmatrix}$$

7. Conclude that without loss of generality, we can assume that $m_{i1} = 0$ for all $i \geq 2$, i.e. that M is of the form:

$$M = \begin{pmatrix} 1 & * & \ldots & * \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix}$$

8. Show that in order to prove that $M \in \mathcal{M}'_n(\mathbf{K})$, without loss of generality, we can assume that M is of the form:

$$M = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & M' & \\ 0 & & & \end{pmatrix}$$

9. Prove that $M \in \mathcal{M}'_n(\mathbf{K})$ and conclude with the following:

Theorem 103 *Given $n \geq 2$, any $n \times n$-matrix with values in $\mathbf{K}$ is a finite product of matrices Q of the following types:*

$$
\begin{array}{ll}
(i) & Qe_1 = \alpha e_1 \ ,\ Qe_j = e_j \ ,\ \forall j = 2, \ldots, n \ ,\ (\alpha \in \mathbf{K}) \\
(ii) & Qe_l = e_k \ ,\ Qe_k = e_l \ ,\ Qe_j = e_j \ ,\ \forall j \neq k, l \ ,\ (k, l \in \mathbf{N}_n) \\
(iii) & Qe_1 = e_1 + e_2 \ ,\ Qe_j = e_j \ ,\ \forall j = 2, \ldots, n
\end{array}
$$

where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbf{K}^n$.

Definition 123 *Let $X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')$ be a measurable map, where $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ are two measurable spaces. Let μ be a (possibly complex) measure on $(\Omega, \mathcal{F})$. Then, we call* **distribution** *of X under μ, or* **image measure** *of μ by X, or even* **law** *of X under μ, the (possibly complex) measure on $(\Omega', \mathcal{F}')$, denoted μ^X, $X(\mu)$ or $\mathcal{L}_\mu(X)$, and defined by:*

$$\forall B \in \mathcal{F}' \ ,\ \mu^X(B) \stackrel{\triangle}{=} \mu(\{X \in B\}) = \mu(X^{-1}(B))$$

Exercise 3. Let $X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')$ be a measurable map, where $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ are two measurable spaces.

1. Let $B \in \mathcal{F}'$. Show that if $(B_n)_{n\geq 1}$ is a measurable partition of B, then $(X^{-1}(B_n))_{n\geq 1}$ is a measurable partition of $X^{-1}(B)$.

2. Show that if μ is a measure on $(\Omega, \mathcal{F})$, μ^X is a well-defined measure on $(\Omega', \mathcal{F}')$.

3. Show that if μ is a complex measure on $(\Omega, \mathcal{F})$, μ^X is a well-defined complex measure on $(\Omega', \mathcal{F}')$.

4. Show that if μ is a complex measure on $(\Omega, \mathcal{F})$, then $|\mu^X| \leq |\mu|^X$.

5. Let $Y : (\Omega', \mathcal{F}') \to (\Omega'', \mathcal{F}'')$ be a measurable map, where $(\Omega'', \mathcal{F}'')$ is another measurable space. Show that for all (possibly complex) measure μ on $(\Omega, \mathcal{F})$, we have:
$$Y(X(\mu)) = (Y \circ X)(\mu) = (\mu^X)^Y = \mu^{(Y \circ X)}$$

Definition 124 *Let μ be a (possibly complex) measure on $\mathbf{R}^n$, $n \geq 1$. We say that μ is* **invariant by translation***, if and only if $\tau_a(\mu) = \mu$ for all $a \in \mathbf{R}^n$, where $\tau_a : \mathbf{R}^n \to \mathbf{R}^n$ is the translation mapping defined by $\tau_a(x) = a + x$, for all $x \in \mathbf{R}^n$.*

Exercise 4. Let μ be a (possibly complex) measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$.

1. Show that $\tau_a : (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n)) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ is measurable.

2. Show $\tau_a(\mu)$ is therefore a well-defined (possibly complex) measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, for all $a \in \mathbf{R}^n$.

3. Show that $\tau_a(dx) = dx$ for all $a \in \mathbf{R}^n$.

4. Show the Lebesgue measure on $\mathbf{R}^n$ is invariant by translation.

EXERCISE 5. Let $k_\alpha : \mathbf{R}^n \to \mathbf{R}^n$ be defined by $k_\alpha(x) = \alpha x$, $\alpha > 0$.

1. Show that $k_\alpha : (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n)) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ is measurable.

2. Show that $k_\alpha(dx) = \alpha^{-n} dx$.

EXERCISE 6. Show the following:

Theorem 104 (Integral Projection 1) *Let $X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')$ be a measurable map, where $(\Omega, \mathcal{F})$, $(\Omega', \mathcal{F}')$ are measurable spaces. Let μ be a measure on $(\Omega, \mathcal{F})$. Then, for all $f : (\Omega', \mathcal{F}') \to [0, +\infty]$ non-negative and measurable, we have:*

$$\int_\Omega f \circ X d\mu = \int_{\Omega'} f dX(\mu)$$

EXERCISE 7. Show the following:

Theorem 105 (Integral Projection 2) *Let $X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')$ be a measurable map, where $(\Omega, \mathcal{F})$, $(\Omega', \mathcal{F}')$ are measurable spaces. Let μ be a measure on $(\Omega, \mathcal{F})$. Then, for all $f : (\Omega', \mathcal{F}') \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ measurable, we have the equivalence:*

$$f \circ X \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \;\Leftrightarrow\; f \in L^1_{\mathbf{C}}(\Omega', \mathcal{F}', X(\mu))$$

in which case, we have:

$$\int_\Omega f \circ X d\mu = \int_{\Omega'} f dX(\mu)$$

EXERCISE 8. Further to theorem (105), suppose μ is in fact a complex measure on $(\Omega, \mathcal{F})$. Show that:

$$\int_{\Omega'} |f| d|X(\mu)| \le \int_\Omega |f \circ X| d|\mu| \tag{1}$$

Conclude with the following:

Theorem 106 (Integral Projection 3) *Let $X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')$ be a measurable map, where $(\Omega, \mathcal{F})$, $(\Omega', \mathcal{F}')$ are measurable spaces. Let μ be a complex measure on $(\Omega, \mathcal{F})$. Then, for all measurable maps $f : (\Omega', \mathcal{F}') \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$, we have:*

$$f \circ X \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \;\Rightarrow\; f \in L^1_{\mathbf{C}}(\Omega', \mathcal{F}', X(\mu))$$

and when the left-hand side of this implication is satisfied:

$$\int_\Omega f \circ X d\mu = \int_{\Omega'} f dX(\mu)$$

Exercise 9. Let $X : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ be a measurable map with distribution $\mu = X(P)$, where $(\Omega, \mathcal{F}, P)$ is a probability space.

1. Show that X is integrable, i.e. $\int |X| dP < +\infty$, if and only if:
$$\int_{-\infty}^{+\infty} |x| d\mu(x) < +\infty$$
2. Show that if X is integrable, then:
$$E[X] = \int_{-\infty}^{+\infty} x d\mu(x)$$
3. Show that:
$$E[X^2] = \int_{-\infty}^{+\infty} x^2 d\mu(x)$$

Exercise 10. Let μ be a locally finite measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, which is invariant by translation. For all $a = (a_1, \ldots, a_n) \in (\mathbf{R}^+)^n$, we define $Q_a = [0, a_1[\times \ldots \times [0, a_n[$, and in particular $Q = Q_{(1,\ldots,1)} = [0, 1[^n$.

1. Show that $\mu(Q_a) < +\infty$ for all $a \in (\mathbf{R}^+)^n$, and $\mu(Q) < +\infty$.
2. Let $p = (p_1, \ldots, p_n)$ where $p_i \geq 1$ is an integer for all i's. Show:
$$Q_p = \biguplus_{\substack{k \in \mathbf{N}^n \\ 0 \leq k_i < p_i}} [k_1, k_1 + 1[\times \ldots \times [k_n, k_n + 1[$$
3. Show that $\mu(Q_p) = p_1 \ldots p_n \mu(Q)$.
4. Let $q_1, \ldots, q_n \geq 1$ be n positive integers. Show that:
$$Q_p = \biguplus_{\substack{k \in \mathbf{N}^n \\ 0 \leq k_i < q_i}} [\frac{k_1 p_1}{q_1}, \frac{(k_1 + 1)p_1}{q_1}[\times \ldots \times [\frac{k_n p_n}{q_n}, \frac{(k_n + 1)p_n}{q_n}[$$
5. Show that $\mu(Q_p) = q_1 \ldots q_n \mu(Q_{(p_1/q_1,\ldots,p_n/q_n)})$
6. Show that $\mu(Q_r) = r_1 \ldots r_n \mu(Q)$, for all $r \in (\mathbf{Q}^+)^n$.
7. Show that $\mu(Q_a) = a_1 \ldots a_n \mu(Q)$, for all $a \in (\mathbf{R}^+)^n$.
8. Show that $\mu(B) = \mu(Q)dx(B)$, for all $B \in \mathcal{C}$, where:
$$\mathcal{C} \stackrel{\triangle}{=} \{[a_1, b_1[\times \ldots \times [a_n, b_n[\ ,\ a_i, b_i \in \mathbf{R}\ ,\ a_i \leq b_i\ ,\ \forall i \in \mathbf{N}^n\}$$
9. Show that $B(\mathbf{R}^n) = \sigma(\mathcal{C})$.
10. Show that $\mu = \mu(Q)dx$, and conclude with the following:

Theorem 107 *Let μ be a locally finite measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$. If μ is invariant by translation, then there exists $\alpha \in \mathbf{R}^+$ such that:*

$$\mu = \alpha dx$$

Exercise 11. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear bijection.

1. Show that T and T^{-1} are continuous.
2. Show that for all $B \subseteq \mathbf{R}^n$, the inverse image $T^{-1}(B) = \{T \in B\}$ coincides with the direct image:
$$T^{-1}(B) \stackrel{\triangle}{=} \{y : \ y = T^{-1}(x) \text{ for some } x \in B\}$$
3. Show that for all $B \subseteq \mathbf{R}^n$, the direct image $T(B)$ coincides with the inverse image $(T^{-1})^{-1}(B) = \{T^{-1} \in B\}$.
4. Let $K \subseteq \mathbf{R}^n$ be compact. Show that $\{T \in K\}$ is compact.
5. Show that $T(dx)$ is a locally finite measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$.
6. Let τ_a be the translation of vector $a \in \mathbf{R}^n$. Show that:
$$T \circ \tau_{T^{-1}(a)} = \tau_a \circ T$$
7. Show that $T(dx)$ is invariant by translation.
8. Show the existence of $\alpha \in \mathbf{R}^+$, such that $T(dx) = \alpha dx$. Show that such constant is unique, and denote it by $\Delta(T)$.
9. Show that $Q = T([0,1]^n) \in \mathcal{B}(\mathbf{R}^n)$ and that we have:
$$\Delta(T)dx(Q) = T(dx)(Q) = 1$$
10. Show that $\Delta(T) \neq 0$.
11. Let $T_1, T_2 : \mathbf{R}^n \to \mathbf{R}^n$ be two linear bijections. Show that:
$$(T_1 \circ T_2)(dx) = \Delta(T_1)\Delta(T_2)dx$$
and conclude that $\Delta(T_1 \circ T_2) = \Delta(T_1)\Delta(T_2)$.

Exercise 12. Let $\alpha \in \mathbf{R} \setminus \{0\}$. Let $H_\alpha : \mathbf{R}^n \to \mathbf{R}^n$ be the linear bijection uniquely defined by $H_\alpha(e_1) = \alpha e_1$, $H_\alpha(e_j) = e_j$ for $j \geq 2$.

1. Show that $H_\alpha(dx)([0,1]^n) = |\alpha|^{-1}$.
2. Conclude that $\Delta(H_\alpha) = |\det H_\alpha|^{-1}$.

Exercise 13. Let $k, l \in \mathbf{N}_n$and $\Sigma : \mathbf{R}^n \to \mathbf{R}^n$ be the linear bijection uniquely defined by $\Sigma(e_k) = e_l$, $\Sigma(e_l) = e_k$, $\Sigma(e_j) = e_j$, for $j \neq k, l$.

1. Show that $\Sigma(dx)([0,1]^n) = 1$.

2. Show that $\Sigma.\Sigma = I_n$. (Identity mapping on $\mathbf{R}^n$).
3. Show that $|\det \Sigma| = 1$.
4. Conclude that $\Delta(\Sigma) = |\det \Sigma|^{-1}$.

Exercise 14. Let $n \geq 2$ and $U : \mathbf{R}^n \to \mathbf{R}^n$ be the linear bijection uniquely defined by $U(e_1) = e_1 + e_2$ and $U(e_j) = e_j$ for $j \geq 2$. Let $Q = [0, 1[^n$.

1. Show that:
$$U^{-1}(Q) = \{x \in \mathbf{R}^n : \ 0 \leq x_1 + x_2 < 1 \ , \ 0 \leq x_i < 1 \ , \ \forall i \neq 2\}$$
2. Define:
$$\begin{aligned} \Omega_1 &\stackrel{\triangle}{=} U^{-1}(Q) \cap \{x \in \mathbf{R}^n : \ x_2 \geq 0\} \\ \Omega_2 &\stackrel{\triangle}{=} U^{-1}(Q) \cap \{x \in \mathbf{R}^n : \ x_2 < 0\} \end{aligned}$$
Show that $\Omega_1, \Omega_2 \in \mathcal{B}(\mathbf{R}^n)$.
3. Let τ_{e_2} be the translation of vector e_2. Draw a picture of Q, Ω_1, Ω_2 and $\tau_{e_2}(\Omega_2)$ in the case when $n = 2$.
4. Show that if $x \in \Omega_1$, then $0 \leq x_2 < 1$.
5. Show that $\Omega_1 \subseteq Q$.
6. Show that if $x \in \tau_{e_2}(\Omega_2)$, then $0 \leq x_2 < 1$.
7. Show that $\tau_{e_2}(\Omega_2) \subseteq Q$.
8. Show that if $x \in Q$ and $x_1 + x_2 < 1$ then $x \in \Omega_1$.
9. Show that if $x \in Q$ and $x_1 + x_2 \geq 1$ then $x \in \tau_{e_2}(\Omega_2)$.
10. Show that if $x \in \tau_{e_2}(\Omega_2)$ then $x_1 + x_2 \geq 1$.
11. Show that $\tau_{e_2}(\Omega_2) \cap \Omega_1 = \emptyset$.
12. Show that $Q = \Omega_1 \uplus \tau_{e_2}(\Omega_2)$.
13. Show that $dx(Q) = dx(U^{-1}(Q))$.
14. Show that $\Delta(U) = 1$.
15. Show that $\Delta(U) = |\det U|^{-1}$.

Exercise 15. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear bijection, $(n \geq 1)$.

1. Show the existence of linear bijections $Q_1, \ldots, Q_p : \mathbf{R}^n \to \mathbf{R}^n$, $p \geq 1$, with $T = Q_1 \circ \ldots \circ Q_p$, $\Delta(Q_i) = |\det Q_i|^{-1}$ for all $i \in \mathbf{N}_p$.
2. Show that $\Delta(T) = |\det T|^{-1}$.

3. Conclude with the following:

Theorem 108 *Let $n \geq 1$ and $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear bijection. Then, the image measure $T(dx)$ of the Lebesgue measure on $\mathbf{R}^n$ is:*

$$T(dx) = |\det T|^{-1} dx$$

EXERCISE 16. Let $f : (\mathbf{R}^2, \mathcal{B}(\mathbf{R}^2)) \to [0, +\infty]$ be a non-negative and measurable map. Let $a, b, c, d \in \mathbf{R}$ such that $ad - bc \neq 0$. Show that:

$$\int_{\mathbf{R}^2} f(ax + by, cx + dy) dxdy = |ad - bc|^{-1} \int_{\mathbf{R}^2} f(x, y) dxdy$$

EXERCISE 17. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear bijection. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$, we have $T(B) \in \mathcal{B}(\mathbf{R}^n)$ and:

$$dx(T(B)) = |\det T| dx(B)$$

EXERCISE 18. Let V be a linear subspace of $\mathbf{R}^n$ and $p = \dim V$. We assume that $1 \leq p \leq n-1$. Let $u_1, \dots, u_p$ be an orthonormal basis of V, and $u_{p+1}, \dots, u_n$ be such that $u_1, \dots, u_n$ is an orthonormal basis of $\mathbf{R}^n$. For $i \in \mathbf{N}_n$, Let $\phi_i : \mathbf{R}^n \to \mathbf{R}$ be defined by $\phi_i(x) = \langle u_i, x \rangle$.

1. Show that all ϕ_i's are continuous.
2. Show that $V = \bigcap_{j=p+1}^n \phi_j^{-1}(\{0\})$.
3. Show that V is a closed subset of $\mathbf{R}^n$.
4. Let $Q = (q_{ij}) \in \mathcal{M}_n(\mathbf{R})$ be the matrix uniquely defined by $Qe_j = u_j$ for all $j \in \mathbf{N}_n$, where $(e_1, \dots, e_n)$ is the canonical basis of $\mathbf{R}^n$. Show that for all $i, j \in \mathbf{N}_n$:
$$\langle u_i, u_j \rangle = \sum_{k=1}^n q_{ki} q_{kj}$$
5. Show that $Q^t.Q = I_n$ and conclude that $|\det Q| = 1$.
6. Show that $dx(\{Q \in V\}) = dx(V)$.
7. Show that $\{Q \in V\} = \mathrm{span}(e_1, \dots, e_p)$.[1]
8. For all $m \geq 1$, we define:
$$E_m \triangleq \overbrace{[-m, m] \times \ldots \times [-m, m]}^{n-1} \times \{0\}$$
Show that $dx(E_m) = 0$ for all $m \geq 1$.
9. Show that $dx(\mathrm{span}(e_1, \dots, e_{n-1})) = 0$.

[1] i.e. the linear subspace of $\mathbf{R}^n$ generated by $e_1, \dots, e_p$.

10. Conclude with the following:

Theorem 109 *Let $n \geq 1$. Any linear subspace V of $\mathbf{R}^n$ is a closed subset of $\mathbf{R}^n$. Moreover, if $\dim V \leq n-1$, then $dx(V) = 0$.*

Solutions to Exercises

Exercise 1.

1. Let $\alpha \in \mathbf{K} \setminus \{0\}$. Then, we have:

$$H_{1/\alpha} \circ H_\alpha e_1 = H_{1/\alpha}(\alpha e_1) = \alpha H_{1/\alpha} e_1 = \alpha(1/\alpha) e_1 = e_1$$

and for all $j \geq 2$, $H_{1/\alpha} \circ H_\alpha e_j = H_{1/\alpha} e_j = e_j$. If I_n denotes the identity matrix of $\mathcal{M}_n(\mathbf{K})$, then I_n and $H_{1/\alpha} \circ H_\alpha$ coincide on the basis $(e_1, \ldots, e_n)$ of $\mathbf{K}^n$. It follows that I_n and $H_{1/\alpha} \circ H_\alpha$ are in fact equal. So H_α is non-singular and $H_\alpha^{-1} = H_{1/\alpha}$.

2. The linear map $\Sigma_{kl} : \mathbf{K}^n \to \mathbf{K}^n$ is defined by $\Sigma_{kl} e_k = e_l$, $\Sigma_{kl} e_l = e_k$ and $\Sigma_{kl} e_j = e_j$ for all $j \notin \{k, l\}$. Hence, it is clear that $\Sigma_{kl} \circ \Sigma_{kl} e_j = e_j$ for all $j \in \mathbf{N}_n$, and consequently $\Sigma_{kl} \circ \Sigma_{kl} = I_n$. So Σ_{kl} is non-singular and $\Sigma_{kl}^{-1} = \Sigma_{kl}$.

3. If $n = 1$, then $U = 1$ and U is indeed non-singular. We assume that $n \geq 2$. Then U is defined by $Ue_1 = e_1 + e_2$ and $Ue_j = e_j$ for all $j \geq 2$. Consider the linear map $U' : \mathbf{K}^n \to \mathbf{K}^n$ defined by $U'e_1 = e_1 - e_2$ and $U'e_j = e_j$ for all $j \geq 2$. Then, we have:

$$U' \circ U e_1 = U'(e_1 + e_2) = U'e_1 + U'e_2 = e_1 - e_2 + e_2 = e_1$$

and it is clear that $U' \circ U e_j = e_j$ for all $j \geq 2$. It follows that $U' \circ U e_j = e_j$ for all $j \in \mathbf{N}_n$ and consequently $U' \circ U = I_n$. We have proved that U is invertible and $U^{-1} = U'$, i.e.:

$$U^{-1} = \begin{pmatrix} 1 & 0 & & \\ -1 & 1 & 0 & \\ & 0 & \ddots & \\ & & & 1 \end{pmatrix}$$

4. Let $M = (m_{ij}) \in \mathcal{M}_n(\mathbf{K})$, and $R_1, \ldots, R_n$ be the rows of M, i.e.

$$M \triangleq \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$$

Specifically, for all $i \in \mathbf{N}_n$, each R_i is the row vector:

$$R_i = (m_{i1}, m_{i2}, \ldots, m_{in})$$

Let $\alpha \in \mathbf{K}$, and consider the matrix $M' \in \mathcal{M}_n(\mathbf{K})$ defined by:

$$M' \triangleq \begin{pmatrix} \alpha R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$$

i.e. $M'e_j = \alpha m_{1j}e_1 + \sum_{i=2}^n m_{ij}e_i$ for all $j \in \mathbf{N}_n$. Then:

$$\begin{aligned} H_\alpha \circ Me_j &= H_\alpha\left(\sum_{i=1}^n m_{ij}e_i\right) \\ &= \sum_{i=1}^n m_{ij}H_\alpha e_i \\ &= m_{1j}H_\alpha e_1 + \sum_{i=2}^n m_{ij}H_\alpha e_i \\ &= \alpha m_{1j}e_1 + \sum_{i=2}^n m_{ij}e_i \\ &= M'e_j \end{aligned}$$

This being true for all $j \in \mathbf{N}_n$, we have proved that $H_\alpha M = M'$, i.e.

$$H_\alpha M = \begin{pmatrix} \alpha R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$$

We conclude that *multiplying M by H_α from the left, amounts to multiplying the first row of M by α.*

5. Let $M = (m_{ij}) \in \mathcal{M}_n(\mathbf{K})$, and $C_1, \ldots, C_n$ be the columns of M:

$$M \stackrel{\triangle}{=} (C_1, C_2, \ldots, C_n)$$

Specifically, for all $j \in \mathbf{N}_n$, each C_j is the column vector:

$$C_j = \begin{pmatrix} m_{1j} \\ m_{2j} \\ \vdots \\ m_{nj} \end{pmatrix}$$

Let $\alpha \in \mathbf{K}$, and consider the matrix M' defined by:

$$M' = (\alpha C_1, C_2, \ldots, C_n)$$

i.e. $M'e_1 = \sum_{i=1}^n \alpha m_{i1}e_i$ and $M'e_j = \sum_{i=1}^n m_{ij}e_i$ for $j \geq 2$:

$$M \circ H_\alpha e_1 = M(\alpha e_1) = \alpha Me_1 = \sum_{i=1}^n \alpha m_{i1}e_i = M'e_1$$

and furthermore, for all $j \geq 2$:

$$M \circ H_\alpha e_j = Me_j = \sum_{i=1}^n m_{ij}e_i = M'e_j$$

So $M \circ H_\alpha e_j = M'e_j$ for all $j \in \mathbf{N}_n$, i.e. $MH_\alpha = M'$. Hence:

$$MH_\alpha = (\alpha C_1, C_2, \ldots, C_n)$$

We conclude that *multiplying M by H_α from the right, amounts to multiplying the first column of M by α.*

6. Let $M = (m_{ij}) \in \mathcal{M}_n(\mathbf{K})$ and $R_1, \ldots, R_n$ be the rows of M, i.e.

$$M \stackrel{\triangle}{=} \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$$

Specifically, for all $i \in \mathbf{N}_n$, R_i is the row vector:

$$R_i = (m_{i1}, m_{i2}, \ldots, m_{in})$$

Let $M' = (m'_{ij}) \in \mathcal{M}_n(\mathbf{K})$ be the matrix defined by:

$$M' \stackrel{\triangle}{=} \begin{pmatrix} R'_1 \\ R'_2 \\ \vdots \\ R'_n \end{pmatrix}$$

where $R'_k = R_l$, $R'_l = R_k$ and $R'_i = R_i$ for all $i \notin \{k,l\}$. In other words, the matrix M' is nothing but the matrix M, where the rows R_k and R_l have been interchanged. Note that for all $i,j \in \mathbf{N}_n$, $m'_{kj} = m_{lj}$, $m'_{lj} = m_{kj}$ and $m'_{ij} = m_{ij}$ for all $i \notin \{k,l\}$. Now, given $j \in \mathbf{N}_n$, we have:

$$\begin{aligned}
\Sigma_{kl} \circ Me_j &= \Sigma_{kl}\left(\sum_{i=1}^n m_{ij}e_i\right) \\
&= \sum_{i=1}^n m_{ij}\Sigma_{kl}e_i \\
&= \sum_{i\neq k,l} m_{ij}e_i + m_{kj}e_l + m_{lj}e_k \\
&= \sum_{i\neq k,l} m'_{ij}e_i + m'_{lj}e_l + m'_{kj}e_k \\
&= \sum_{i=1}^n m'_{ij}e_i = M'e_j
\end{aligned}$$

This being true for all $j \in \mathbf{N}_n$, $\Sigma_{kl}M = M'$. We conclude that *multiplying M by Σ_{kl} from the left, amounts to interchanging the rows R_l and R_k of M.*

7. Let $M = (m_{ij}) \in \mathcal{M}_n(\mathbf{K})$, and $C_1, \ldots, C_n$ be the columns of M:

$$M \stackrel{\triangle}{=} (C_1, C_2, \ldots, C_n)$$

Specifically, for all $j \in \mathbf{N}_n$, each C_j is the column vector:

$$C_j = \begin{pmatrix} m_{1j} \\ m_{2j} \\ \vdots \\ m_{nj} \end{pmatrix}$$

Let $M' = (m'_{ij}) \in \mathcal{M}_n(\mathbf{K})$ be the matrix defined by:

$$M' \stackrel{\triangle}{=} (C'_1, C'_2, \ldots, C'_n)$$

where $C'_k = C_l$, $C'_l = C_k$ and $C'_j = C_j$ for all $j \notin \{k, l\}$. In other words, the matrix M' is nothing but the matrix M, where the columns C_k and C_l have been interchanged. For all $i, j \in \mathbf{N}_n$, $m'_{ik} = m_{il}$, $m'_{il} = m_{ik}$ and $m'_{ij} = m_{ij}$ for all $j \notin \{k, l\}$. Now:

$$\begin{aligned} M \circ \Sigma_{kl} e_k &= M e_l \\ &= \sum_{i=1}^n m_{il} e_i \\ &= \sum_{i=1}^n m'_{ik} e_i = M' e_k \end{aligned}$$

and similarly $M \circ \Sigma_{kl} e_l = M' e_l$. Furthermore, if $j \neq k, l$:

$$\begin{aligned} M \circ \Sigma_{kl} e_j &= M e_j \\ &= \sum_{i=1}^n m_{ij} e_i \\ &= \sum_{i=1}^n m'_{ij} e_i = M' e_j \end{aligned}$$

It follows that $M \circ \Sigma_{kl} e_j = M' e_j$ for all $j \in \mathbf{N}_n$. We conclude that $M \Sigma_{kl} = M'$ and consequently, *multiplying M by Σ_{kl} from the right, amounts to interchanging the columns C_l and C_k of M.*

8. Let $M = (m_{ij}) \in \mathcal{M}_n(\mathbf{K})$ and $R_1, \ldots, R_n$ be the rows of M, i.e.

$$M \stackrel{\triangle}{=} \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$$

Specifically, for all $i \in \mathbf{N}_n$, R_i is the row vector:

$$R_i = (m_{i1}, m_{i2}, \ldots, m_{in})$$

Let $M' = (m'_{ij}) \in \mathcal{M}_n(\mathbf{K})$ be the matrix defined by:

$$M' \stackrel{\triangle}{=} \begin{pmatrix} R_1 \\ R_2 - R_1 \\ \vdots \\ R_n \end{pmatrix}$$

Specifically, M' is exactly the matrix M, where the second row R_2 has been replaced by $R_2 - R_1$, i.e. where the first row R_1 has been subtracted from the second row R_2. Recall from 3. that U^{-1} is given by $U^{-1}e_1 = e_1 - e_2$ and $U^{-1}e_j = e_j$ for all $j \geq 2$. Note that for all $i, j \in \mathbf{N}_n$, we have $m'_{ij} = m_{ij}$ if $i \neq 2$, and $m'_{2j} = m_{2j} - m_{1j}$. Now for all $j \in \mathbf{N}_n$:

$$\begin{aligned} U^{-1}Me_j &= U^{-1}\left(\sum_{i=1}^n m_{ij}e_i\right) \\ &= \sum_{i=1}^n m_{ij}U^{-1}e_i \\ &= m_{1j}(e_1 - e_2) + \sum_{i=2}^n m_{ij}e_i \\ &= \sum_{i\neq 2} m_{ij}e_i + (m_{2j} - m_{1j})e_2 \\ &= \sum_{i=1}^n m'_{ij}e_i = M'e_j \end{aligned}$$

It follows that $U^{-1}M = M'$, and we conclude that *multiplying* M *by* U^{-1} *from the left, amounts to subtracting* R_1 *from* R_2.

9. Let $M = (m_{ij}) \in \mathcal{M}_n(\mathbf{K})$, and $C_1, \ldots, C_n$ be the columns of M:

$$M \stackrel{\triangle}{=} (C_1, C_2, \ldots, C_n)$$

Specifically, for all $j \in \mathbf{N}_n$, each C_j is the column vector:

$$C_j = \begin{pmatrix} m_{1j} \\ m_{2j} \\ \vdots \\ m_{nj} \end{pmatrix}$$

Let $M' = (m'_{ij}) \in \mathcal{M}_n(\mathbf{K})$ be the matrix defined by:

$$M' \stackrel{\triangle}{=} (C_1 - C_2, C_2, \ldots, C_n)$$

Specifically, M' is exactly the matrix M, where the second column C_2 has been subtracted from the first column C_1. For all $i, j \in \mathbf{N}_n$, we have

$m'_{ij} = m_{ij}$ if $j \neq 1$ and $m'_{i1} = m_{i1} - m_{i2}$. Furthermore:

$$\begin{aligned} MU^{-1}e_1 &= M(e_1 - e_2) \\ &= Me_1 - Me_2 \\ &= \sum_{i=1}^n m_{i1}e_i - \sum_{i=1}^n m_{i2}e_i \\ &= \sum_{i=1}^n (m_{i1} - m_{i2})e_i \\ &= \sum_{i=1}^n m'_{i1}e_i = M'e_1 \end{aligned}$$

and for all $j \geq 2$:

$$\begin{aligned} MU^{-1}e_j &= Me_j \\ &= \sum_{i=1}^n m_{ij}e_i \\ &= \sum_{i=1}^n m'_{ij}e_i = M'e_j \end{aligned}$$

Having proved that $MU^{-1}e_j = M'e_j$ for all $j \in \mathbf{N}_n$, we conclude that $MU^{-1} = M'$, or equivalently that *multiplying M by U^{-1} from the right, amounts to subtracting C_2 from C_1*.

10. Let $U' = \Sigma_{12}U^{-1}\Sigma_{12}$. Let $C_1, \ldots, C_2$ be the column vectors of $M \in \mathcal{M}_n(\mathbf{K})$. It follows from 7. and 9. that:

$$\begin{aligned} MU' &= M\Sigma_{12}U^{-1}\Sigma_{12} \\ &= (C_1, C_2, \ldots, C_n)\Sigma_{12}U^{-1}\Sigma_{12} \\ &= (C_2, C_1, \ldots, C_n)U^{-1}\Sigma_{12} \\ &= (C_2 - C_1, C_1, \ldots, C_n)\Sigma_{12} \\ &= (C_1, C_2 - C_1, \ldots, C_n) \end{aligned}$$

We conclude that *multiplying M by U' from the right, amounts to subtracting C_1 from C_2*.

11. Suppose $n = 1$. It is clear that $\mathcal{M}'_n(\mathbf{K}) \subseteq \mathcal{M}_n(\mathbf{K})$ for all $n \geq 1$, and in particular $\mathcal{M}'_1(\mathbf{K}) \subseteq \mathcal{M}_1(\mathbf{K})$. Suppose $M \in \mathcal{M}_1(\mathbf{K})$. Then $M = (\alpha)$ for some $\alpha \in \mathbf{K}$. However, $(\alpha) = H_\alpha$ (one-dimensional). Hence, defining $Q_1 = H_\alpha$, we have $Q_1 \in \mathcal{N}_1(\mathbf{K})$ with $M = Q_1$. In particular, M is a finite product of elements of $\mathcal{N}_1(\mathbf{K})$. So $M \in \mathcal{M}'_1(\mathbf{K})$ and we have proved the equality $\mathcal{M}_1(\mathbf{K}) = \mathcal{M}'_1(\mathbf{K})$.

Exercise 1

Exercise 2.

1. Our induction hypothesis is $\mathcal{M}_{n-1}(\mathbf{K}) = \mathcal{M}'_{n-1}(\mathbf{K})$, $n \geq 2$. For all $n \geq 1$, $O_n \in \mathcal{M}_n(\mathbf{K})$ denotes the matrix with all entries equal to $0 \in \mathbf{K}$. Since $O_{n-1} \in \mathcal{M}_{n-1}(\mathbf{K}) = \mathcal{M}'_{n-1}(\mathbf{K})$, O_{n-1} is a finite product of elements of $\mathcal{N}_{n-1}(\mathbf{K})$. Hence, there exist $p \geq 1$ and $Q'_1, \ldots, Q'_p$ elements of $\mathcal{N}_{n-1}(\mathbf{K})$ such that:
$$O_{n-1} = Q'_1 \ldots Q'_p$$

2. Given $k \in \{1, \ldots, p\} = \mathbf{N}_p$, we define $Q_k \in \mathcal{M}_n(\mathbf{K})$ by:
$$Q_k \stackrel{\triangle}{=} \begin{pmatrix} & & & 0 \\ & Q'_k & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}$$
Since $Q'_k \in \mathcal{N}_{n-1}(\mathbf{K})$, Q'_k can be of three different forms: If Q'_k is of the form H_α (of dimension $n-1$) for some $\alpha \in \mathbf{K}$, it is clear that $Q_k = H_\alpha$ (of dimension n). If Q'_k is of the form Σ_{lm} for some $l, m \in \mathbf{N}_{n-1}$, then $Q'_k e_l = e_m$, $Q'_k e_m = e_l$ and $Q'_k e_j = e_j$ for all $j \in \mathbf{N}_{n-1} \setminus \{l, m\}$. Hence, it is clear that $Q_k e_l = e_m$, $Q_k e_m = e_l$ and $Q_k e_j = e_j$ for all $j \in \mathbf{N}_n \setminus \{l, m\}$. So Q_k is of the form Σ_{lm} (of dimension n) for some $l, m \in \mathbf{N}_n$ (in fact, for some $l, m \in \mathbf{N}_{n-1}$). Note that we have used the same notation $e_1, \ldots, e_{n-1}$ and $e_1, \ldots, e_n$ to denote successively the canonical basis of $\mathbf{K}^{n-1}$ and $\mathbf{K}^n$. Now, if $Q'_k = U$ (of dimension $n-1$), it is clear that $Q_k = U$ (of dimension n) in the case when $n-1 \geq 2$. In the case when $n-1 = 1$, we have $Q'_k = (1)$ and consequently $Q_k = I_2 = H_1$ (of dimension 2). In any case, we see that Q_k is an element of $\mathcal{N}_{n-1}(\mathbf{K})$. Now, using 6. and 7. together with block matrix multiplication, we obtain:
$$\begin{aligned}
\Sigma_{1n} Q_1 \ldots Q_p \Sigma_{1n} &= \Sigma_{1n} \cdot \begin{pmatrix} & & & 0 \\ & Q'_1 \ldots Q'_p & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix} \cdot \Sigma_{1n} \\
&= \Sigma_{1n} \cdot \begin{pmatrix} & & & 0 \\ & O_{n-1} & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix} \cdot \Sigma_{1n} \\
&= \Sigma_{1n} \cdot \begin{pmatrix} 0 & & & \\ \vdots & & O_{n-1} & \\ 0 & & & \\ 1 & 0 & \ldots & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & O_{n-1} & \\ 0 & & & \end{pmatrix}
\end{aligned}$$

which is exactly what we intended to prove.

3. Having proved that:

$$\Sigma_{1n}.Q_1.\ldots.Q_p.\Sigma_{1n} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & O_{n-1} & \\ 0 & & & \end{pmatrix}$$

since H_0 can be written as:

$$H_0 = \begin{pmatrix} 0 & & & \\ & 1 & 0 & \\ & 0 & \ddots & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \end{pmatrix}$$

we obtain:

$$H_0 \cdot \Sigma_{1n}.Q_1.\ldots.Q_p.\Sigma_{1n} = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & O_{n-1} & \\ 0 & & & \end{pmatrix} = O_n$$

We have been able to express O_n as a finite product of elements of $\mathcal{N}_n(\mathbf{K})$. We conclude that $O_n \in \mathcal{M}'_n(\mathbf{K})$.

4. Let $M = (m_{ij}) \in \mathcal{M}_n(\mathbf{K})$. We assume that $M \neq O_n$. Then, there exist $k, l \in \mathbf{N}_n$ such that $m_{kl} \neq 0$. From 7. of exercise (1), multiplying M by Σ_{1l} from the right, amounts to interchanging column l with column 1. So m_{kl} appears in the matrix $M\Sigma_{1l}$ as the k-th element of the first column. Multiplying $M\Sigma_{1l}$ by Σ_{1k} from the left, amounts to interchanging row k with row 1. So m_{kl} now appears in the matrix $\Sigma_{1k}M\Sigma_{1l}$ at the intersection of the first row and the first column, i.e. at the top left position. In other words, $\Sigma_{1k}M\Sigma_{1l}$ is of the form:

$$\Sigma_{1k}M\Sigma_{1l} = \begin{pmatrix} m_{kl} & * & \ldots & * \\ * & & & \\ \vdots & & * & \\ * & & & \end{pmatrix}$$

Multiplying by $H^{-1}_{m_{kl}} = H_{1/m_{kl}}$ from the left, amounts to multiplying the first row by $1/m_{kl}$. We conclude that:

$$H^{-1}_{m_{kl}}\Sigma_{1k}M\Sigma_{1l} = \begin{pmatrix} 1 & * & \ldots & * \\ * & & & \\ \vdots & & * & \\ * & & & \end{pmatrix}$$

5. Suppose we have proved $H^{-1}_{m_{kl}}\Sigma_{1k}M\Sigma_{1l} \in \mathcal{M}'_n(\mathbf{K})$. Then this matrix is a finite product of elements of $\mathcal{N}_n(\mathbf{K})$. In other words, there exist $p \geq 1$ and $Q_1, \ldots, Q_p$ elements of $\mathcal{N}_n(\mathbf{K})$ with:
$$H^{-1}_{m_{kl}}\Sigma_{1k}M\Sigma_{1l} = Q_1 \ldots Q_p$$
Since $\Sigma^{-1}_{1k} = \Sigma_{1k}$ and $\Sigma^{-1}_{1l} = \Sigma_{1l}$, we obtain:
$$M = \Sigma_{1k}H_{m_{kl}}Q_1 \ldots Q_p\Sigma_{1l}$$
So M is therefore also a finite product of elements of $\mathcal{N}_n(\mathbf{K})$, i.e. $M \in \mathcal{M}'_n(\mathbf{K})$. Hence, in order to prove that $M \in \mathcal{M}'_n(\mathbf{K})$ it is sufficient to prove that $H^{-1}_{m_{kl}}\Sigma_{1k}M\Sigma_{1l}$ is an element of $\mathcal{M}'_n(\mathbf{K})$. It follows from 4. that without loss of generality, we may assume that $m_{11} = 1$.

6. Let $i \in \{2, \ldots, n\}$ and suppose $m_{i1} \neq 0$. So M is of the form:
$$M = \begin{pmatrix} 1 & * & \ldots & * \\ * & & & \\ m_{i1} & \leftarrow i & * & \\ * & & & \end{pmatrix}$$
with $m_{i1} \neq 0$. Since $H^{-1}_{1/m_{i1}} = H_{m_{i1}}$, multiplying M by $H^{-1}_{1/m_{i1}}$ from the left amounts to multiplying the first row of M by m_{i1}. So $H^{-1}_{1/m_{i1}}M$ is of the form:
$$H^{-1}_{1/m_{i1}}M = \begin{pmatrix} m_{i1} & * & \ldots & * \\ * & & & \\ m_{i1} & \leftarrow i & * & \\ * & & & \end{pmatrix}$$
Multiplying by Σ_{2i} from the left amounts to interchanging row 2 with row i. Multiplying by U^{-1} from the left amounts to subtracting row 1 from row 2. Multiplying once more by Σ_{2i} from the left amounts to switching back row 2 and row i. It follows that $\Sigma_{2i}U^{-1}\Sigma_{2i}H^{-1}_{1/m_{i1}}M$ is of the form:
$$\Sigma_{2i}U^{-1}\Sigma_{2i}H^{-1}_{1/m_{i1}}M = \begin{pmatrix} m_{i1} & * & \ldots & * \\ * & & & \\ 0 & \leftarrow i & * & \\ * & & & \end{pmatrix}$$
Multiplying once more by $H^{-1}_{m_{i1}} = H_{1/m_{i1}}$ from the left amounts to multiplying the first row by $1/m_{i1}$. We conclude that:
$$H^{-1}_{m_{i1}}\Sigma_{2i}U^{-1}\Sigma_{2i}H^{-1}_{1/m_{i1}}M = \begin{pmatrix} 1 & * & \ldots & * \\ * & & & \\ 0 & \leftarrow i & * & \\ * & & & \end{pmatrix}$$

7. If we prove that the matrix:

$$H_{m_{i1}}^{-1}\Sigma_{2i}U^{-1}\Sigma_{2i}H_{1/m_{i1}}^{-1}M = \begin{pmatrix} 1 & * & \dots & * \\ * & & & \\ 0 & \leftarrow i & * & \\ * & & & \end{pmatrix}$$

is a finite product of elements of $\mathcal{N}_n(\mathbf{K})$, then clearly M is also a finite product of elements of $\mathcal{N}_n(\mathbf{K})$. Hence in order to show that $M \in \mathcal{M}'_n(\mathbf{K})$, without loss of generality we may assume that $m_{i1} = 0$. This being true of all $i \in \{2, \dots, n\}$, without loss of generality we may assume that M is of the form:

$$M = \begin{pmatrix} 1 & * & \dots & * \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix}$$

8. So we now want to prove that $M \in \mathcal{M}'_n(\mathbf{K})$, where:

$$M = \begin{pmatrix} 1 & * & \dots & * \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix}$$

Let $j \in \{2, \dots, n\}$ and suppose that $m_{1j} \neq 0$. From 5. of exercise (1), multiplying M by $H_{1/m_{1j}}^{-1} = H_{m_{1j}}$ from the right, amounts to multiplying the first column of M by m_{1j}. So $MH_{1/m_{1j}}^{-1}$ is of the form:

$$MH_{1/m_{1j}}^{-1} = \begin{pmatrix} m_{1j} & * & m_{1j} & * \\ 0 & & j\uparrow & \\ \vdots & & * & \\ 0 & & & \end{pmatrix}$$

Multiplying by Σ_{2j} from the right amounts to interchanging column 2 with column j. From 10. of exercise (1), multiplying by $U' = \Sigma_{12}U^{-1}\Sigma_{12}$ from the right amounts to subtracting column 1 from column 2. Multiplying by Σ_{2j} once more from the right, amounts to switching back column 2 and column j. It follows that $MH_{1/m_{1j}}^{-1}\Sigma_{2j}U'\Sigma_{2j}$ is of the form:

$$MH_{1/m_{1j}}^{-1}\Sigma_{2j}U'\Sigma_{2j} = \begin{pmatrix} m_{1j} & * & 0 & * \\ 0 & & j\uparrow & \\ \vdots & & * & \\ 0 & & & \end{pmatrix}$$

Multiplying once more by $H^{-1}_{m_{1j}} = H_{1/m_{1j}}$ from the right:

$$MH^{-1}_{1/m_{1j}}\Sigma_{2j}U'\Sigma_{2j}H^{-1}_{m_{1j}} = \begin{pmatrix} 1 & * & 0 & * \\ 0 & & j\uparrow & \\ \vdots & & * & \\ 0 & & & \end{pmatrix}$$

Since $U' = \Sigma_{12}U^{-1}\Sigma_{12}$, it is clear that in order to prove that M is a finite product of elements of $\mathcal{N}_n(\mathbf{K})$, it is sufficient to prove that the above matrix is itself a finite product of elements of $\mathcal{N}_n(\mathbf{K})$. Hence, in order to prove that $M \in \mathcal{M}'_n(\mathbf{K})$, without loss of generality we may assume that $m_{1j} = 0$. This being true for all $j \in \{2, \ldots, n\}$, without loss of generality we may assume that M is of the form:

$$M = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & M' & \\ 0 & & & \end{pmatrix}$$

where $M' \in \mathcal{M}_{n-1}(\mathbf{K})$.

9. So we now assume that $M \in \mathcal{M}_n(\mathbf{K})$ is of the form:

$$M = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & M' & \\ 0 & & & \end{pmatrix}$$

and we shall prove that $M \in \mathcal{M}'_n(\mathbf{K})$, i.e. that M can be expressed as a finite product of elements of $\mathcal{N}_n(\mathbf{K})$. Now since $M' \in \mathcal{M}_{n-1}(\mathbf{K})$, and $\mathcal{M}_{n-1}(\mathbf{K}) = \mathcal{M}'_{n-1}(\mathbf{K})$ being true from our induction hypothesis, M' can be expressed as a finite product of elements of $\mathcal{N}_{n-1}(\mathbf{K})$. Hence, there exist $p \geq 1$ and $Q'_1, \ldots, Q'_p$ elements of $\mathcal{N}_{n-1}(\mathbf{K})$ such that:

$$M' = Q'_1 \ldots Q'_p$$

For all $k \in \mathbf{N}_p$, we define:

$$Q_k \stackrel{\triangle}{=} \begin{pmatrix} & & & 0 \\ & Q'_k & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}$$

Following an argument identical to that contained in 2., each Q_k is an element of $\mathcal{N}_n(\mathbf{K})$. Furthermore, we have:

$$Q_1 \ldots Q_p \quad = \quad \begin{pmatrix} & & & 0 \\ & Q'_1 \ldots Q'_p & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} & & & 0 \\ & M' & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

and consequently:

$$\Sigma_{1n} Q_1 \dots Q_p \Sigma_{1n} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & M' & \\ 0 & & & \end{pmatrix} = M$$

It follows that M is indeed a finite product of elements of $\mathcal{N}_n(\mathbf{K})$, and we have proved that $M \in \mathcal{M}'_n(\mathbf{K})$. In 11. of exercise (1), we have proved that $\mathcal{M}_1(\mathbf{K}) = \mathcal{M}'_1(\mathbf{K})$. Having assumed that $n \geq 2$ and $\mathcal{M}_{n-1}(\mathbf{K}) = \mathcal{M}'_{n-1}(\mathbf{K})$, we have shown that $O_n \in \mathcal{M}'_n(\mathbf{K})$, and furthermore that if $M \neq O_n$, then M is also an element of $\mathcal{M}'_n(\mathbf{K})$. This shows that the equality $\mathcal{M}_n(\mathbf{K}) = \mathcal{M}'_n(\mathbf{K})$ holds, and completes our induction argument. We conclude that $\mathcal{M}_n(\mathbf{K}) = \mathcal{M}'_n(\mathbf{K})$ is true for all $n \geq 1$. In particular, it is true for all $n \geq 2$, which is the statement of theorem (103).

Exercise 2

Exercise 3.

1. Let $B \in \mathcal{F}'$ and $(B_n)_{n\geq 1}$ be a measurable partition of B, i.e from definition (91), a sequence of pairwise disjoint elements of $\mathcal{F}'$ such that $B = \uplus_{n\geq 1} B_n$. Then, we claim that $(X^{-1}(B_n))_{n\geq 1}$ is a measurable partition of $X^{-1}(B)$. Since X is measurable, $X^{-1}(B)$ and each $X^{-1}(B_n)$ is an element of $\mathcal{F}$. So we only need to prove that:

$$X^{-1}(B) = \biguplus_{n=1}^{+\infty} X^{-1}(B_n)$$

Since $B_n \subseteq B$ for all $n \geq 1$, it is clear that $X^{-1}(B_n) \subseteq X^{-1}(B)$, which establishes the inclusion $\supseteq$. Let $\omega \in X^{-1}(B)$. Then $X(\omega) \in B = \cup_{n\geq 1} B_n$. There exists $n \geq 1$ such that $X(\omega) \in B_n$, i.e. $\omega \in X^{-1}(B_n)$. This proves the inclusion $\subseteq$. In order to show that the $X^{-1}(B_n)$'s are pairwise disjoint, suppose we have $\omega \in X^{-1}(B_n) \cap X^{-1}(B_m)$. Then $X(\omega) \in B_n \cap B_m$, and since the B_n's are pairwise disjoint, we conclude that $n = m$.

2. Let μ be a measure on $(\Omega, \mathcal{F})$. Then $\mu : \mathcal{F} \to [0, +\infty]$ is a map such that $\mu(\emptyset) = 0$, and which is countably additive. Since X is measurable, for all $B \in \mathcal{F}'$, $X^{-1}(B)$ is an element of $\mathcal{F}$, and:

$$\mu^X(B) \stackrel{\triangle}{=} \mu(X^{-1}(B))$$

is therefore well-defined. So $\mu^X : \mathcal{F}' \to [0, +\infty]$ is a well-defined map. Since $X^{-1}(\emptyset) = \emptyset$, it is clear that $\mu^X(\emptyset) = 0$. To show that μ^X is a measure on $(\Omega', \mathcal{F}')$, we only need to show that μ^X is countably additive. Let $(B_n)_{n\geq 1}$ be a sequence of pairwise disjoint elements of $\mathcal{F}'$, and $B = \uplus_{n\geq 1} B_n$. Then:

$$X^{-1}(B) = \biguplus_{n=1}^{+\infty} X^{-1}(B_n)$$

and consequently, μ being countable additive:

$$\begin{aligned} \mu^X(B) &= \mu(X^{-1}(B)) \\ &= \sum_{n=1}^{+\infty} \mu(X^{-1}(B_n)) \\ &= \sum_{n=1}^{+\infty} \mu^X(B_n) \end{aligned}$$

So μ^X is countably additive, and we have proved that μ^X is indeed a well-defined measure on $(\Omega', \mathcal{F}')$.

3. Suppose that μ is a complex measure on $(\Omega, \mathcal{F})$. Then from definition (92), $\mu : \mathcal{F} \to \mathbf{C}$ is a map such that for any $B \in \mathcal{F}$ and $(B_n)_{n\geq 1}$ measurable partition of B, the series $\sum_{n\geq 1} \mu(B_n)$ converges to $\mu(B)$. Since X is measurable, for all $B \in \mathcal{F}'$, $X^{-1}(B) \in \mathcal{F}$ and consequently:

$$\mu^X(B) \stackrel{\triangle}{=} \mu(X^{-1}(B))$$

is well-defined. So $\mu^X : \mathcal{F}' \to \mathbf{C}$ is a well-defined map. Let $B \in \mathcal{F}'$ and $(B_n)_{n\geq 1}$ be a measurable partition of B. Then $(X^{-1}(B_n))_{n\geq 1}$ is a measurable partition of $X^{-1}(B)$, and so:

$$\begin{aligned} \mu^X(B) &= \mu(X^{-1}(B)) \\ &= \lim_{N\to+\infty} \sum_{n=1}^{N} \mu(X^{-1}(B_n)) \\ &= \lim_{N\to+\infty} \sum_{n=1}^{N} \mu^X(B_n) \end{aligned}$$

Hence, the series $\sum_{n\geq 1} \mu^X(B_n)$ converges to $\mu^X(B)$, and μ^X is indeed a well-defined complex measure on $(\Omega', \mathcal{F}')$.

4. Suppose μ is a complex measure on $(\Omega, \mathcal{F})$. Let $B \in \mathcal{F}'$ and $(B_n)_{n\geq 1}$ be a measurable partition of B. Then, $(X^{-1}(B_n))_{n\geq 1}$ is a measurable partition of $X^{-1}(B)$. From definition (94), since $|\mu|(X^{-1}(B))$ is an upper-bound of all sums $\sum_{n\geq 1} |\mu(E_n)|$, as $(E_n)_{n\geq 1}$ ranges through all measurable partitions of $X^{-1}(B)$:

$$\begin{aligned} \sum_{n=1}^{+\infty} |\mu^X(B_n)| &= \sum_{n=1}^{+\infty} |\mu(X^{-1}(B_n))| \\ &\leq |\mu|(X^{-1}(B)) = |\mu|^X(B) \end{aligned}$$

So $|\mu|^X(B)$ is an upper-bound of all sums $\sum_{n\geq 1}|\mu^X(B_n)|$, as $(B_n)_{n\geq 1}$ ranges through all measurable partitions of B. Since $|\mu^X|(B)$ is the smallest of such upper-bounds, we obtain:

$$|\mu^X|(B) \leq |\mu|^X(B)$$

This being true for all $B \in \mathcal{F}'$, we have $|\mu^X| \leq |\mu|^X$.

5. Let $Y : (\Omega', \mathcal{F}') \to (\Omega'', \mathcal{F}'')$ be a measurable map, where $(\Omega'', \mathcal{F}'')$ is another measurable space. Let μ be a (possibly complex) measure on $(\Omega, \mathcal{F})$. Then $X(\mu)$ is a well-defined (possibly complex) measure on $(\Omega', \mathcal{F}')$. So $Y(X(\mu))$ is a well-defined (possibly complex) measure on $(\Omega'', \mathcal{F}'')$. For all $B \in \mathcal{F}''$:

$$\begin{aligned} Y(X(\mu))(B) &= X(\mu)(Y^{-1}(B)) \\ &= \mu(X^{-1}(Y^{-1}(B))) \\ &= \mu((Y \circ X)^{-1}(B)) \\ &= (Y \circ X)(\mu)(B) \end{aligned}$$

This being true for all $B \in \mathcal{F}''$, $Y(X(\mu)) = (Y \circ X)(\mu)$. From definition (123), we obtain immediately:

$$(\mu^X)^Y = Y(\mu^X) = Y(X(\mu)) = (Y \circ X)(\mu) = \mu^{(Y \circ X)}$$

Exercise 3

Exercise 4.

1. Let $a \in \mathbf{R}^n$ and $\tau_a : \mathbf{R}^n \to \mathbf{R}^n$ be the associated translation mapping. Since $\|\tau_a(x) - \tau_a(y)\| = \|x - y\|$ for all $x, y \in \mathbf{R}^n$, it is clear that τ_a is continuous. It is therefore Borel measurable.

2. Let μ be a (possibly complex) measure on $\mathbf{R}^n$. Let $a \in \mathbf{R}^n$. Since $\tau_a : \mathbf{R}^n \to \mathbf{R}^n$ is measurable, $\tau_a(\mu)$ is a well-defined (possibly complex) measure on $\mathbf{R}^n$.

3. Let $a \in \mathbf{R}^n$ and $u, v \in \mathbf{R}^n$ with $u_i \leq v_i$ for all $i \in \mathbf{N}_n$. Then:

$$\begin{aligned} \tau_a(dx)\left(\prod_{i=1}^n [u_i, v_i]\right) &= dx\left(\tau_a^{-1}\left(\prod_{i=1}^n [u_i, v_i]\right)\right) \\ &= dx\left(\prod_{i=1}^n [u_i - a_i, v_i - a_i]\right) \\ &= \prod_{i=1}^n (v_i - u_i) \\ &= dx\left(\prod_{i=1}^n [u_i, v_i]\right) \end{aligned}$$

From the uniqueness property of definition (63), $\tau_a(dx) = dx$.

4. Having proved that $\tau_a(dx) = dx$ for all $a \in \mathbf{R}^n$, we conclude from definition (124) that the Lebesgue measure dx on $\mathbf{R}^n$ is invariant by translation.

Exercise 4

Exercise 5.

1. Let $\alpha > 0$, and $k_\alpha : \mathbf{R}^n \to \mathbf{R}^n$ be defined by $k_\alpha(x) = \alpha x$. Since $\|k_\alpha(x) - k_\alpha(y)\| = \alpha\|x - y\|$ for all $x, y \in \mathbf{R}^n$, it is clear that k_α is continuous and consequently Borel measurable.

2. Since k_α is measurable, $k_\alpha(dx)$ is a well-defined measure on $\mathbf{R}^n$, and so is $\alpha^n k_\alpha(dx)$. Let $u, v \in \mathbf{R}^n$ with $u_i \leq v_i$ for all $i \in \mathbf{N}_n$:

$$\begin{aligned}
\alpha^n k_\alpha(dx)\left(\prod_{i=1}^n [u_i, v_i]\right) &= \alpha^n dx\left(k_\alpha^{-1}\left(\prod_{i=1}^n [u_i, v_i]\right)\right) \\
&= \alpha^n dx\left(\prod_{i=1}^n \left[\frac{u_i}{\alpha}, \frac{v_i}{\alpha}\right]\right) \\
&= \alpha^n \prod_{i=1}^n \left(\frac{v_i}{\alpha} - \frac{u_i}{\alpha}\right) \\
&= \prod_{i=1}^n (v_i - u_i) \\
&= dx\left(\prod_{i=1}^n [u_i, v_i]\right)
\end{aligned}$$

From the uniqueness property of definition (63), $\alpha^n k_\alpha(dx) = dx$. It follows that $k_\alpha(dx) = \alpha^{-n} dx$.

Exercise 5

Exercise 6. Let $X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')$ be a measurable map, where $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ are measurable spaces. Let μ be a measure on $(\Omega, \mathcal{F})$. Let $f : (\Omega', \mathcal{F}') \to [0, +\infty]$ be a non-negative and measurable map. We claim that:

$$\int_\Omega f \circ X d\mu = \int_{\Omega'} f dX(\mu) \tag{2}$$

Note that X being measurable, $X(\mu)$ is a well-defined measure on $(\Omega', \mathcal{F}')$ and consequently the right-hand-side of (2) is perfectly meaningful. Furthermore, $f \circ X$ is a non-negative and measurable map on $(\Omega, \mathcal{F})$ and the left-hand-side of (2) is also perfectly meaningful. In the case when $f = 1_A$ for some $A \in \mathcal{F}'$, equation (2) reduces to:

$$\begin{aligned}
\int_\Omega f \circ X d\mu &= \int_\Omega 1_A \circ X d\mu \\
&= \int_\Omega 1_{X^{-1}(A)} d\mu
\end{aligned}$$

$$\begin{aligned}
&= \mu(X^{-1}(A)) \\
&= X(\mu)(A) \\
&= \int_{\Omega'} 1_A dX(\mu) \\
&= \int_{\Omega'} f dX(\mu)
\end{aligned}$$

which is true by virtue of $X(\mu)(A) = \mu(X^{-1}(A))$ of definition (123). When $f = \sum_{i=1}^n \alpha_i 1_{A_i}$ is a simple function on $(\Omega', \mathcal{F}')$, we have:

$$\begin{aligned}
\int_\Omega f \circ X d\mu &= \int_\Omega \left(\sum_{i=1}^n \alpha_i 1_{A_i}\right) \circ X d\mu \\
&= \int_\Omega \left(\sum_{i=1}^n \alpha_i 1_{A_i} \circ X\right) d\mu \\
&= \sum_{i=1}^n \alpha_i \int_\Omega 1_{A_i} \circ X d\mu \\
&= \sum_{i=1}^n \alpha_i \int_{\Omega'} 1_{A_i} dX(\mu) \\
&= \int_{\Omega'} \left(\sum_{i=1}^n \alpha_i 1_{A_i}\right) dX(\mu) \\
&= \int_{\Omega'} f dX(\mu)
\end{aligned}$$

Hence equation (2) is also true in the case when f is a simple function on $(\Omega', \mathcal{F}')$. We now assume that f is an arbitrary non-negative and measurable function on $(\Omega', \mathcal{F}')$. From theorem (18), there exists a sequence $(s_n)_{n\geq 1}$ of simple functions on $(\Omega', \mathcal{F}')$ such that $s_n \uparrow f$, i.e. $s_n \leq s_{n+1} \leq f$ for all $n \geq 1$ and $s_n(\omega) \to f(\omega)$ for all $\omega \in \Omega'$. Then it is clear that $s_n \circ X \uparrow f \circ X$, and from the monotone convergence theorem (19), we obtain:

$$\begin{aligned}
\int_\Omega f \circ X d\mu &= \lim_{n\to+\infty} \int_\Omega s_n \circ X d\mu \\
&= \lim_{n\to+\infty} \int_{\Omega'} s_n dX(\mu) \\
&= \int_{\Omega'} f dX(\mu)
\end{aligned}$$

This completes the proof of theorem (104).

Exercise 6

Exercise 7. Let $X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')$ be a measurable map, where $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ are measurable spaces. Let μ be a measure on $(\Omega, \mathcal{F})$. Let $f : (\Omega', \mathcal{F}') \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ be a measurable map. Then, the map $f \circ X : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is also measurable. Applying theorem (104) to the non-negative and measurable map $|f|$, we obtain:

$$\begin{aligned}\int_\Omega |f\circ X|d\mu &= \int_\Omega |f|\circ X d\mu \\ &= \int_{\Omega'} |f|dX(\mu)\end{aligned}$$

It follows that $\int_\Omega |f\circ X|d\mu < +\infty \Leftrightarrow \int_{\Omega'} |f|dX(\mu) < +\infty$, or equivalently, all maps involved being measurable:

$$f\circ X \in L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu) \;\Leftrightarrow\; f\in L^1_{\mathbf{C}}(\Omega',\mathcal{F}',X(\mu))$$

We now assume that $f\in L^1_{\mathbf{C}}(\Omega',\mathcal{F}',X(\mu))$. Let $u = Re(f)$ and $v = Im(f)$. Then $f = u^+ - u^- + i(v^+ - v^-)$, and applying theorem (104) to each non-negative and measurable map $u^\pm, v^\pm$, we obtain:

$$\begin{aligned}\int_\Omega f\circ X d\mu &= \int_\Omega [u^+ - u^- + i(v^+ - v^-)]\circ X d\mu \\ &= \int_\Omega u^+\circ X d\mu - \int_\Omega u^-\circ X d\mu \\ &+ i\left(\int_\Omega v^+\circ X d\mu - \int_\Omega v^-\circ X d\mu\right) \\ &= \int_{\Omega'} u^+ dX(\mu) - \int_{\Omega'} u^- dX(\mu) \\ &+ i\left(\int_{\Omega'} v^+ dX(\mu) - \int_{\Omega'} v^- dX(\mu)\right) \\ &= \int_{\Omega'} [u^+ - u^- + i(v^+ - v^-)]dX(\mu) \\ &= \int_{\Omega'} f dX(\mu)\end{aligned}$$

Note that this derivation is perfectly legitimate, as all the integrals involved are finite. This completes the proof of theorem (105).

Exercise 7

Exercise 8. Let $X : (\Omega,\mathcal{F}) \to (\Omega',\mathcal{F}')$ be a measurable map, where $(\Omega,\mathcal{F})$ and $(\Omega',\mathcal{F}')$ are measurable spaces. Let μ be a complex measure on $(\Omega,\mathcal{F})$. Let $f : (\Omega',\mathcal{F}') \to (\mathbf{C},\mathcal{B}(\mathbf{C}))$ be measurable. From 4. of exercise (3), we have $|\mu^X| \leq |\mu|^X$, or equivalently $|X(\mu)| \leq X(|\mu|)$. Using exercise (18) of Tutorial 12 together with theorem (104):

$$\begin{aligned}\int_{\Omega'} |f|d|X(\mu)| &\leq \int_{\Omega'} |f|dX(|\mu|) \\ &= \int_\Omega |f|\circ X d|\mu| \\ &= \int_\Omega |f\circ X|d|\mu|\end{aligned}$$

So $\int_\Omega |f\circ X|d|\mu| < +\infty \Rightarrow \int_{\Omega'} |f|d|X(\mu)| < +\infty$ and consequently:

$$f\circ X \in L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu) \;\Rightarrow\; f \in L^1_{\mathbf{C}}(\Omega',\mathcal{F}',X(\mu))$$

We now assume that $f\circ X \in L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu)$. Let $\mu_1 = Re(\mu)$ and $\mu_2 = Im(\mu)$. Then, we have $\mu = \mu_1^+ - \mu_1^- + i(\mu_2^+ - \mu_2^-)$, and from exercise (19) of Tutorial 12, $f\circ X \in L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu_k^\pm)$, $k=1,2$, with:

$$\begin{aligned}\int_\Omega f\circ X d\mu &= \int_\Omega f\circ X d\mu_1^+ - \int_\Omega f\circ X d\mu_1^- \\ &+ i\left(\int_\Omega f\circ X d\mu_2^+ - \int_\Omega f\circ X d\mu_2^-\right)\end{aligned} \tag{3}$$

Applying theorem (105) to each measure $\mu_k^\pm$, we obtain:

$$\int_\Omega f\circ X d\mu_k^\pm = \int_{\Omega'} f dX(\mu_k^\pm)\ ,\ k=1,2 \tag{4}$$

Moreover, for all $B\in\mathcal{F}'$, we have:

$$\begin{aligned}X(\mu)(B) &= \mu(X^{-1}(B)) \\ &= \mu_1^+(X^{-1}(B)) - \mu_1^-(X^{-1}(B)) \\ &+ i(\mu_2^+(X^{-1}(B)) - \mu_2^-(X^{-1}(B))) \\ &= X(\mu_1^+)(B) - X(\mu_1^-)(B) + i(X(\mu_2^+)(B) - X(\mu_2^-)(B))\end{aligned}$$

and consequently:

$$X(\mu) = X(\mu_1^+) - X(\mu_1^-) + i(X(\mu_2^+) - X(\mu_2^-))$$

Since $f\in L^1_{\mathbf{C}}(\Omega',\mathcal{F}',X(\mu_k^\pm))$, from exercise (17) of Tutorial 12:

$$\begin{aligned}\int_{\Omega'} f dX(\mu) &= \int_{\Omega'} f dX(\mu_1^+) - \int_{\Omega'} f dX(\mu_1^-) \\ &+ i\left(\int_{\Omega'} f dX(\mu_2^+) - \int_{\Omega'} f dX(\mu_2^-)\right)\end{aligned} \tag{5}$$

From (3), (4) and (5), we conclude that:

$$\int_\Omega f\circ X d\mu = \int_{\Omega'} f dX(\mu)$$

which completes the proof of theorem (106).

Exercise 8

Exercise 9.

1. Let $X : (\Omega,\mathcal{F}) \to (\mathbf{R},\mathcal{B}(\mathbf{R}))$ be a random variable with distribution $\mu = X(P)$ under P, where $(\Omega,\mathcal{F},P)$ is a probability space. Recall that the notions of probability space, random variable and expectation are defined

in (70), (71) and (72) respectively. Let $i : \mathbf{R} \to \mathbf{R}$ be the identity mapping. Applying theorem (104), we have:

$$\begin{aligned}\int_\Omega |X|dP &= \int_\Omega |i \circ X|dP \\ &= \int_\Omega |i| \circ XdP \\ &= \int_{\mathbf{R}} |i|dX(P) \\ &= \int_{-\infty}^{+\infty} |x|d\mu(x)\end{aligned}$$

So X is integrable, if and only if $\int_{\mathbf{R}} |x|d\mu(x) < +\infty$.

2. If $\int |X|dP < +\infty$, applying theorem (105) we obtain:

$$\begin{aligned}E[X] = \int_\Omega XdP &= \int_\Omega i \circ XdP \\ &= \int_{\mathbf{R}} idX(P) = \int_{-\infty}^{+\infty} xd\mu(x)\end{aligned}$$

3. Let $f : x \to x^2$. From theorem (104), we have:

$$\begin{aligned}E[X^2] = \int_\Omega X^2dP &= \int_\Omega f \circ XdP \\ &= \int_{\mathbf{R}} fdX(P) = \int_{-\infty}^{+\infty} x^2d\mu(x)\end{aligned}$$

Exercise 9

Exercise 10.

1. Let μ be a locally finite measure on $\mathbf{R}^n$, which is invariant by translation. Given $a \in \mathbf{R}^n$, let $Q_a = [0, a_1[\times \ldots \times [0, a_n[$. Let $K_a = [0, a_1] \times \ldots \times [0, a_n]$. Then K_a is a closed subset of $\mathbf{R}^n$. Indeed, it can be written as $K_a = \cap_{i=1}^n p_i^{-1}([0, a_i])$, where $p_i : \mathbf{R}^n \to \mathbf{R}$ denotes the i-th canonical projection, which is a continuous map. Since $[0, a_i]$ is closed in $\mathbf{R}$, each $p_i^{-1}([0, a_i])$ is closed in $\mathbf{R}^n$, and K_a is closed. Moreover, for all $x, y \in K_a$:

$$\|x - y\| \leq \|x\| + \|y\| \leq 2\|a\|$$

Taking the supremum as $x, y \in K_a$, we obtain $\delta(K_a) \leq 2\|a\|$, and in particular $\delta(K_a) < +\infty$, where $\delta(K_a)$ is the diameter of K_a, as defined in (68). So K_a is a closed and bounded subset of $\mathbf{R}^n$. From theorem (48), K_a is a compact subset of $\mathbf{R}^n$. Hence, from exercise (10) of Tutorial 13, since μ is locally finite, we have $\mu(K_a) < +\infty$. We conclude from $Q_a \subseteq K_a$ that:

$$\mu(Q_a) \leq \mu(K_a) < +\infty$$

In particular, if $Q = Q_{(1,\ldots,1)}$ then $\mu(Q) < +\infty$.

2. Let $p = (p_1, \ldots, p_n)$ where $p_i \in \mathbf{N}^*$ for all $i \in \mathbf{N}_n$. We claim:

$$Q_p = \biguplus_{\substack{k \in \mathbf{N}^n \\ 0 \le k_i < p_i}} [k_1, k_1 + 1[\times \ldots \times [k_n, k_n + 1[$$

Let $k \in \mathbf{N}^n$ with $0 \le k_i < p_i$ for all $i \in \mathbf{N}_n$. Let $x \in \mathbf{R}^n$ and suppose that $k_i \le x_i < k_i + 1$ for all $i \in \mathbf{N}_n$. Then, we have:

$$0 \le k_i \le x_i < k_i + 1 \le p_i \ , \ \forall i \in \mathbf{N}_n$$

So in particular $x \in Q_p$. This shows the inclusion $\supseteq$. To show the reverse inclusion, suppose $x \in Q_p$. Given $i \in \mathbf{N}_n$, consider the set $X_i = \{k \in \mathbf{N} : 0 \le x_i < k+1\}$. Since $0 \le x_i < p_i$ and $p_i \ge 1$, it is clear that $p_i - 1 \in X_i$. So X_i is a non-empty subset of $\mathbf{N}$ which therefore has a smallest element $k_i \le p_i - 1$. Defining $k = (k_1, \ldots, k_n) \in \mathbf{N}^n$, we have $0 \le k_i < p_i$ for all $i \in \mathbf{N}_n$, and furthermore:

$$k_i \le x_i < k_i + 1 \ , \ \forall i \in \mathbf{N}_n$$

This shows the inclusion $\subseteq$. It remains to show that the above union is indeed a union of pairwise disjoint sets. Let $k, k' \in \mathbf{N}^n$ and suppose that $x \in \mathbf{R}^n$ is such that:

$$x \in \left(\prod_{i=1}^n [k_i, k_i + 1[\right) \bigcap \left(\prod_{i=1}^n [k'_i, k'_i + 1[\right)$$

Then for all $i \in \mathbf{N}_n$, $x_i \in [k_i, k_i + 1[\cap[k'_i, k'_i + 1[$ and consequently $k_i = k'_i$. So $k = k'$.

3. For all $k \in \mathbf{N}^n$ with $0 \le k_i < p_i$, define:

$$A_k = [k_1, k_1 + 1[\times \ldots \times [k_n, k_n + 1[$$

Let $\tau_k : \mathbf{R}^n \to \mathbf{R}^n$ be the translation mapping of vector k, defined by $\tau_k(x) = k + x$ for all $x \in \mathbf{R}^n$. Since μ is invariant by translation, $\tau_k(\mu) = \mu$ and consequently:

$$\begin{aligned} \mu(A_k) &= \tau_k(\mu)(A_k) \\ &= \mu(\tau_k^{-1}(A_k)) \\ &= \mu(\{\tau_k \in A_k\}) \\ &= \mu(\{x : k_i \le k_i + x_i < k_i + 1, \forall i \in \mathbf{N}^n\}) \\ &= \mu(\{x : 0 \le x_i < 1, \forall i \in \mathbf{N}_n\}) \\ &= \mu(Q) \end{aligned}$$

Having proved in 2 that $Q_p = \uplus_k A_k$, we obtain:

$$\mu(Q_p) = \sum_k \mu(A_k) = \sum_k \mu(Q) = p_1 \ldots p_n \mu(Q)$$

where we have used the fact that:

$$\text{card}\{k \in \mathbf{N}^n : 0 \leq k_i < p_i \ , \ \forall i \in \mathbf{N}_n\} = p_1 \dots p_n$$

4. Let $q_1, \dots, q_n \geq 1$ be positive integers. We claim that:

$$Q_p = \biguplus_{\substack{k \in \mathbf{N}^n \\ 0 \leq k_i < q_i}} [\frac{k_1 p_1}{q_1}, \frac{(k_1+1)p_1}{q_1}[\times \dots \times [\frac{k_n p_n}{q_n}, \frac{(k_n+1)p_n}{q_n}[$$

Let $k \in \mathbf{N}^n$ with $0 \leq k_i < q_i$ for all $i \in \mathbf{N}_n$. Let $x \in \mathbf{R}^n$ with:

$$\frac{k_i p_i}{q_i} \leq x_i < \frac{(k_i+1)p_i}{q_i} \ , \ \forall i \in \mathbf{N}_n$$

Then in particular $0 \leq x_i < p_i$ for all i's and consequently $x \in Q_p$. This shows the inclusion $\supseteq$. To show the reverse inclusion, suppose $x \in Q_p$. Given $i \in \mathbf{N}_n$, consider the set:

$$X_i = \left\{ k \in \mathbf{N} : x_i < \frac{(k+1)p_i}{q_i} \right\}$$

Since $0 \leq x_i < p_i$ and $q_i \geq 1$, it is clear that $q_i - 1 \in X_i$. So X_i is a non-empty subset of $\mathbf{N}$, which therefore has a smallest element $k_i \leq q_i - 1$. Defining $k = (k_1, \dots, k_n) \in \mathbf{N}^n$, it is clear that $0 \leq k_i < q_i$ for all $i \in \mathbf{N}_n$ and furthermore:

$$\frac{k_i p_i}{q_i} \leq x_i < \frac{(k_i+1)p_i}{q_i} \ , \ \forall i \in \mathbf{N}_n$$

This shows the inclusion $\subseteq$. It remains to show that the above union is indeed a union a pairwise disjoint sets. But if $k, k' \in \mathbf{N}^n$ are such that there exists $x \in \mathbf{R}^n$ with:

$$x_i \in \left[\frac{k_i p_i}{q_i}, \frac{(k_i+1)p_i}{q_i}\right[\bigcap \left[\frac{k'_i p_i}{q_i}, \frac{(k'_i+1)p_i}{q_i}\right[$$

for all $i \in \mathbf{N}_n$, then $k_i = k'_i$ for all i's and consequently $k = k'$.

5. Given $i \in \mathbf{N}_n$, define $r_i = p_i/q_i$. Let $r = (r_1, \dots, r_n)$. Given $k \in \mathbf{N}^n$ with $0 \leq k_i < q_i$ for all $i \in \mathbf{N}_n$, define:

$$A_k = [k_1 r_1, (k_1+1)r_1[\times \dots \times [k_n r_n, (k_n+1)r_n[$$

Let $\tau : \mathbf{R}^n \to \mathbf{R}^n$ be the translation mapping associated with the vector $u = (k_1 r_1, \dots, k_n r_n)$, and defined by $\tau(x) = u + x$ for all $x \in \mathbf{R}^n$. Since μ is invariant by translation, we have $\tau(\mu) = \mu$, and consequently:

$$\begin{aligned}
\mu(A_k) &= \tau(\mu)(A_k) \\
&= \mu(\tau^{-1}(A_k)) \\
&= \mu(\{\tau \in A_k\}) \\
&= \mu(\{x : k_i r_i \leq k_i r_i + x_i < (k_i+1)r_i, \forall i \in \mathbf{N}_n\})
\end{aligned}$$

$$\begin{aligned} &= \mu(\{x : 0 \leq x_i < r_i, \forall i \in \mathbf{N}_n\}) \\ &= \mu(Q_r) \end{aligned}$$

Having proved in 4. that $Q_p = \uplus_k A_k$, we obtain:

$$\mu(Q_p) = \sum_k \mu(A_k) = \sum_k \mu(Q_r) = q_1 \ldots q_n \mu(Q_r)$$

where we have used the fact that:

$$\mathrm{card}\{k \in \mathbf{N}^n : 0 \leq k_i < q_i \ , \ \forall i \in \mathbf{N}_n\} = q_1 \ldots q_n$$

Hence, we have proved that:

$$\mu(Q_p) = q_1 \ldots q_n \mu(Q_{(p_1/q_1,\ldots,p_n/q_n)})$$

6. Let $r \in (\mathbf{Q}^+)^n$. We claim that:

$$\mu(Q_r) = r_1 \ldots r_n \mu(Q) \tag{6}$$

If $r_i = 0$ for some $i \in \mathbf{N}_n$, then it is clear that $Q_r = \emptyset$ and (6) is satisfied. So we assume that $r_i > 0$ for all $i \in \mathbf{N}_n$. There exist integers $p_1, \ldots, p_n \geq 1$ and $q_1, \ldots, q_n \geq 1$ such that $r_i = p_i/q_i$ for all $i \in \mathbf{N}_n$. Using 5. and 3. we obtain:

$$\mu(Q_r) = \frac{\mu(Q_p)}{q_1 \ldots q_n} = \frac{p_1 \ldots p_n}{q_1 \ldots q_n}\mu(Q) = r_1 \ldots r_n \mu(Q)$$

which establishes our claim of equation (6).

7. Let $a \in (\mathbf{R}^+)^n$. We claim that:

$$\mu(Q_a) = a_1 \ldots a_n \mu(Q) \tag{7}$$

If $a_i = 0$ for some $i \in \mathbf{N}_n$, then (7) is obviously true. So we assume that $a_i > 0$ for all $i \in \mathbf{N}_n$. Let $(r^p)_{p\geq 1}$ be a sequence in $(\mathbf{Q}^+)^n$ such that $r_i^p \uparrow\uparrow a_i$ for all $i \in \mathbf{N}_n$, i.e. $r_i^p \leq r_i^{p+1} < a_i$ for all $p \geq 1$ and $r_i^p \to a_i$ as $p \to +\infty$. The map $\phi : \mathbf{R}^n \to \mathbf{R}$ defined by $\phi(x) = x_1 \ldots x_n$ can be written as $\phi = p_1 \ldots p_n$ where $p_i : \mathbf{R}^n \to \mathbf{R}$ is the i-th canonical projection. Since each p_i is continuous, ϕ is itself continuous. Furthermore, since $r_i^p \to a_i$ for all $i \in \mathbf{N}_n$, we have $r^p \to a$ with respect to the product topology of $\mathbf{R}^n$ (which is also the usual topology of $\mathbf{R}^n$). Hence:

$$\lim_{p\to+\infty} r_1^p \ldots r_n^p = \lim_{p\to+\infty} \phi(r^p) = \phi(a) = a_1 \ldots a_n \tag{8}$$

We now claim that $Q_{r^p} \uparrow Q_a$. Since $r_i^p \leq r_i^{p+1}$ for all $i \in \mathbf{N}_n$ and $p \geq 1$, it is clear that $Q_{r^p} \subseteq Q_{r^{p+1}}$ for all $p \geq 1$. So we only need to prove that $Q_a = \cup_{p\geq 1} Q_{r^p}$. From $r_i^p < a_i$ (and in particular $r_i^p \leq a_i$) for all $i \in \mathbf{N}_n$ and $p \geq 1$, we obtain $Q_{r^p} \subseteq Q_a$ for all $p \geq 1$. This shows the inclusion $\supseteq$. To show the reverse inclusion, let $x \in Q_a$. Given $i \in \mathbf{N}_n$, we have $0 \leq x_i < a_i$. Since $r_i^p \to a_i$ as $p \to +\infty$, there exist $N_i \geq 1$ such that:

$$p \geq N_i \ \Rightarrow \ x_i < r_i^p < a_i$$

Taking $p = \max(N_1, \ldots, N_n)$ we obtain $0 \leq x_i < r_i^p$ for all $i \in \mathbf{N}_n$, and consequently $x \in Q_{r^p}$. This shows the inclusion $\subseteq$. Having proved that $Q_{r^p} \uparrow Q_a$, from theorem (7) we have:

$$\lim_{p \to +\infty} \mu(Q_{r^p}) = \mu(Q_a) \tag{9}$$

Using 6. together with (8) and (9) we obtain:

$$\begin{aligned} \mu(Q_a) &= \lim_{p \to +\infty} \mu(Q_{r^p}) \\ &= \lim_{p \to +\infty} r_1^p \ldots r_n^p \mu(Q) \\ &= a_1 \ldots a_n \mu(Q) \end{aligned}$$

which establishes our claim of equation (7). Note that the third equality is legitimate from $\mu(Q) < +\infty$ and the continuity of the map $\psi : \mathbf{R}^+ \to \mathbf{R}$ defined by $\psi(x) = x\mu(Q)$. If we had $\mu(Q) = +\infty$, the conclusion would remain valid (the sequence $r_1^p \ldots r_n^p$ is non-decreasing), but it would no longer be true that ψ (with values in $[0, +\infty]$) is continuous, (recall that $(1/p) \cdot (+\infty)$ does not converge to $0 \cdot (+\infty)$ as $p \to +\infty$).

8. We define the set of subsets of $\mathbf{R}^n$:

$$\mathcal{C} \stackrel{\triangle}{=} \{[a_1, b_1[\times \ldots \times [a_n, b_n[\ ,\ a_i, b_i \in \mathbf{R}\ ,\ a_i \leq b_i\ ,\ \forall i \in \mathbf{N}^n\}$$

Let $B = [a_1, b_1[\times \ldots \times [a_n, b_n[\in \mathcal{C}$. Let $a = (a_1, \ldots, a_n) \in \mathbf{R}^n$ and $b = (b_1, \ldots, b_n) \in \mathbf{R}^n$. Let $c = b - a \in (\mathbf{R}^+)^n$. Let $\tau_a : \mathbf{R}^n \to \mathbf{R}^n$ be the translation mapping of vector a, defined by $\tau_a(x) = a + x$ for all $x \in \mathbf{R}^n$. Since μ is invariant by translation, we have $\tau_a(\mu) = \mu$. Using 7. we obtain:

$$\begin{aligned} \mu(B) &= \tau_a(\mu)(B) \\ &= \mu(\tau_a^{-1}(B)) \\ &= \mu(\{\tau_a \in B\}) \\ &= \mu(\{x : a_i \leq a_i + x_i < b_i, \forall i \in \mathbf{N}_n\}) \\ &= \mu(\{x : 0 \leq x_i < c_i, \forall i \in \mathbf{N}_n\}) \\ &= \mu(Q_c) \\ &= c_1 \ldots c_n \mu(Q) \\ &= \mu(Q) \prod_{i=1}^n (b_i - a_i) \\ &= \mu(Q) \prod_{i=1}^n dx^i(]a_i, b_i]) \\ &= \mu(Q) \prod_{i=1}^n dx^i([a_i, b_i[) \\ &= \mu(Q) dx^1 \otimes \ldots \otimes dx^n(B) \\ &= \mu(Q) dx(B) \end{aligned}$$

So we have proved that $\mu(B) = \mu(Q)dx(B)$ for all $B \in \mathcal{C}$. Note that in obtaining this equality, we have refrained from writing directly:

$$\prod_{i=1}^{n}(b_i - a_i) = dx\left(\prod_{i=1}^{n}[a_i, b_i[\right) = dx(B) \tag{10}$$

as this equality has not been proved anywhere in the Tutorials. Indeed, definition (63) of the Lebesgue measure on $\mathbf{R}^n$, defines it as the unique measure with the property (given $a, b \ldots$):

$$\prod_{i=1}^{n}(b_i - a_i) = dx\left(\prod_{i=1}^{n}[a_i, b_i]\right)$$

which is not quite the same as (10). However, if dx^i denotes the Lebesgue measure on $\mathbf{R}$, then it is clear that:

$$dx^i([a_i, b_i]) = dx^i(]a_i, b_i]) = dx^i([a_i, b_i[)$$

and furthermore, it is not difficult from the uniqueness property of definition (63) to establish the fact that the Lebesgue measure dx on $\mathbf{R}^n$ is the product measure $dx = dx^1 \otimes \ldots \otimes dx^n$.

9. Let $\mathcal{C}_1 = \{[a, b[\colon a, b \in \mathbf{R}\}$. It is by now a standard exercise to show that $\mathcal{B}(\mathbf{R}) = \sigma(\mathcal{C}_1)$. Let $\mathcal{C}_1^{\amalg n}$ be the n-fold product $\mathcal{C}_1 \amalg \ldots \amalg \mathcal{C}_1$, i.e. the set of rectangles, as per definition (52):

$$\mathcal{C}_1^{\amalg n} = \{A_1 \times \ldots \times A_n : A_i \in \mathcal{C}_1 \cup \{\mathbf{R}\} \ , \ \forall i \in \mathbf{N}_n\}$$

Since $\mathbf{R}$ is separable (has a countable base), from exercise (18) of Tutorial 6, we have $\mathcal{B}(\mathbf{R}^n) = \mathcal{B}(\mathbf{R})^{\otimes n}$ and consequently from theorem (26):

$$\mathcal{B}(\mathbf{R}^n) = \mathcal{B}(\mathbf{R})^{\otimes n} = \sigma(\mathcal{C}_1)^{\otimes n} = \sigma(\mathcal{C}_1^{\amalg n})$$

Hence, in order to prove that $\mathcal{B}(\mathbf{R}^n) = \sigma(\mathcal{C})$, we only need to show that $\sigma(\mathcal{C}) = \sigma(\mathcal{C}_1^{\amalg n})$. It is clear that $\mathcal{C} \subseteq \mathcal{C}_1^{\amalg n}$ which establishes the inclusion $\subseteq$. To show the reverse inclusion, it is sufficient to prove that $\mathcal{C}_1^{\amalg n} \subseteq \sigma(\mathcal{C})$. Let $B = A_1 \times \ldots \times A_n$ be a rectangle of $\mathcal{C}_1^{\amalg n}$. Suppose $A_1 = \mathbf{R}$. Then, we have:

$$B = \bigcup_{p=1}^{+\infty}[-p, p[\times A_2 \times \ldots \times A_n$$

and in order to prove that $B \in \sigma(\mathcal{C})$, it is sufficient to prove that each $[-p, p[\times A_2 \times \ldots \times A_n$ is an element of $\sigma(\mathcal{C})$. Hence, without loss of generality, we may assume that $A_1 \in \mathcal{C}_1$. Likewise, we may assume that $A_2 \in \mathcal{C}_1$, and in fact we may assume without loss of generality that $A_i \in \mathcal{C}_1$ for all $i \in \mathbf{N}_n$, in which case $B \in \mathcal{C} \subseteq \sigma(\mathcal{C})$. This completes our proof, and $\mathcal{B}(\mathbf{R}^n) = \sigma(\mathcal{C})$.

10. Given $p \geq 1$ we define:

$$\mathcal{D}_p = \{B \in \mathcal{B}(\mathbf{R}^n) : \mu(B \cap [-p, p[^n) = \mu(Q)dx(B \cap [-p, p[^n)\}$$

Having proved in 8. that $\mu(B) = \mu(Q)dx(B)$ for all $B \in \mathcal{C}$, since $\mathcal{C}$ is closed under finite intersection and $[-p,p[^n \in \mathcal{C}$, it is clear that $\mathcal{C} \subseteq \mathcal{D}_p$ and $\mathbf{R}^n \in \mathcal{D}_p$. Furthermore, if $A, B \in \mathcal{D}_p$ are such that $A \subseteq B$, then:

$$\begin{aligned}\mu((B \setminus A) \cap [-p,p[^n) &= \mu(B \cap [-p,p[^n) - \mu(A \cap [-p,p[^n) \\ &= \mu(Q)dx(B \cap [-p,p[^n) \\ &- \mu(Q)dx(A \cap [-p,p[^n) \\ &= \mu(Q)dx((B \setminus A) \cap [-p,p[^n)\end{aligned}$$

So $B \setminus A \in \mathcal{D}_p$. Note that the above derivation is legitimate, as all the quantities involved are finite since $\mu(Q) < +\infty$. This is a very important point, and is in fact the very reason why we have *localized* the problem on $[-p,p[^n$ by defining $\mathcal{D}_p$, rather than considering directly:

$$\mathcal{D} = \{B \in \mathcal{B}(\mathbf{R}^n) : \mu(B) = \mu(Q)dx(B)\}$$

for which the property $B \setminus A \in \mathcal{D}$ whenever $A, B \in \mathcal{D}$, $A \subseteq B$, may not be easy to establish, if at all true. Let $(B_k)_{k \geq 1}$ be a sequence of elements of $\mathcal{D}_p$ such that $B_k \uparrow B$. From theorem (7):

$$\begin{aligned}\mu(B \cap [-p,p[^n) &= \lim_{k \to +\infty} \mu(B_k \cap [-p,p[^n) \\ &= \lim_{k \to +\infty} \mu(Q)dx(B_k \cap [-p,p[^n) \\ &= \mu(Q) \lim_{k \to +\infty} dx(B_k \cap [-p,p[^n) \\ &= \mu(Q)dx(B \cap [-p,p[^n)\end{aligned}$$

So $B \in \mathcal{D}_p$, and we have proved that $\mathcal{D}_p$ is a Dynkin system on $\mathbf{R}^n$. Since $\mathcal{C} \subseteq \mathcal{D}_p$ and $\mathcal{C}$ is closed under finite intersection, from the Dynkin system theorem (1), we obtain $\sigma(\mathcal{C}) \subseteq \mathcal{D}_p$. Having proved in 9. that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbf{R}^n)$, it follows that $\mathcal{B}(\mathbf{R}^n) \subseteq \mathcal{D}_p$ for all $p \geq 1$. Hence, given $B \in \mathcal{B}(\mathbf{R}^n)$, using theorem (7):

$$\begin{aligned}\mu(B) &= \lim_{p \to +\infty} \mu(B \cap [-p,p[^n) \\ &= \lim_{p \to +\infty} \mu(Q)dx(B \cap [-p,p[^n) \\ &= \mu(Q) \lim_{p \to +\infty} dx(B \cap [-p,p[^n) \\ &= \mu(Q)dx(B)\end{aligned}$$

So $\mu = \mu(Q)dx$. Given a locally finite measure μ on $\mathbf{R}^n$, which is invariant by translation, we have found $\alpha = \mu(Q) \in \mathbf{R}^+$, such that $\mu = \alpha dx$. This completes the proof of theorem (107).

Exercise 10

Exercise 11.

1. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear bijection. In particular, T is a linear map defined on a finite dimensional normed space. So T is continuous.

Likewise, T^{-1} is a linear map defined on a finite dimensional normed space, so T^{-1} is continuous. The fact that a linear map defined on a finite dimensional normed space is continuous, has not yet been proved in these Tutorials (we have not even defined what a *normed space* is, see Tutorial 18). For those not familiar with the result, the proof in the case $\mathbf{R}^n$ (together with its usual inner-product) goes as follows: Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbf{R}^n$ and $x, y \in \mathbf{R}^n$. We have:

$$\begin{aligned}\|T(x) - T(y)\| &= \left\| T\left(\sum_{i=1}^n x_i e_i\right) - T\left(\sum_{i=1}^n y_i e_i\right)\right\| \\ &= \left\|\sum_{i=1}^n (x_i - y_i) T(e_i)\right\| \\ &\leq \sum_{i=1}^n |x_i - y_i| \cdot \|T(e_i)\| \\ &\leq \left(\sum_{i=1}^n \|T(e_i)\|^2\right)^{1/2} \left(\sum_{i=1}^n |x_i - y_i|^2\right)^{1/2} \\ &= M\|x - y\|\end{aligned}$$

where $M = (\sum_{i=1}^n \|T(e_i)\|^2)^{1/2}$, and we have used the Cauchy-Schwarz inequality (50). Having proved the existence of $M \in \mathbf{R}^+$ such that $\|T(x) - T(y)\| \leq M\|x - y\|$ for all $x, y \in \mathbf{R}^n$, it is clear that T is continuous. Similarly, there exists $M' \in \mathbf{R}^+$ such that $\|T^{-1}(x) - T^{-1}(y)\| \leq M'\|x - y\|$ for all $x, y \in \mathbf{R}^n$. So T^{-1} is continuous.

2. Let $B \subseteq \mathbf{R}^n$. The notation $T^{-1}(B)$ is potentially ambiguous, as it may refer to the inverse image of B by T as defined in (26), or the direct image of B by T^{-1} as defined in (25). Let $S = T^{-1}$, and let $S(B)$ denote the direct image, whereas $T^{-1}(B)$ denotes the inverse image. We claim that $T^{-1}(B) = S(B)$. Indeed, suppose that $x \in T^{-1}(B)$. Then $T(x) \in B$. Let $y = T(x)$. Then $y \in B$ and $S(y) = T^{-1}(T(x)) = x$. So $x \in S(B)$. This shows that $T^{-1}(B) \subseteq S(B)$. To show the reverse inclusion, suppose $x \in S(B)$. There exists $y \in B$ such that $x = S(y)$. So $T(x) = T(S(y)) = y$. So $T(x) \in B$, and $x \in T^{-1}(B)$. This shows that $S(B) \subseteq T^{-1}(B)$. We have proved that $T^{-1}(B) = S(B)$, and it follows that whether we view $T^{-1}(B)$ as an inverse image (that of B by T) or a direct image (that of B by T^{-1}) makes no difference, as the two sets are in fact equal. The notation $T^{-1}(B)$ is no longer ambiguous.

3. Let $B \subseteq \mathbf{R}^n$. Since $T : \mathbf{R}^n \to \mathbf{R}^n$ is a linear bijection, T^{-1} is also a linear bijection. Applying 2. to T^{-1}, it follows that the direct image $T(B)$ of B by $T = (T^{-1})^{-1}$ coincides with the inverse image $(T^{-1})^{-1}(B)$ of B by T^{-1}, i.e. $T(B) = (T^{-1})^{-1}(B)$.

4. Let $K \subseteq \mathbf{R}^n$ be a compact subset of $\mathbf{R}^n$. $\{T \in K\} = T^{-1}(K)$ denotes the inverse image of K by T. However from 2. it can also be viewed as

the direct image of K by T^{-1}. Having proved that $T^{-1} : \mathbf{R}^n \to \mathbf{R}^n$ is continuous and K being compact, it follows from exercise (8) of Tutorial 8 that $T^{-1}(K)$ is a compact subset of $\mathbf{R}^n$. We conclude that $\{T \in K\}$ is a compact subset of $\mathbf{R}^n$.

5. The Lebesgue measure dx on $\mathbf{R}^n$ is clearly locally finite, as can be seen from definition (102). Indeed, given $x \in \mathbf{R}^n$, the set $U = \Pi_{i=1}^n]x_i - 1, x_i + 1[$ is an open neighborhood of x with finite Lebesgue measure ($dx(U) = 2^n < +\infty$). From exercise (10) of Tutorial 13, if K' is a compact subset of $\mathbf{R}^n$, then we have $dx(K') < +\infty$. Furthermore, $\mathbf{R}^n$ is locally compact, as can be seen from definition (105). Indeed, given $x \in \mathbf{R}^n$, x has an open neighborhood with compact closure: taking U as above, the closure $K = \bar{U}$ is closed and bounded, and therefore compact from theorem (48). Having proved in 4. that $K' = \{T \in K\}$ is itself compact, it follows that:

$$T(dx)(U) \leq T(dx)(K) = dx(\{T \in K\}) = dx(K') < +\infty$$

Given $x \in \mathbf{R}^n$, we have shown the existence of U open, such that $x \in U$ and $T(dx)(U) < +\infty$. We conclude from definition (102) that $T(dx)$ (which is well-defined since T is continuous, hence Borel measurable) is a locally finite measure on $\mathbf{R}^n$.

6. Given $a \in \mathbf{R}^n$, let $\tau_a : \mathbf{R}^n \to \mathbf{R}^n$ be the translation mapping of vector a, defined by $\tau_a(x) = a + x$ for all $x \in \mathbf{R}^n$. We have:

$$\begin{aligned} T \circ \tau_{T^{-1}(a)}(x) &= T(T^{-1}(a) + x) \\ &= T(T^{-1}(a)) + T(x) \\ &= a + T(x) \\ &= \tau_a(T(x)) = \tau_a \circ T(x) \end{aligned}$$

This being true for all $x \in \mathbf{R}^n$, $T \circ \tau_{T^{-1}(a)} = \tau_a \circ T$.

7. Using 6. together with 5. of exercise (3), we have:

$$\begin{aligned} \tau_a(T(dx)) &= (\tau_a \circ T)(dx) \\ &= (T \circ \tau_{T^{-1}(a)})(dx) \\ &= T(\tau_{T^{-1}(a)}(dx)) = T(dx) \end{aligned}$$

where the last equality stems from the fact that the Lebesgue measure dx is invariant by translation. Having proved that $\tau_a(T(dx)) = T(dx)$ for all $a \in \mathbf{R}^n$, we conclude that $T(dx)$ is itself invariant by translation.

8. From 5. $T(dx)$ is a locally finite measure on $\mathbf{R}^n$. From 7. it is invariant by translation. It follows from theorem (107) that there exists $\alpha \in \mathbf{R}^+$ such that $T(dx) = \alpha dx$. Suppose β is another element of $\mathbf{R}^+$ such that $T(dx) = \beta dx$. Then:

$$\alpha = \alpha dx([0,1]^n) = \beta dx([0,1]^n) = \beta$$

Hence, α is unique and we denote it $\Delta(T)$, so that $\Delta(T)$ is the unique element of $\mathbf{R}^+$ such that $T(dx) = \Delta(T)dx$.

9. Let $Q = T([0,1]^n)$. Then Q is the direct image of $[0,1]^n$ by T. However from 3. it can also be viewed as the inverse image $(T^{-1})^{-1}([0,1]^n)$ of $[0,1]^n$ by T^{-1}. Since T^{-1} is continuous, in particular it is Borel measurable. It follows from $[0,1]^n \in \mathcal{B}(\mathbf{R}^n)$ that $(T^{-1})^{-1}([0,1]^n) \in \mathcal{B}(\mathbf{R}^n)$. So $Q \in \mathcal{B}(\mathbf{R}^n)$. Furthermore, denoting $S = T^{-1}$, we have:

$$\begin{aligned}\Delta(T)dx(Q) &= T(dx)(Q)\\ &= dx(T^{-1}(Q))\\ &= dx(T^{-1}(T([0,1]^n)))\\ &= dx(S(T([0,1]^n)))\\ &= dx((S\circ T)([0,1]^n))\\ &= dx([0,1]^n) = 1\end{aligned}$$

10. Since $\Delta(T)dx(Q) = 1$ for some $Q \in \mathcal{B}(\mathbf{R}^n)$, $\Delta(T) \neq 0$.

11. Let $T_1, T_2 : \mathbf{R}^n \to \mathbf{R}^n$ be two linear bijections. If $B \in \mathcal{B}(\mathbf{R}^n)$:

$$\begin{aligned}(T_1\circ T_2)(dx)(B) &= T_1(T_2(dx))(B)\\ &= T_1(\Delta(T_2)dx)(B)\\ &= (\Delta(T_2)dx)(T_1^{-1}(B))\\ &= \Delta(T_2)dx(T_1^{-1}(B))\\ &= \Delta(T_2)T_1(dx)(B)\\ &= \Delta(T_2)(\Delta(T_1)dx(B))\\ &= \Delta(T_1)\Delta(T_2)dx(B)\end{aligned}$$

This being true for all $B \in \mathcal{B}(\mathbf{R}^n)$, we have:

$$(T_1\circ T_2)(dx) = \Delta(T_1)\Delta(T_2)dx$$

Since $\Delta(T_1\circ T_2)$ is the unique element of $\mathbf{R}^+$ with the property $(T_1 \circ T_2)(dx) = \Delta(T_1\circ T_2)dx$, we conclude that:

$$\Delta(T_1\circ T_2) = \Delta(T_1)\Delta(T_2)$$

Exercise 11

Exercise 12.

1. Let $\alpha \in \mathbf{R}\setminus\{0\}$ and $H_\alpha : \mathbf{R}^n \to \mathbf{R}^n$ be the linear bijection defined by $H_\alpha e_1 = \alpha e_1$ and $H_\alpha e_j = e_j$ for $j \geq 2$, where $e_1,\ldots,e_n$ is the canonical basis of $\mathbf{R}^n$. If $\alpha > 0$, we have:

$$\begin{aligned}H_\alpha(dx)([0,1]^n) &= dx(H_\alpha^{-1}([0,1]^n))\\ &= dx(\{x : H_\alpha x \in [0,1]^n\})\\ &= dx\left(\left\{x : \sum_{j=1}^n x_j H_\alpha e_j \in [0,1]^n\right\}\right)\end{aligned}$$

$$\begin{aligned} &= dx(\{x : (\alpha x_1, x_2, \ldots, x_n) \in [0,1]^n\}) \\ &= dx([0, \alpha^{-1}] \times [0,1]^{n-1}) = \alpha^{-1} \end{aligned}$$

If $\alpha < 0$, we have similarly:

$$H_\alpha(dx)([0,1]^n) = dx([\alpha^{-1}, 0] \times [0,1]^{n-1}) = -\alpha^{-1}$$

In any case we obtain $H_\alpha(dx)([0,1]^n) = |\alpha|^{-1}$.

2. The determinant $\det H_\alpha$ of H_α has not been defined in these Tutorials. Until we do so, we will have to accept that:

$$\det H_\alpha = \det \begin{pmatrix} \alpha & & & \\ & 1 & 0 & \\ & 0 & \ddots & \\ & & & 1 \end{pmatrix} = \alpha$$

This being granted, using 1. we have:

$$\begin{aligned} \Delta(H_\alpha) &= \Delta(H_\alpha) dx([0,1]^n) \\ &= H_\alpha(dx)([0,1]^n) \\ &= |\alpha|^{-1} = |\det H_\alpha|^{-1} \end{aligned}$$

Exercise 12

Exercise 13.

1. Let $k, l \in \mathbf{N}_n$ and $\Sigma : \mathbf{R}^n \to \mathbf{R}^n$ be the linear bijection defined by $\Sigma e_k = e_l$, $\Sigma e_l = e_k$ and $\Sigma e_j = e_j$ for $j \neq k, l$, where $e_1, \ldots, e_n$ is the canonical basis of $\mathbf{R}^n$. Let $\sigma : \mathbf{N}_n \to \mathbf{N}_n$ be the permutation of $\mathbf{N}_n$ defined by $\sigma(k) = l$, $\sigma(l) = k$ and $\sigma(j) = j$ for $j \neq k, l$. Then $\Sigma e_j = e_{\sigma(j)}$ for all $j \in \mathbf{N}_n$. We have:

$$\begin{aligned} \Sigma(dx)([0,1]^n) &= dx(\Sigma^{-1}([0,1]^n)) \\ &= dx(\{x : \Sigma x \in [0,1]^n\}) \\ &= dx\left(\left\{x : \sum_{j=1}^n x_j \Sigma e_j \in [0,1]^n\right\}\right) \\ &= dx\left(\left\{x : \sum_{j=1}^n x_{\sigma^{-1}(j)} \Sigma e_{\sigma^{-1}(j)} \in [0,1]^n\right\}\right) \\ &= dx(\{x : (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}) \in [0,1]^n\}) \\ &= dx([0,1]^n) = 1 \end{aligned}$$

2. Since $\Sigma \cdot \Sigma e_j = e_j$ for all $j \in \mathbf{N}_n$, we have $\Sigma \cdot \Sigma = I_n$.

3. Until we have a Tutorial on the determinant, we shall have to accept that given $A, B \in \mathcal{M}_n(\mathbf{K})$, we have:

$$\det AB = \det A \det B$$

This being granted, using 2. we obtain:

$$1 = \det I_n = \det \Sigma\Sigma = (\det \Sigma)^2$$

from which we conclude that $|\det \Sigma| = 1$.

4. Using 1. we have:

$$\begin{aligned}\Delta(\Sigma) &= \Delta(\Sigma)dx([0,1]^n)\\ &= \Sigma(dx)([0,1]^n)\\ &= 1 = |\det \Sigma|^{-1}\end{aligned}$$

Exercise 13

Exercise 14.

1. Let $n \geq 2$ and $U : \mathbf{R}^n \to \mathbf{R}^n$ be the linear bijection defined by $Ue_1 = e_1 + e_2$ and $Ue_j = e_j$ for $j \geq 2$, where $e_1, \ldots, e_n$ is the canonical basis of $\mathbf{R}^n$. Let $Q = [0,1[^n$. Given $x \in \mathbf{R}^n$, we have:

$$\begin{aligned}Ux &= U\left(\sum_{j=1}^n x_j e_j\right)\\ &= \sum_{j=1}^n x_j U e_j\\ &= x_1(e_1+e_2) + \sum_{j=2}^n x_j e_j\\ &= (x_1, x_1+x_2, x_3, \ldots, x_n)\end{aligned}$$

Since $U^{-1}(Q) = \{x \in \mathbf{R}^n : Ux \in [0,1[^n\}$ we conclude that:

$$U^{-1}(Q) = \{x \in \mathbf{R}^n :\ 0 \leq x_1 + x_2 < 1\ ,\ 0 \leq x_i < 1\ ,\ \forall i \neq 2\}$$

2. We define:

$$\begin{aligned}\Omega_1 &\stackrel{\triangle}{=} U^{-1}(Q) \cap \{x \in \mathbf{R}^n :\ x_2 \geq 0\}\\ \Omega_2 &\stackrel{\triangle}{=} U^{-1}(Q) \cap \{x \in \mathbf{R}^n :\ x_2 < 0\}\end{aligned}$$

Given $i \in \mathbf{N}_n$, let $p_i : \mathbf{R}^n \to \mathbf{R}$ be the i-th canonical projection. Then each p_i is continuous and therefore Borel measurable. From 1. we obtain:

$$U^{-1}(Q) = (p_1+p_2)^{-1}([0,1[) \cap \left(\bigcap_{i\neq 2} p_i^{-1}([0,1[)\right)$$

So it is clear that $U^{-1}(Q) \in \mathcal{B}(\mathbf{R}^n)$. From:

$$\begin{aligned}\Omega_1 &= U^{-1}(Q) \cap p_2^{-1}([0,+\infty[)\\ \Omega_2 &= U^{-1}(Q) \cap p_2^{-1}(]-\infty,0[)\end{aligned}$$

we conclude that $\Omega_1, \Omega_2 \in \mathcal{B}(\mathbf{R}^n)$.

3. It is impossible for me to draw a picture with Latex. Some people can do it, but I can't. A picture is not a proof of anything, and is therefore not essential. However, if you have spent the time drawing it, it should be clear to you that $\{\Omega_1, \tau_{e_2}(\Omega_2)\}$ forms a partition of Q, which we shall prove formally in this exercise.

4. Suppose $x \in \Omega_1$. Then $x_2 \geq 0$ and furthermore $x \in U^{-1}(Q)$. So $0 \leq x_1 + x_2 < 1$ while $0 \leq x_1 < 1$. Hence, we have:

$$0 \leq x_2 \leq x_1 + x_2 < 1$$

   We have proved that $x \in \Omega_1 \Rightarrow 0 \leq x_2 < 1$.

5. If $x \in \Omega_1$ then in particular $x \in U^{-1}(Q)$. So $0 \leq x_i < 1$ for all $i \in \mathbf{N}_n$, $i \neq 2$. However from 4. we have $0 \leq x_2 < 1$. It follows that $0 \leq x_i < 1$ for all $i \in \mathbf{N}_n$. So $x \in Q = [0, 1[^n$. We have proved that $\Omega_1 \subseteq Q$.

6. Suppose $x \in \tau_{e_2}(\Omega_2)$. There exists $y \in \Omega_2$ such that $x = \tau_{e_2}(y) = e_2 + y$. In particular, $x_1 = y_1$ and $x_2 = 1 + y_2$ for some $y \in \Omega_2$. The fact that $y \in \Omega_2$ implies in particular that $y_2 < 0$ and $y \in U^{-1}(Q)$. So $0 \leq y_1 < 1$ and $0 \leq y_1 + y_2 < 1$. Hence:

$$0 \leq y_1 + y_2 < 1 + y_2 = x_2 < 1 + 0 = 1$$

   We have proved that $x \in \tau_{e_2}(\Omega_2) \Rightarrow 0 \leq x_2 < 1$. In fact, we have proved the stronger inequality $0 < x_2 < 1$, but we shall not need it.

7. Suppose $x \in \tau_{e_2}(\Omega_2)$. There exists $y \in \Omega_2$ such that $x = \tau_{e_2}(y) = e_2 + y$. So $x_2 = 1 + y_2$ and $x_i = y_i$ for all $i \neq 2$. The fact that $y \in \Omega_2$ implies in particular that $y \in U^{-1}(Q)$. So $0 \leq y_i < 1$ for all $i \neq 2$ and consequently $0 \leq x_i < 1$ for all $i \neq 2$. However, we have proved in 6. that $0 \leq x_2 < 1$. So $0 \leq x_i < 1$ for all $i \in \mathbf{N}_n$, i.e. $x \in Q = [0, 1[^n$. We have proved that $\tau_{e_2}(\Omega_2) \subseteq Q$.

8. Suppose $x \in Q$ and $x_1 + x_2 < 1$. Then for all $i \in \mathbf{N}_n$, we have $0 \leq x_i < 1$ and furthermore $x_1 + x_2 < 1$. In particular, we have $x_2 \geq 0$ and $0 \leq x_1 + x_2 < 1$, while $0 \leq x_i < 1$ for all $i \neq 2$. So $x \in U^{-1}(Q)$ while $x_2 \geq 0$, i.e. $x \in \Omega_1$. We have proved that $x \in Q$ and $x_1 + x_2 < 1$ implies that $x \in \Omega_1$.

9. Suppose $x \in Q$ and $x_1 + x_2 \geq 1$. Then for all $i \in \mathbf{N}_n$ we have $0 \leq x_i < 1$ and furthermore $x_1 + x_2 \geq 1$. Define $y = (x_1, -1 + x_2, x_3, \ldots, x_n)$. Then it is clear that $e_2 + y = x$. So $x = \tau_{e_2}(y)$. We claim that $y \in \Omega_2$. From $x_2 < 1$ we obtain $y_2 = -1 + x_2 < 0$. Furthermore, for all $i \neq 2$ we have $x_i = y_i$ and consequently $0 \leq y_i < 1$. Finally, from $x_1 + x_2 \geq 1$, we obtain:

$$0 \leq x_1 + x_2 - 1 = y_1 + y_2 < 1 + 0 = 1$$

   Hence, we see that $y \in U^{-1}(Q)$ while $y_2 < 0$. So $y \in \Omega_2$ and since $x = \tau_{e_2}(y)$, we have $x \in \tau_{e_2}(\Omega_2)$. We have proved that $x \in Q$ and $x_1 + x_2 \geq 1$ implies that $x \in \tau_{e_2}(\Omega_2)$.

10. Suppose $x \in \tau_{e_2}(\Omega_2)$. There exists $y \in \Omega_2$ such that $x = \tau_{e_2}(y) = e_2 + y$. In particular, $x_1 = y_1$ and $x_2 = 1 + y_2$ for some $y \in \Omega_2$. The fact that $y \in \Omega_2$ implies that $y \in U^{-1}(Q)$ and $0 \leq y_1 + y_2 < 1$. Hence, we have:

$$1 \leq 1 + y_1 + y_2 = x_1 + x_2$$

We have proved that $x \in \tau_{e_2}(\Omega_2) \Rightarrow x_1 + x_2 \geq 1$.

11. Suppose $x \in \tau_{e_2}(\Omega_2) \cap \Omega_1$. From $x \in \Omega_1$ we have in particular $x \in U^{-1}(Q)$ and consequently $x_1 + x_2 < 1$. From $x \in \tau_{e_2}(\Omega_2)$ using 10. we have $x_1 + x_2 \geq 1$. This is a contradiction. We have proved that $\tau_{e_2}(\Omega_2) \cap \Omega_1 = \emptyset$.

12. From 5. we have $\Omega_1 \subseteq Q$ while from 7. we have $\tau_{e_2}(\Omega_2) \subseteq Q$. This shows that $\Omega_1 \cup \tau_{e_2}(\Omega_2) \subseteq Q$. To show the reverse inclusion, suppose $x \in Q$. If $x_1 + x_2 < 1$ from 8. we have $x \in \Omega_1$. If $x_1 + x_2 \geq 1$ from 9. we have $x \in \tau_{e_2}(\Omega_2)$. In any case, we have $x \in \Omega_1 \cup \tau_{e_2}(\Omega_2)$. This shows that $Q \subseteq \Omega_1 \cup \tau_{e_2}(\Omega_2)$, and we have proved that $Q = \Omega_1 \cup \tau_{e_2}(\Omega_2)$. Having proved that Ω_1 and $\tau_{e_2}(\Omega_2)$ are disjoint, we conclude that $Q = \Omega_1 \uplus \tau_{e_2}(\Omega_2)$.

13. Noting that $\tau_{e_2}(\Omega_2) = \tau^{-1}_{-e_2}(\Omega_2) \in \mathcal{B}(\mathbf{R}^n)$, we have:

$$\begin{aligned}
dx(Q) &= dx(\Omega_1 \uplus \tau_{e_2}(\Omega_2)) \\
&= dx(\Omega_1) + dx(\tau_{e_2}(\Omega_2)) \\
&= dx(\Omega_1) + dx(\Omega_2) \\
&= dx(U^{-1}(Q) \cap \{x_2 \geq 0\}) + dx(U^{-1}(Q) \cap \{x_2 < 0\}) \\
&= dx(U^{-1}(Q))
\end{aligned}$$

where the third equality stems from the fact that the Lebesgue measure dx is invariant by translation.

14. It follows from 13. that:

$$\Delta(U) = \Delta(U)dx(Q) = U(dx)(Q) = dx(U^{-1}(Q)) = dx(Q) = 1$$

15. Until we have a Tutorial on determinants, we shall accept:

$$\det U = \det \begin{pmatrix} 1 & 0 & & \\ 1 & 1 & 0 & \\ & 0 & \ddots & \\ & & & 1 \end{pmatrix} = 1$$

This being granted, we conclude from 14. that:

$$\Delta(U) = 1 = |\det U|^{-1}$$

Exercise 14

Exercise 15.

1. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear bijection where $n \geq 1$. If $n = 1$ then T is of the form $T = H_\alpha$ as defined in exercise (12), where $\alpha \neq 0$. In particular, we have $\Delta(T) = |\det T|^{-1}$. We now assume that $n \geq 2$. From theorem (103), there exist $p \geq 1$ and $Q_1, \ldots, Q_p \in \mathcal{M}_n(\mathbf{R})$ such that:
$$T = Q_1 \circ \ldots \circ Q_p \tag{11}$$
and each Q_i is of the form H_α of exercise (12), or of the form Σ of exercise (13), or is equal to U as defined in exercise (14). From (11) we obtain $\det T = \det Q_1 \ldots \det Q_p$ and since T is a bijection, $\det T \neq 0$. It follows that $\det Q_i \neq 0$ for all $i \in \mathbf{N}_p$, and in particular that $\alpha \neq 0$ whenever Q_i is of the form $Q_i = H_\alpha$ of exercise (12). This shows that exercise (12) can be applied as much as exercise (13) and exercise (14), from which we see that $\Delta(Q_i) = |\det Q_i|^{-1}$ for all $i \in \mathbf{N}_p$. We have proved that T can be decomposed as (11), where each $Q_i : \mathbf{R}^n \to \mathbf{R}^n$ is a linear bijection satisfying $\Delta(Q_i) = |\det Q_i|^{-1}$ for all $i \in \mathbf{N}_p$.

2. Using 11. of exercise (11), we obtain:
$$\begin{aligned}
\Delta(T) &= \Delta(Q_1 \circ \ldots \circ Q_p) \\
&= \Delta(Q_1) \ldots \Delta(Q_p) \\
&= |\det Q_1|^{-1} \ldots |\det Q_p|^{-1} \\
&= |\det Q_1 \ldots \det Q_p|^{-1} \\
&= |\det(Q_1 \ldots Q_p)|^{-1} \\
&= |\det T|^{-1}
\end{aligned}$$

3. Given $n \geq 1$ and a linear bijection $T : \mathbf{R}^n \to \mathbf{R}^n$, we have proved in exercise (11) that $T(dx) = \Delta(T)dx$ for a unique constant $\Delta(T) \in \mathbf{R}^+$. However, it follows from 2. that $\Delta(T) = |\det T|^{-1}$. So $T(dx) = |\det T|^{-1}dx$, which completes the proof of theorem (108).

Exercise 15

Exercise 16. Let $f : (\mathbf{R}^2, \mathcal{B}(\mathbf{R}^2)) \to [0, +\infty]$ be a non-negative and measurable map. Let $a, b, c, d \in \mathbf{R}$ be such that $ad - bc \neq 0$. Let $T \in \mathcal{M}_2(\mathbf{R})$ be defined by:
$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
Then $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear map, and $\det T = ad - bc \neq 0$. So T is a linear bijection. Using theorem (104) with theorem (108):
$$\begin{aligned}
\int_{\mathbf{R}^2} f(ax + by, cx + dy)dxdy &= \int_{\mathbf{R}^2} f \circ T(x, y)dxdy \\
&= \int_{\mathbf{R}^2} f \circ T dx \\
&= \int_{\mathbf{R}^2} fT(dx)
\end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbf{R}^2} f(|\det T|^{-1} dx) \\
&= |\det T|^{-1} \int_{\mathbf{R}^2} f dx \\
&= |ad - bc|^{-1} \int_{\mathbf{R}^2} f(x,y) dx dy
\end{aligned}$$

where the fifth equality stems from exercise (18) of Tutorial 12.

Exercise 16

Exercise 17. Let $B \in \mathcal{B}(\mathbf{R}^n)$ and $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear bijection. From 3. of exercise (11), the direct image $T(B)$ is also the inverse image $(T^{-1})^{-1}(B)$ of B by T^{-1}. Since T^{-1} is continuous, in particular it is Borel measurable, and consequently $T(B) \in \mathcal{B}(\mathbf{R}^n)$. From $TT^{-1} = I_n$, we obtain $\det T \det T^{-1} = 1$, and it follows that $\det T^{-1} = (\det T)^{-1}$. Applying theorem (108) to T^{-1}, we obtain:

$$\begin{aligned}
dx(T(B)) &= dx((T^{-1})^{-1}(B)) \\
&= T^{-1}(dx)(B) \\
&= |\det T^{-1}|^{-1} dx(B) \\
&= |(\det T)^{-1}|^{-1} dx(B) \\
&= |\det T| dx(B)
\end{aligned}$$

Exercise 17

Exercise 18.

1. Let V be a linear subspace of $\mathbf{R}^n$, and $p = \dim V$. We assume that $1 \leq p \leq n-1$. Let $u_1, \ldots, u_p$ be an orthonormal basis of V, and $u_{p+1}, \ldots, u_n$ be such that $u_1, \ldots u_n$ is an orthonormal basis of $\mathbf{R}^n$. Note that the existence of an orthonormal basis of V, and the fact that such basis can be extended to an orthonormal basis of $\mathbf{R}^n$, has not been proved in these Tutorials. So we shall have to accept it for the time being. Given $i \in \mathbf{N}_n$, we define $\phi_i : \mathbf{R}^n \to \mathbf{R}$ by $\phi_i(x) = \langle u_i, x \rangle$ for all $x \in \mathbf{R}^n$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner-product of $\mathbf{R}^n$. From the Cauchy-Schwarz inequality (50), for all $x, y \in \mathbf{R}^n$, we have:

$$\begin{aligned}
|\phi_i(x) - \phi_i(y)| &= |\langle u_i, x \rangle - \langle u_i, y \rangle| \\
&= |\langle u_i, x - y \rangle| \\
&\leq \|u_i\| \cdot \|x - y\|
\end{aligned}$$

   So it is clear that $\phi_i : \mathbf{R}^n \to \mathbf{R}$ is continuous.

2. Let $x \in \mathbf{R}^n$. Since $u_1, \ldots, u_n$ is a basis of $\mathbf{R}^n$, there exists a unique $(\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$ such that:

$$x = \alpha_1 u_1 + \ldots + \alpha_n u_n$$

Now suppose that $x \in \cap_{j=p+1}^n \phi_j^{-1}(\{0\})$. Then for all $j \geq p+1$ we have $\phi_j(x) = 0$, i.e.:

$$\begin{aligned} 0 &= \phi_j(x) \\ &= \langle u_j, x \rangle \\ &= \langle u_j, \alpha_1 u_1 + \ldots + \alpha_n u_n \rangle \\ &= \sum_{i=1}^n \alpha_i \langle u_j, u_i \rangle \\ &= \alpha_j \langle u_j, u_j \rangle \\ &= \alpha_j \end{aligned}$$

where we have used the fact that $u_1, \ldots, u_n$ is an orthonormal basis of $\mathbf{R}^n$. Since $\alpha_j = 0$ for all $j \geq p+1$, we obtain $x = \alpha_1 u_1 + \ldots + \alpha_p u_p \in V$. This shows that $\cap_{j=p+1}^n \phi_j^{-1}(\{0\}) \subseteq V$. To show the reverse inclusion, suppose $x \in V$. Since $u_1, \ldots, u_p$ is a basis of V, there exists $\alpha_1, \ldots, \alpha_p \in \mathbf{R}$ such that $x = \alpha_1 u_1 + \ldots + \alpha_p u_p$, and since $u_1, \ldots, u_n$ is orthogonal, it is clear that $\langle u_j, x \rangle = 0$ for all $j \geq p+1$. Hence, we have $x \in \cap_{j=p+1}^n \phi_j^{-1}(\{0\})$ and we have proved that $V \subseteq \cap_{j=p+1}^n \phi_j^{-1}(\{0\})$. We conclude that $V = \cap_{j=p+1}^n \phi_j^{-1}(\{0\})$.

3. Since ϕ_j is continuous for all $j \in \mathbf{N}_n$, in particular $\phi_j^{-1}(\{0\})$ is a closed subset of $\mathbf{R}^n$ for all $j \in \mathbf{N}_n$. It follows from 2. that $V = \cap_{j=p+1}^n \phi_j^{-1}(\{0\})$ is a closed subset of $\mathbf{R}^n$.

4. Let $Q = (q_{ij}) \in \mathcal{M}_n(\mathbf{R})$ be the matrix defined by $Qe_j = u_j$ for all $j \in \mathbf{N}_n$, where $e_1, \ldots, e_n$ is the canonical basis of $\mathbf{R}^n$. For all $i, j \in \mathbf{N}_n$, we have:

$$\begin{aligned} \langle u_i, u_j \rangle &= \langle Qe_i, Qe_j \rangle \\ &= \left\langle \sum_{k=1}^n q_{ki} e_k, \sum_{l=1}^n q_{lj} e_l \right\rangle \\ &= \sum_{k=1}^n \sum_{l=1}^n q_{ki} q_{lj} \langle e_k, e_l \rangle \\ &= \sum_{k=1}^n q_{ki} q_{kj} \langle e_k, e_k \rangle \\ &= \sum_{k=1}^n q_{ki} q_{kj} \end{aligned}$$

5. Using 4. for all $i, j \in \mathbf{N}_n$, we obtain:

$$\begin{aligned} (Q^t Q)_{ij} &= \sum_{k=1}^n (Q^t)_{ik} (Q)_{kj} \\ &= \sum_{k=1}^n q_{ki} q_{kj} \end{aligned}$$

$$= \quad \langle u_i, u_j \rangle = (I_n)_{ij}$$

This being true for all $i, j \in \mathbf{N}_n$, $Q^t \cdot Q = I_n$. Accepting the fact that $\det Q^t = \det Q$, we obtain:

$$1 = \det I_n = \det Q^t \cdot Q = \det Q^t \det Q = (\det Q)^2$$

We conclude that $|\det Q| = 1$.

6. Applying theorem (108) to Q, we obtain:

$$\begin{aligned} dx(\{Q \in V\}) &= Q(dx)(V) \\ &= |\det Q|^{-1} dx(V) = dx(V) \end{aligned}$$

7. Let $\mathrm{span}(e_1, \ldots, e_p)$ denote the linear subspace of $\mathbf{R}^n$ generated by $e_1, \ldots, e_p$, i.e. the set:

$$\mathrm{span}(e_1, \ldots, e_p) = \{\alpha_1 e_1 + \ldots + \alpha_p e_p : \alpha_i \in \mathbf{R}, \forall i \in \mathbf{N}_p\}$$

We claim that $\{Q \in V\} = \mathrm{span}(e_1, \ldots, e_p)$. Let $x \in \{Q \in V\}$. Then $Q(x) \in V$. Given $j \in \{p+1, \ldots, n\}$, it follows from 2. that $\phi_j(Q(x)) = 0$, i.e.:

$$\begin{aligned} 0 &= \phi_j(Q(x)) \\ &= \langle u_j, x_1 Q e_1 + \ldots + x_n Q e_n \rangle \\ &= \langle u_j, x_1 u_1 + \ldots + x_n u_n \rangle \\ &= x_j \langle u_j, u_j \rangle = x_j \end{aligned}$$

So $x_j = 0$ for all $j \geq p+1$ and consequently:

$$x = \sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{p} x_i e_i \in \mathrm{span}(e_1, \ldots, e_p)$$

This shows the inclusion $\subseteq$. To show the reverse inclusion, suppose $x \in \mathrm{span}(e_1, \ldots, e_p)$. Then $x_j = 0$ for all $j \geq p+1$, and going back through the preceding calculation, it is clear that $\phi_j(Q(x)) = 0$ for all $j \geq p+1$. So $Q(x) \in \cap_{j=p+1}^{n} \phi_j^{-1}(\{0\}) = V$, i.e. $x \in \{Q \in V\}$. This shows the inclusion $\supseteq$, and we have proved that $\{Q \in V\} = \mathrm{span}(e_1, \ldots, e_p)$.

8. Let $m \geq 1$ be an integer. We define:

$$E_m \stackrel{\triangle}{=} \overbrace{[-m, m] \times \ldots \times [-m, m]}^{n-1} \times \{0\}$$

It is clear from definition (63) that $dx(E_m) = 0$ for all $m \geq 1$.

9. Since $E_m \uparrow \mathrm{span}(e_1, \ldots, e_{n-1})$, i.e. $E_m \subseteq E_{m+1}$ for all $m \geq 1$ and $\cup_{m \geq 1} E_m = \mathrm{span}(e_1, \ldots, e_{n-1})$, from theorem (7) we obtain:

$$dx(\mathrm{span}(e_1, \ldots, e_{n-1})) = \lim_{m \to +\infty} dx(E_m) = 0$$

10. Using 6. and 7. together with 9. we have:

$$\begin{aligned} dx(V) &= dx(\{Q \in V\}) = dx(\mathrm{span}(e_1, \ldots, e_p)) \\ &\leq dx(\mathrm{span}(e_1, \ldots, e_{n-1})) = 0 \end{aligned}$$

This completes the proof of theorem (109) in the case when $1 \leq \dim V \leq n-1$. The case $\dim V = 0$, i.e. $V = \{0\}$ is clear.

Exercise 18