8. Jensen inequality

Definition 64 *Let $a, b \in \bar{\mathbf{R}}$, with $a < b$. Let $\phi :]a, b[\to \mathbf{R}$ be an $\mathbf{R}$-valued function. We say that ϕ is a* **convex function***, if and only if, for all $x, y \in]a, b[$ and $t \in [0, 1]$, we have:*

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$

Exercise 1. Let $a, b \in \bar{\mathbf{R}}$, with $a < b$. Let $\phi :]a, b[\to \mathbf{R}$ be a map.

1. Show that $\phi :]a, b[\to \mathbf{R}$ is convex, if and only if for all $x_1, \ldots, x_n$ in $]a, b[$ and $\alpha_1, \ldots, \alpha_n$ in $\mathbf{R}^+$ with $\alpha_1 + \ldots + \alpha_n = 1$, $n \geq 1$, we have:
$$\phi(\alpha_1 x_1 + \ldots + \alpha_n x_n) \leq \alpha_1 \phi(x_1) + \ldots \alpha_n \phi(x_n)$$
2. Show that $\phi :]a, b[\to \mathbf{R}$ is convex, if and only if for all x, y, z with $a < x < y < z < b$ we have:
$$\phi(y) \leq \frac{z-y}{z-x}\phi(x) + \frac{y-x}{z-x}\phi(z)$$
3. Show that $\phi :]a, b[\to \mathbf{R}$ is convex if and only if for all x, y, z with $a < x < y < z < b$, we have:
$$\frac{\phi(y) - \phi(x)}{y - x} \leq \frac{\phi(z) - \phi(y)}{z - y}$$
4. Let $\phi :]a, b[\to \mathbf{R}$ be convex. Let $x_0 \in]a, b[$, and $u, u', v, v' \in]a, b[$ be such that $u < u' < x_0 < v < v'$. Show that for all $x \in]x_0, v[$:
$$\frac{\phi(u') - \phi(u)}{u' - u} \leq \frac{\phi(x) - \phi(x_0)}{x - x_0} \leq \frac{\phi(v') - \phi(v)}{v' - v}$$
and deduce that $\lim_{x \downarrow\downarrow x_0} \phi(x) = \phi(x_0)$
5. Show that if $\phi :]a, b[\to \mathbf{R}$ is convex, then ϕ is continuous.
6. Define $\phi : [0, 1] \to \mathbf{R}$ by $\phi(0) = 1$ and $\phi(x) = 0$ for all $x \in]0, 1]$. Show that $\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$, $\forall x, y, t \in [0, 1]$, but that ϕ fails to be continuous on $[0, 1]$.

Definition 65 *Let $(\Omega, \mathcal{T})$ be a topological space. We say that $(\Omega, \mathcal{T})$ is a* **compact topological space** *if and only if, for all family $(V_i)_{i \in I}$ of open sets in Ω, such that $\Omega = \cup_{i \in I} V_i$, there exists a finite subset $\{i_1, \ldots, i_n\}$ of I such that $\Omega = V_{i_1} \cup \ldots \cup V_{i_n}$.*

In short, we say that $(\Omega, \mathcal{T})$ is compact if and only if, from any open covering of Ω, one can extract a finite sub-covering.

Definition 66 *Let $(\Omega, \mathcal{T})$ be a topological space, and $K \subseteq \Omega$. We say that K is a* **compact subset** *of Ω, if and only if the induced topological space $(K, \mathcal{T}_{|K})$ is a compact topological space.*

EXERCISE 2. Let $(\Omega, \mathcal{T})$ be a topological space.

1. Show that if $(\Omega, \mathcal{T})$ is compact, it is a compact subset of itself.
2. Show that $\emptyset$ is a compact subset of Ω.
3. Show that if $\Omega' \subseteq \Omega$ and K is a compact subset of Ω', then K is also a compact subset of Ω.
4. Show that if $(V_i)_{i\in I}$ is a family of open sets in Ω such that $K \subseteq \cup_{i\in I} V_i$, then $K = \cup_{i\in I}(V_i \cap K)$ and $V_i \cap K$ is open in K for all $i \in I$.
5. Show that $K \subseteq \Omega$ is a compact subset of Ω, if and only if for any family $(V_i)_{i\in I}$ of open sets in Ω such that $K \subseteq \cup_{i\in I} V_i$, there is a finite subset $\{i_1, \ldots, i_n\}$ of I such that $K \subseteq V_{i_1} \cup \ldots \cup V_{i_n}$.
6. Show that if $(\Omega, \mathcal{T})$ is compact and K is closed in Ω, then K is a compact subset of Ω.

EXERCISE 3. Let $a, b \in \mathbf{R}$, $a < b$. Let $(V_i)_{i\in I}$ be a family of open sets in $\mathbf{R}$ such that $[a, b] \subseteq \cup_{i\in I} V_i$. We define A as the set of all $x \in [a, b]$ such that $[a, x]$ can be covered by a finite number of V_i's. Let $c = \sup A$.

1. Show that $a \in A$.
2. Show that there is $\epsilon > 0$ such that $a + \epsilon \in A$.
3. Show that $a < c \leq b$.
4. Show the existence of $i_0 \in I$ and c', c'' with $a < c' < c < c''$, such that $]c', c''] \subseteq V_{i_0}$.
5. Show that $[a, c']$ can be covered by a finite number of V_i's.
6. Show that $[a, c'']$ can be covered by a finite number of V_i's.
7. Show that $b \wedge c'' \leq c$ and conclude that $c = b$.
8. Show that $[a, b]$ is a compact subset of $\mathbf{R}$.

Theorem 34 *Let $a, b \in \mathbf{R}$, $a < b$. The closed interval $[a, b]$ is a compact subset of $\mathbf{R}$.*

Definition 67 *Let $(\Omega, \mathcal{T})$ be a topological space. We say that $(\Omega, \mathcal{T})$ is a* **Hausdorff topological space***, if and only if for all $x, y \in \Omega$ with $x \neq y$, there exists open sets U and V in Ω, such that:*

$$x \in U \ , \ y \in V \ , \ U \cap V = \emptyset$$

EXERCISE 4. Let $(\Omega, \mathcal{T})$ be a topological space.

1. Show that if $(\Omega, \mathcal{T})$ is Hausdorff and $\Omega' \subseteq \Omega$, then the induced topological space $(\Omega', \mathcal{T}_{|\Omega'})$ is itself Hausdorff.

2. Show that if $(\Omega, \mathcal{T})$ is metrizable, then it is Hausdorff.

3. Show that any subset of $\bar{\mathbf{R}}$ is Hausdorff.

4. Let $(\Omega_i, \mathcal{T}_i)_{i\in I}$ be a family of Hausdorff topological spaces. Show that the product topological space $\Pi_{i\in I}\Omega_i$ is Hausdorff.

Exercise 5. Let $(\Omega, \mathcal{T})$ be a Hausdorff topological space. Let K be a compact subset of Ω and suppose there exists $y \in K^c$.

1. Show that for all $x \in K$, there are open sets V_x, W_x in Ω, such that $y \in V_x, x \in W_x$ and $V_x \cap W_x = \emptyset$.

2. Show that there exists a finite subset $\{x_1, \ldots, x_n\}$ of K such that $K \subseteq W^y$ where $W^y = W_{x_1} \cup \ldots \cup W_{x_n}$.

3. Let $V^y = V_{x_1} \cap \ldots \cap V_{x_n}$. Show that V^y is open and $V^y \cap W^y = \emptyset$.

4. Show that $y \in V^y \subseteq K^c$.

5. Show that $K^c = \cup_{y\in K^c} V^y$

6. Show that K is closed in Ω.

Theorem 35 *Let $(\Omega, \mathcal{T})$ be a Hausdorff topological space. For all $K \subseteq \Omega$, if K is a compact subset, then it is closed.*

Definition 68 *Let (E, d) be a metric space. For all $A \subseteq E$, we call* **diameter** *of A with respect to d, the element of $\bar{\mathbf{R}}$ denoted $\delta(A)$, defined as $\delta(A) = \sup\{d(x, y) \ : \ x, y \in A\}$, with the convention that $\delta(\emptyset) = -\infty$.*

Definition 69 *Let (E, d) be a metric space, and $A \subseteq E$. We say that A is* **bounded***, if and only if $\delta(A) < +\infty$.*

Exercise 6. Let (E, d) be a metric space. Let $A \subseteq E$.

1. Show that $\delta(A) = 0$ if and only if $A = \{x\}$ for some $x \in E$.

2. Let $\phi : \mathbf{R} \to]-1, 1[$ be an increasing homeomorphism. Define $d''(x, y) = |x - y|$ and $d'(x, y) = |\phi(x) - \phi(y)|$, for all $x, y \in \mathbf{R}$. Show that d' is a metric on $\mathbf{R}$ inducing the usual topology on $\mathbf{R}$. Show that $\mathbf{R}$ is bounded with respect to d' but not with respect to d''.

3. Show that if $K \subseteq E$ is a compact subset of E, for all $\epsilon > 0$, there is a finite subset $\{x_1, \ldots, x_n\}$ of K such that:
$$K \subseteq B(x_1, \epsilon) \cup \ldots \cup B(x_n, \epsilon)$$

4. Show that any compact subset of any metrizable topological space $(\Omega, \mathcal{T})$, is bounded with respect to any metric inducing the topology $\mathcal{T}$.

Exercise 7. Suppose K is a closed subset of $\mathbf{R}$ which is bounded with respect to the usual metric on $\mathbf{R}$.

1. Show that there exists $M \in \mathbf{R}^+$ such that $K \subseteq [-M, M]$.
2. Show that K is also closed in $[-M, M]$.
3. Show that K is a compact subset of $[-M, M]$.
4. Show that K is a compact subset of $\mathbf{R}$.
5. Show that any compact subset of $\mathbf{R}$ is closed and bounded.
6. Show the following:

Theorem 36 *A subset of* $\mathbf{R}$ *is compact if and only if it is closed, and bounded with respect to the usual metric on* $\mathbf{R}$*.*

Exercise 8. Let $(\Omega, \mathcal{T})$ and $(S, \mathcal{T}_S)$ be two topological spaces. Let $f : (\Omega, \mathcal{T}) \to (S, \mathcal{T}_S)$ be a continuous map.

1. Show that if $(W_i)_{i \in I}$ is an open covering of $f(\Omega)$, then the family $(f^{-1}(W_i))_{i \in I}$ is an open covering of Ω.
2. Show that if $(\Omega, \mathcal{T})$ is a compact topological space, then $f(\Omega)$ is a compact subset of $(S, \mathcal{T}_S)$.

Exercise 9.

1. Show that $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is a compact topological space.
2. Show that any compact subset of $\mathbf{R}$ is a compact subset of $\bar{\mathbf{R}}$.
3. Show that a subset of $\bar{\mathbf{R}}$ is compact if and only if it is closed.
4. Let A be a non-empty subset of $\bar{\mathbf{R}}$, and let $\alpha = \sup A$. Show that if $\alpha \neq -\infty$, then for all $U \in \mathcal{T}_{\bar{\mathbf{R}}}$ with $\alpha \in U$, there exists $\beta \in \mathbf{R}$ with $\beta < \alpha$ and $]\beta, \alpha] \subseteq U$. Conclude that $\alpha \in \bar{A}$.
5. Show that if A is a non-empty closed subset of $\bar{\mathbf{R}}$, then we have $\sup A \in A$ and $\inf A \in A$.
6. Consider $A = \{x \in \mathbf{R} \ , \ \sin(x) = 0\}$. Show that A is closed in $\mathbf{R}$, but that $\sup A \notin A$ and $\inf A \notin A$.
7. Show that if A is a non-empty, closed and bounded subset of $\mathbf{R}$, then $\sup A \in A$ and $\inf A \in A$.

Exercise 10. Let $(\Omega, \mathcal{T})$ be a compact, non-empty topological space. Let $f : (\Omega, \mathcal{T}) \to (\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ be a continuous map.

1. Show that if $f(\Omega) \subseteq \mathbf{R}$, the continuity of f with respect to $\mathcal{T}_{\bar{\mathbf{R}}}$ is equivalent to the continuity of f with respect to $\mathcal{T}_{\mathbf{R}}$.

2. Show the following:

Theorem 37 *Let $f : (\Omega, \mathcal{T}) \to (\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ be a continuous map, where $(\Omega, \mathcal{T})$ is a non-empty topological space. Then, if $(\Omega, \mathcal{T})$ is compact, f attains its maximum and minimum, i.e. there exist $x_m, x_M \in \Omega$, such that:*

$$f(x_m) = \inf_{x \in \Omega} f(x) \ , \ f(x_M) = \sup_{x \in \Omega} f(x)$$

Exercise 11. Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$, and differentiable on $]a, b[$, with $f(a) = f(b)$.

1. Show that if $c \in]a, b[$ and $f(c) = \sup_{x \in [a,b]} f(x)$, then $f'(c) = 0$.

2. Show the following:

Theorem 38 (Rolle) *Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$, and differentiable on $]a, b[$, with $f(a) = f(b)$. Then, there exists $c \in]a, b[$ such that $f'(c) = 0$.*

Exercise 12. Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on $]a, b[$. Define:

$$h(x) \stackrel{\triangle}{=} f(x) - (x - a)\frac{f(b) - f(a)}{b - a}$$

1. Show that h is continuous on $[a, b]$ and differentiable on $]a, b[$.

2. Show the existence of $c \in]a, b[$ such that:

$$f(b) - f(a) = (b - a)f'(c)$$

Exercise 13. Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a, b] \to \mathbf{R}$ be a map. Let $n \geq 0$. We assume that f is of class C^n on $[a, b]$, and that $f^{(n+1)}$ exists on $]a, b[$. Define:

$$h(x) \stackrel{\triangle}{=} f(b) - f(x) - \sum_{k=1}^{n} \frac{(b - x)^k}{k!} f^{(k)}(x) - \alpha \frac{(b - x)^{n+1}}{(n + 1)!}$$

where α is chosen such that $h(a) = 0$.

1. Show that h is continuous on $[a, b]$ and differentiable on $]a, b[$.

2. Show that for all $x \in]a,b[$:

$$h'(x) = \frac{(b-x)^n}{n!}(\alpha - f^{(n+1)}(x))$$

3. Prove the following:

Theorem 39 (Taylor-Lagrange) *Let $a, b \in \mathbf{R}$, $a < b$, and $n \geq 0$. Let $f : [a,b] \to \mathbf{R}$ be a map of class C^n on $[a,b]$ such that $f^{(n+1)}$ exists on $]a,b[$. Then, there exists $c \in]a,b[$ such that:*

$$f(b) - f(a) = \sum_{k=1}^{n} \frac{(b-a)^k}{k!} f^{(k)}(a) + \frac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

Exercise 14. Let $a, b \in \bar{\mathbf{R}}$, $a < b$ and $\phi :]a,b[\to \mathbf{R}$ be differentiable.

1. Show that if ϕ is convex, then for all $x, y \in]a,b[$, $x < y$, we have:

$$\phi'(x) \leq \phi'(y)$$

2. Show that if $x, y, z \in]a,b[$ with $x < y < z$, there are $c_1, c_2 \in]a,b[$, with $c_1 < c_2$ and:

$$\begin{array}{rcl} \phi(y) - \phi(x) & = & \phi'(c_1)(y-x) \\ \phi(z) - \phi(y) & = & \phi'(c_2)(z-y) \end{array}$$

3. Show conversely that if ϕ' is non-decreasing, then ϕ is convex.

4. Show that $x \to e^x$ is convex on $\mathbf{R}$.

5. Show that $x \to -\ln(x)$ is convex on $]0, +\infty[$.

Definition 70 *we say that a finite measure space $(\Omega, \mathcal{F}, P)$ is a* **probability space***, if and only if $P(\Omega) = 1$.*

Definition 71 *Let $(\Omega, \mathcal{F}, P)$ be a probability space, and (S, Σ) be a measurable space. We call* **random variable** *w.r. to (S, Σ), any measurable map $X : (\Omega, \mathcal{F}) \to (S, \Sigma)$.*

Definition 72 *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let X be a non-negative random variable, or an element of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, P)$. We call* **expectation** *of X, denoted $E[X]$, the integral:*

$$E[X] \stackrel{\triangle}{=} \int_{\Omega} X dP$$

Exercise 15. Let $a, b \in \bar{\mathbf{R}}$, $a < b$ and $\phi :]a,b[\to \mathbf{R}$ be a convex map. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$ be such that $X(\Omega) \subseteq]a,b[$.

1. Show that $\phi \circ X : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is measurable.

2. Show that $\phi \circ X \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$, if and only if $E[|\phi \circ X|] < +\infty$.

3. Show that if $E[X] = a$, then $a \in \mathbf{R}$ and $X = a$ P-a.s.

4. Show that if $E[X] = b$, then $b \in \mathbf{R}$ and $X = b$ P-a.s.

5. Let $m = E[X]$. Show that $m \in]a, b[$.

6. Define:
$$\beta \stackrel{\triangle}{=} \sup_{x \in]a,m[} \frac{\phi(m) - \phi(x)}{m - x}$$
Show that $\beta \in \mathbf{R}$ and that for all $z \in]m, b[$, we have:
$$\beta \leq \frac{\phi(z) - \phi(m)}{z - m}$$

7. Show that for all $x \in]a, b[$, we have $\phi(m) + \beta(x - m) \leq \phi(x)$.

8. Show that for all $\omega \in \Omega$, $\phi(m) + \beta(X(\omega) - m) \leq \phi(X(\omega))$.

9. Show that if $\phi \circ X \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$ then $\phi(m) \leq E[\phi \circ X]$.

Theorem 40 (Jensen inequality) *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $a, b \in \bar{\mathbf{R}}$, $a < b$ and $\phi :]a, b[\to \mathbf{R}$ be a convex map. Suppose that $X \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$ is such that $X(\Omega) \subseteq]a, b[$ and such that $\phi \circ X \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$. Then:*
$$\phi(E[X]) \leq E[\phi \circ X]$$

Solutions to Exercises

Exercise 1.

1. Let $\phi :]a,b[\to \mathbf{R}$ be convex. Given $n \geq 1$, let H_n be the property that for all $x_1, \ldots, x_n$ in $]a,b[$, and $\alpha_1, \ldots, \alpha_n$ in $\mathbf{R}^+$ such that $\alpha_1 + \ldots + \alpha_n = 1$, we have:
$$\phi(\alpha_1 x_1 + \ldots + \alpha_n x_n) \leq \alpha_1 \phi(x_1) + \ldots + \alpha_n \phi(x_n) \tag{1}$$
H_1 is obviously true. Since ϕ is convex, H_2 is also true. Given $n \geq 3$, suppose that H_{n-1} has been proved. Let $x_1, \ldots, x_n$ in $]a,b[$ and $\alpha_1, \ldots, \alpha_n$ in $\mathbf{R}^+$ be such that $\alpha_1 + \ldots + \alpha_n = 1$. Define $t = \alpha_1 + \ldots + \alpha_{n-1}$. If $t = 0$, then $\alpha_i = 0$ for all $i \in \{1, \ldots, n-1\}$, and $\alpha_n = 1$. So (1) is clearly satisfied. Suppose $t \neq 0$. From our induction hypothesis H_{n-1}, we obtain:
$$\phi((\alpha_1 x_1 + \ldots + \alpha_{n-1} x_{n-1})/t) \leq (\alpha_1 \phi(x_1) + \ldots + \alpha_{n-1} \phi(x_{n-1}))/t$$
i.e. $t\phi(x) \leq \alpha_1 \phi(x_1) + \ldots + \alpha_{n-1}\phi(x_{n-1})$, where x has been defined as $x = (\alpha_1 x_1 + \ldots + \alpha_{n-1} x_{n-1})/t$. Note that x is an element of $]a,b[$. Let $y = x_n$. Since by assumption, ϕ is convex and $t \in [0,1]$, we have:
$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$
and thus:
$$\phi(tx + (1-t)y) \leq \alpha_1 \phi(x_1) + \ldots + \alpha_{n-1}\phi(x_{n-1}) + (1-t)\phi(y)$$
Since $1 - t = \alpha_n$, we see that (1) is therefore satisfied, which proves that H_n is true. This induction argument shows that H_n is true for all $n \geq 1$, whenever ϕ is convex. Conversely, if H_n is true for all $n \geq 1$, then in particular H_2 is true, and ϕ is immediately convex.

2. Let $\phi :]a,b[\to \mathbf{R}$ be convex, and x, y, z with $a < x < y < z < b$. Let $t = (z-y)/(z-x)$. Then $t \in]0,1[$ and $1 - t = (y-x)/(z-x)$. Moreover, we have $y = tx + (1-t)z$. ϕ being convex, we obtain:
$$\phi(y) \leq \frac{z-y}{z-x}\phi(x) + \frac{y-x}{z-x}\phi(z) \tag{2}$$
Conversely, suppose $\phi :]a,b[\to \mathbf{R}$ is a map such that (2) holds for all x, y, z with $a < x < y < z < b$. Let $x, z \in]a,b[$ and $t \in [0,1]$. Without loss of generality, we can assume that $x \leq z$. If $t = 0$, $t = 1$, or $x = z$, then we immediately have:
$$\phi(tx + (1-t)z) \leq t\phi(x) + (1-t)\phi(z) \tag{3}$$
Assume that $x < z$ and $t \in]0,1[$. Define $y = tx + (1-t)z$. Then, $x < y < z$. Moreover, it is easy to check that $(z-y)/(z-x) = t$ and $(y-x)/(z-x) = 1-t$. From (2), we conclude that (3) is also satisfied. Hence, we see that ϕ is convex. We have proved that a map $\phi :]a,b[\to \mathbf{R}$ is convex, if and only if inequality (2) holds, whenever $a < x < y < z < b$.

3. From the previous question, $\phi :]a,b[\to \mathbf{R}$ is convex, if and only if for all x,y,z with $a < x < y < z < b$, we have:
$$\phi(y) \leq \frac{z-y}{z-x}\phi(x) + \frac{y-x}{z-x}\phi(z)$$
which is equivalent to:
$$\frac{\phi(y)-\phi(x)}{y-x} \leq \frac{\phi(z)-\phi(y)}{z-y} \tag{4}$$

4. Let $\phi :]a,b[\to \mathbf{R}$ be convex. Let $x_0 \in]a,b[$ and u,u',v,v' in $]a,b[$ such that $u < u' < x_0 < v < v'$. Let $x \in]x_0, v[$. Using inequality (4), we obtain:
$$\frac{\phi(u')-\phi(u)}{u'-u} \leq \frac{\phi(x_0)-\phi(u')}{x_0-u'} \leq \frac{\phi(x)-\phi(x_0)}{x-x_0}$$
and furthermore:
$$\frac{\phi(x)-\phi(x_0)}{x-x_0} \leq \frac{\phi(v)-\phi(x)}{v-x} \leq \frac{\phi(v')-\phi(v)}{v'-v}$$
So, in particular:
$$\frac{\phi(u')-\phi(u)}{u'-u} \leq \frac{\phi(x)-\phi(x_0)}{x-x_0} \leq \frac{\phi(v')-\phi(v)}{v'-v}$$
It follows that there exist $\alpha, \beta \in \mathbf{R}$, such that for all $x \in]x_0, v[$:
$$\alpha(x-x_0) \leq \phi(x) - \phi(x_0) \leq \beta(x-x_0)$$
We conclude that the right-hand limit, $\lim_{x\downarrow\downarrow x_0} \phi(x)$ exists, and is equal to $\phi(x_0)$.

5. Similarly to 4., for all $x \in]u', x_0[$, we have:
$$\frac{\phi(u')-\phi(u)}{u'-u} \leq \frac{\phi(x_0)-\phi(x)}{x_0-x} \leq \frac{\phi(v')-\phi(v)}{v'-v}$$
So there exist $\alpha, \beta \in \mathbf{R}$, such that for all $x \in]u', x_0[$:
$$\alpha(x_0-x) \leq \phi(x_0) - \phi(x) \leq \beta(x_0-x)$$
We conclude that the left-hand limit, $\lim_{x\uparrow\uparrow x_0} \phi(x)$ exists, and is equal to $\phi(x_0)$. Finally, from:
$$\lim_{x\downarrow\downarrow x_0} \phi(x) = \phi(x_0) = \lim_{x\uparrow\uparrow x_0} \phi(x)$$
ϕ is continuous on x_0. This being true for all $x_0 \in]a,b[$, we have proved that $\phi :]a,b[\to \mathbf{R}$ is a continuous map.

6. Let $\phi : [0,1] \to \mathbf{R}$ be defined by $\phi(0) = 1$, and $\phi(x) = 0$ for all $x \in]0,1]$. The fact that:
$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$

for all $t, x, y \in [0, 1]$, is clear. Yet, ϕ obviously fails to be continuous on $[0, 1]$. The purpose of this question is to emphasize an important point: in definition (64), we have restricted a convex function to be defined on some open interval $]a, b[$ (it needs to be an interval, as $\phi(tx + (1-t)y)$ needs to be meaningful). If instead, we had allowed a convex function to be defined on some closed interval $[a, b]$, it would not necessarily be continuous.

Exercise 1

Exercise 2.

1. Let $(\Omega, \mathcal{T})$ be a compact topological space. The induced topological space $(\Omega, \mathcal{T}_{|\Omega})$ is nothing but $(\Omega, \mathcal{T})$ itself. So $(\Omega, \mathcal{T}_{|\Omega})$ is compact, and Ω is therefore a compact subset of itself.

2. The induced topology $\mathcal{T}_{|\emptyset}$ is defined by $\mathcal{T}_{|\emptyset} = \{A \cap \emptyset \ : \ A \in \mathcal{T}\}$. So $\mathcal{T}_{|\emptyset} = \{\emptyset\}$. The topological space $(\emptyset, \{\emptyset\})$ being compact, we see that $\emptyset$ is a compact subset of Ω.

3. Let $(\Omega, \mathcal{T})$ be a topological space and $\Omega' \subseteq \Omega$. Let K be a compact subset of Ω'. Then $K \subseteq \Omega'$, and the topological space $(K, (\mathcal{T}_{|\Omega'})_{|K})$ is compact. However, the induced topology $(\mathcal{T}_{|\Omega'})_{|K}$ coincide with the induced topology $\mathcal{T}_{|K}$. It follows that $(K, \mathcal{T}_{|K})$ is a compact topological space, and K is therefore a compact subset of Ω.

4. Let $(V_i)_{i\in I}$ be a family of open sets in Ω, such that $K \subseteq \cup_{i\in I} V_i$. If $x \in K$, then $x \in V_i \cap K$ for some $i \in I$. Conversely, if $x \in V_i \cap K$ for some $i \in I$, then $x \in K$. So $K = \cup_{i\in I} V_i \cap K$. By definition (23) of the induced topology, each $V_i \cap K$ is an element of $\mathcal{T}_{|K}$, i.e. each $V_i \cap K$ is open in K.

5. Let $(\Omega, \mathcal{T})$ be a topological space, and $K \subseteq \Omega$. Suppose K is a compact subset of Ω. Let $(V_i)_{i\in I}$ be a family of open sets in Ω, such that $K \subseteq \cup_{i\in I} V_i$. From 4., $K = \cup_{i\in I} V_i \cap K$, and each $V_i \cap K$ is an open set in K. By assumption, the topological space $(K, \mathcal{T}_{|K})$ is compact. From definition (65), it follows that there exists $\{i_1, \ldots, i_n\}$ finite subset of I, such that:

   $$K = (V_{i_1} \cap K) \cup \ldots \cup (V_{i_n} \cap K) = (V_{i_1} \cup \ldots \cup V_{i_n}) \cap K$$

   In particular, $K \subseteq V_{i_1} \cup \ldots \cup V_{i_n}$. Conversely, suppose that $K \subseteq \Omega$ has the property that for any family $(V_i)_{i\in I}$ of open sets in Ω, such that $K \subseteq \cup_{i\in I} V_i$, there exists $\{i_1, \ldots, i_n\}$ finite subset of I such that $K \subseteq V_{i_1} \cup \ldots \cup V_{i_n}$. We claim that K is a compact subset of Ω. Indeed, let $(W_i)_{i\in I}$ be a family of open sets in K such that $K = \cup_{i\in I} W_i$. Since each W_i lies in $\mathcal{T}_{|K}$, for all $i \in I$, there exists $V_i \in \mathcal{T}$ such that $W_i = V_i \cap K$. So $K = \cup_{i\in I} V_i \cap K$, and in particular $K \subseteq \cup_{i\in I} V_i$. By assumption, there exists $\{i_1, \ldots, i_n\}$ finite subset of I, such that $K \subseteq V_{i_1} \cup \ldots \cup V_{i_n}$, and therefore $K = (V_{i_1} \cup \ldots \cup V_{i_n}) \cap K = W_{i_1} \cup \ldots \cup W_{i_n}$. From definition (65), we conclude that $(K, \mathcal{T}_{|K})$ is compact, i.e. K is a compact subset of Ω.

We have proved that $K \subseteq \Omega$ is a compact subset of Ω, if and only if for any family $(V_i)_{i\in I}$ of open sets in Ω such that $K \subseteq \cup_{i\in I} V_i$, there exists $\{i_1,\ldots,i_n\}$ finite subset of I, such that $K \subseteq V_{i_1} \cup \ldots \cup V_{i_n}$.

6. Let $(\Omega, \mathcal{T})$ be a compact topological space. Let $K \subseteq \Omega$, and suppose that K is closed in Ω. Let $(V_i)_{i\in I}$ be a family of open sets in Ω, such that $K \subseteq \cup_{i\in I} V_i$. For all $x \in \Omega$, either $x \in K^c$ or $x \in V_i$ for some $i \in I$ (or both). So $\Omega = (\cup_{i\in I} V_i) \cup K^c$. Since K^c is assumed to be open in Ω, and $(\Omega, \mathcal{T})$ is compact, from definition (65), there exists $\{i_1,\ldots,i_n\}$ finite subset of I, such that $\Omega = V_{i_1} \cup \ldots \cup V_{i_n}$, or $\Omega = (V_{i_1} \cup \ldots \cup V_{i_n}) \cup K^c$. In any case, we have $K \subseteq V_{i_1} \cup \ldots \cup V_{i_n}$. Hence, given a family $(V_i)_{i\in I}$ of open sets in Ω, such that $K \subseteq \cup_{i\in I} V_i$, we have found a finite subset $\{i_1,\ldots,i_n\}$ of I, such that $K \subseteq V_{i_1} \cup \ldots \cup V_{i_n}$. From 5., we conclude that K is a compact subset of Ω. We have proved that any closed subset of a compact topological space, is itself compact (is a compact subset of it).

Exercise 2

Exercise 3.

1. By assumption, $[a,b] \subseteq \cup_{i\in I} V_i$ and in particular, there exists $i \in I$ such that $a \in V_i$. So $\{a\} = [a,a]$ can be covered by a finite number of V_i's. We have proved that $a \in A$.

2. Since $a \in V_i$ for some i, and V_i is open in $\mathbf{R}$, there exists $\epsilon > 0$ such that $[a, a+\epsilon] \subseteq V_i$. Since $a < b$, by choosing ϵ small enough, we can ensure that $a+\epsilon \in [a,b]$. Hence, we have found $\epsilon > 0$, such that $a+\epsilon \in [a,b]$, and $[a,a+\epsilon]$ is covered by a finite number of V_i's. So we have found $\epsilon > 0$, such that $a+\epsilon \in A$.

3. Since $c = \sup A$, c is an upper-bound of A. From 2., there exists $\epsilon > 0$, such that $a+\epsilon \in A$. So $a+\epsilon \leq c$ and in particular, $a < c$. By definition, A is a subset of $[a,b]$. So b is an upper-bound of A. c being the smallest of such upper-bounds, we have $c \leq b$. We have proved that $a < c \leq b$.

4. From 3., $c \in]a,b] \subseteq \cup_{i\in I} V_i$. There exists $i_0 \in I$ with $c \in V_{i_0}$. V_{i_0} being open in $\mathbf{R}$, there exist c', c'' such that $c' < c < c''$ and $]c', c''] \subseteq V_{i_0}$. Moreover, since $a < c$, it is possible to choose c' such that $a < c'$. We have proved the existence of $i_0 \in I$ and c', c'', with $a < c' < c < c''$ and $]c', c''] \subseteq V_{i_0}$.

5. Since $c' < c$ and c is the smallest of all upper-bounds of A, c' cannot be such upper-bound. There exists $x \in A$, such that $c' < x$. Since $x \in A$, $[a,x]$ can be covered by a finite number of V_i's. From $[a,c'] \subseteq [a,x]$, we conclude that $[a,c']$ can also be covered by a finite number of V_i's.

6. From $[a,c''] = [a,c'] \cup]c',c'']$, $]c',c''] \subseteq V_{i_0}$ and the fact that $[a,c']$ can be covered by a finite number of V_i's, we conclude that $[a,c'']$ can also be covered by a finite number of V_i's.

7. Since $[a, b \wedge c''] \subseteq [a, c'']$, it follows from 6. that $[a, b \wedge c'']$ can be covered by a finite number of V_i's. Moreover, since $b \wedge c'' \in [a, b]$, we see that $b \wedge c'' \in A$. Hence, we have $b \wedge c'' \leq c$. We know from 3. that $c \leq b$. Suppose we had $c < b$. Since $c < c''$, this would imply that $c < b \wedge c''$, which is a contradiction. It follows that $b = c$.

8. From 7., we have $[a, b] = [a, c] \subseteq [a, c'']$. From 6., $[a, c'']$ can be covered by a finite number of V_i's. It follows that $[a, b]$ can also be covered by a finite number of V_i's. In other words, there exists a finite subset $\{i_1, \ldots, i_n\}$ of I, such that $[a, b] \subseteq V_{i_1} \cup \ldots \cup V_{i_n}$. Having assumed that $[a, b] \subseteq \cup_{i \in I} V_i$, for an arbitrary family $(V_i)_{i \in I}$ of open sets in $\mathbf{R}$, we have shown the existence of a finite subset $\{i_1, \ldots, i_n\}$ of I, such that $[a, b] \subseteq V_{i_1} \cup \ldots \cup V_{i_n}$. From exercise (2), we see that $[a, b]$ is a compact subset of $\mathbf{R}$.

Exercise 3

Exercise 4.

1. Let $(\Omega, \mathcal{T})$ be a Hausdorff topological space, and $\Omega' \subseteq \Omega$. Let $x, y \in \Omega'$ with $x \neq y$. In particular, $x, y \in \Omega$ with $x \neq y$. Since $(\Omega, \mathcal{T})$ is Hausdorff, there exist two open sets U, V in Ω, such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Define $U' = U \cap \Omega'$ and $V' = V \cap \Omega'$. Then U' and V' are elements of the induced topology $\mathcal{T}_{|\Omega'}$ and furthermore, we have $x \in U'$, $y \in V'$ and $U' \cap V' = \emptyset$. Given two distinct elements x, y of Ω', we have found two disjoint open sets U', V' in Ω', containing x and y respectively. This shows that the induced topological space $(\Omega', \mathcal{T}_{|\Omega'})$ is Hausdorff.

2. Let $(\Omega, \mathcal{T})$ be a metrizable topological space. Let d be a metric on Ω, inducing the topology $\mathcal{T}$ on Ω. Let $x, y \in \Omega$ with $x \neq y$. Define $\epsilon = d(x, y)/2 > 0$, $U = B(x, \epsilon)$ and $V = B(y, \epsilon)$. Then, U, V are open sets in Ω, with $x \in U$ and $y \in V$. Furthermore, if $z \in B(x, \epsilon)$, then $d(x, z) < d(x, y)/2$ and consequently:
$$d(x, y) \leq d(x, z) + d(z, y) < d(x, y)/2 + d(z, y)$$
from which we see that $d(z, y) > d(x, y)/2 = \epsilon$. So $z \notin B(y, \epsilon)$, and we have proved that $U \cap V = \emptyset$. Given two distinct elements x, y of Ω, we have found two disjoint open sets U, V in Ω, containing x and y respectively. This shows that the metrizable topological space $(\Omega, \mathcal{T})$ is Hausdorff.

3. From theorem (13), the topological space $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is metrizable. It follows from 2. that $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is Hausdorff. From 1., any subset of $\bar{\mathbf{R}}$ (together with its induced topology) is a Hausdorff topological space.

4. Let $(\Omega_i, \mathcal{T}_i)_{i \in I}$ be a family of Hausdorff topological spaces. Let $\Omega = \Pi_{i \in I} \Omega_i$ and $\mathcal{T} = \odot_{i \in I} \mathcal{T}_i$ be the product topology on Ω [definition (56)]. Let $x, y \in \Omega$ with $x \neq y$. There exists $i_0 \in I$ such that $x(i_0) \neq y(i_0)$. Since $(\Omega_{i_0}, \mathcal{T}_{i_0})$ is Hausdorff, there exist U_{i_0}, V_{i_0} open sets in Ω_{i_0}, such that $x(i_0) \in U_{i_0}$, $y(i_0) \in V_{i_0}$ and $U_{i_0} \cap V_{i_0} = \emptyset$. Define $U = U_{i_0} \times \Pi_{i \in I \setminus \{i_0\}} \Omega_i$ and

$V = V_{i_0} \times \Pi_{i\in I\setminus\{i_0\}}\Omega_i$. Then $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Furthermore, U and V are rectangles of the family of topologies $(\mathcal{T}_i)_{i\in I}$ [definition (52)], and therefore belong to the product topology $\odot_{i\in I}\mathcal{T}_i = \mathcal{T}$. Given two distinct elements x, y in Ω, we have found two disjoint open sets U, V in Ω, containing x and y respectively. This shows that the product topological space $(\Omega, \mathcal{T})$ is Hausdorff.

Exercise 4

Exercise 5.

1. Let $x \in K$. Since by assumption, $y \in K^c$, we have $x \neq y$. The topological space $(\Omega, \mathcal{T})$ being Hausdorff, there exist open sets V_x and W_x in Ω, such that $y \in V_x$, $x \in W_x$ and $V_x \cap W_x = \emptyset$.

2. For all $x \in K$, we have $x \in W_x$. In particular, $K \subseteq \cup_{x\in K} W_x$. K being a compact subset of Ω, and $(W_x)_{x\in K}$ being a family of open sets in Ω, there exists $\{x_1, \ldots, x_n\}$ finite subset of K, such that $K \subseteq W_{x_1} \cup \ldots \cup W_{x_n}$, i.e. $K \subseteq W^y = W_{x_1} \cup \ldots \cup W_{x_n}$.

3. Let $V^y = V_{x_1} \cap \ldots \cap V_{x_n}$. All V_x's being open in Ω, V^y is a finite intersection of open sets in Ω, and is therefore open in Ω. Suppose that $x \in V^y \cap W^y$. Then, there exists $i \in \{1, \ldots, n\}$ such that $x \in W_{x_i}$. Since $V^y \subseteq V_{x_i}$, we see that $x \in W_{x_i} \cap V_{x_i}$, which contradicts that fact that $W_{x_i} \cap V_{x_i} = \emptyset$. It follows that $V^y \cap W^y = \emptyset$.

4. By construction, $y \in V_{x_i}$ for all $i \in \{1, \ldots, n\}$. It follows that $y \in V_{x_1} \cap \ldots \cap V_{x_n} = V^y$. Furthermore from 2., $K \subseteq W^y$ and from 3., $V^y \cap W^y = \emptyset$. It follows that for all $x \in V^y$, $x \notin K$. So $V^y \subseteq K^c$. We have proved that $y \in V^y \subseteq K^c$.

5. So far, for all $y \in K^c$, we have shown the existence of an open set V^y in Ω, such that $y \in V^y \subseteq K^c$. It is clear that $\cup_{y\in K^c} V^y \subseteq K^c$. Conversely, for all $y \in K^c$, we have $y \in V^y$. So $K^c \subseteq \cup_{y\in K^c} V^y$. We have proved that $K^c = \cup_{y\in K^c} V^y$.

6. From 5., K^c is a union of open sets in Ω, and is therefore open in Ω. We conclude that K is a closed subset of Ω. The purpose of this exercise is to prove theorem (35).

Exercise 5

Exercise 6.

1. Suppose $A = \{x\}$ for some $x \in E$. Then $\delta(A) = \sup\{0\} = 0$. Conversely, suppose $\delta(A) = 0$. Then $A \neq \emptyset$, since otherwise we would have $\delta(A) = -\infty$. Suppose A had two distinct elements x and y, We would have $0 < d(x, y) \leq \delta(A)$, contradicting the assumption that $\delta(A) = 0$. It follows that A has only one element. We have proved that $\delta(A) = 0$, if and only if $A = \{x\}$ for some $x \in E$.

2. let $\phi : \mathbf{R} \to]-1,1[$ be an increasing homeomorphism. Let $d'(x,y) = |\phi(x)-\phi(y)|$. Since ϕ is injective, $d'(x,y)=0$ is equivalent to $x=y$. So d' is clearly a metric on $\mathbf{R}$. Let A be open for the usual topology on $\mathbf{R}$, i.e. $A \in \mathcal{T}_{\mathbf{R}}$. ϕ being a homeomorphism, ϕ^{-1} is continuous, and therefore $\phi(A)$ is open in $]-1,1[$. It follows that $\phi(A)$ is also open in $\mathbf{R}$. Let $x \in A$. Then $\phi(x) \in \phi(A)$, and there exists $\epsilon > 0$ such that $|\phi(x) - z| < \epsilon \Rightarrow z \in \phi(A)$. Let $y \in \mathbf{R}$ be such that $d'(x,y) < \epsilon$. Then $|\phi(x) - \phi(y)| < \epsilon$ and therefore $\phi(y) \in \phi(A)$. ϕ being injective, we see that $y \in A$. We have found $\epsilon > 0$, such that $d'(x,y) < \epsilon \Rightarrow y \in A$. This shows that A is open with respect to the metric topology induced by d', i.e. $A \in \mathcal{T}_{d'}$. This being true for all $A \in \mathcal{T}_{\mathbf{R}}$, we have $\mathcal{T}_{\mathbf{R}} \subseteq \mathcal{T}_{d'}$. Conversely, let $A \in \mathcal{T}_{d'}$. Let $x \in A$. There exists $\epsilon > 0$, such that $d'(x,y) < \epsilon \Rightarrow y \in A$. However, ϕ being continuous, there exists $\eta > 0$, such that $|x-y| < \eta \Rightarrow d'(x,y) < \epsilon$. Hence, we see that $|x-y| < \eta \Rightarrow y \in A$. This shows that A is open with respect to the usual topology on $\mathbf{R}$, i.e. $A \in \mathcal{T}_{\mathbf{R}}$. This being true for all $A \in \mathcal{T}_{d'}$, we have $\mathcal{T}_{d'} \subseteq \mathcal{T}_{\mathbf{R}}$, and finally $\mathcal{T}_{d'} = \mathcal{T}_{\mathbf{R}}$. We conclude that the metric d' induces the usual topology on $\mathbf{R}$. Let $\delta'(\mathbf{R})$ be the diameter of $\mathbf{R}$ with respect to the metric d'. For all $x,y \in \mathbf{R}$, we have $d'(x,y) \leq 2$. It follows that $\delta'(\mathbf{R}) \leq 2$ and in particular $\delta'(\mathbf{R}) < +\infty$. So $\mathbf{R}$ is bounded with respect to the metric d'. However, if d'' denotes the usual metric on $\mathbf{R}$, and $\delta''(\mathbf{R})$ the diameter of $\mathbf{R}$ with respect to d'', then it is clear that $\delta''(\mathbf{R}) = +\infty$. So $\mathbf{R}$ is not bounded with respect to the usual metric on $\mathbf{R}$.

3. Let K be a compact subset of E. Let $\epsilon > 0$. We clearly have $K \subseteq \cup_{x\in K} B(x,\epsilon)$. The family $(B(x,\epsilon))_{x\in K}$ being a family of open sets in E, from exercise (2), there exists $\{x_1,\ldots,x_n\}$ finite subset of K, such that $K \subseteq B(x_1,\epsilon) \cup \ldots \cup B(x_n,\epsilon)$.

4. Let $(\Omega,\mathcal{T})$ be a metrizable topological space. Let d be an arbitrary metric inducing the topology $\mathcal{T}$. Let K be a compact subset of Ω. Taking $\epsilon = 1$ in 3., there exists $\{x_1,\ldots,x_n\}$ finite subset of K, such that $K \subseteq B(x_1,1) \cup \ldots \cup B(x_n,1)$. Let $x,y \in K$. There exists $i,j \in \{1,\ldots,n\}$ such that $x \in B(x_i,1)$ and $y \in B(x_j,1)$. It follows that:

$$d(x,y) \leq d(x,x_i) + d(x_i,x_j) + d(x_j,y) \leq 2 + M$$

where $M = \max_{i,j} d(x_i,x_j)$. Hence, we see that $\delta(K) \leq 2 + M$, where $\delta(K)$ is the diameter of K with respect to the metric d. In particular, $\delta(K) < +\infty$, and K is bounded with respect to the metric d. This is true for all d inducing $\mathcal{T}$.

Exercise 6

Exercise 7.

1. Since K is bounded with respect to the usual metric on $\mathbf{R}$, we have $\delta(K) < +\infty$. If $K = \emptyset$, then $K \subseteq [-M,M]$ for any $M \in \mathbf{R}^+$. Suppose $K \neq \emptyset$. Then $\delta(K) \in \mathbf{R}^+$, and for all $x,y \in K$, we have $|x-y| \leq \delta(K)$. Let

$y_0 \in K$. For all $x \in K$, we have $|x| \leq \delta(K) + |y_0|$. So $K \subseteq [-M, M]$, with $M = \delta(K) + |y_0|$.

2. Let K' denote the complement of K in $[-M, M]$. We have $K' = [-M, M] \cap K^c$, where K^c is the complement of K in $\mathbf{R}$. Since by assumption K is closed in $\mathbf{R}$, K^c is open in $\mathbf{R}$. It follows that $[-M, M] \cap K^c$ is open with respect to the induced topology on $[-M, M]$. So K' is open in $[-M, M]$, and we conclude that K is closed in $[-M, M]$.

3. From theorem (34), $[-M, M]$ is a compact subset of $\mathbf{R}$. From 2., K is a closed subset of $[-M, M]$. From exercise (2)[6.], we conclude that K is a compact subset of $[-M, M]$.

4. From 3., K is a compact subset of $[-M, M]$. It follows from exercise (2)[3.], that K is also a compact subset of $\mathbf{R}$. We have proved that any closed and bounded subset of $\mathbf{R}$, is also a compact subset of $\mathbf{R}$.

5. Let K be a compact subset of $\mathbf{R}$. Since $(\mathbf{R}, \mathcal{T}_{\mathbf{R}})$ is Hausdorff, from theorem (35), K is a closed subset of $\mathbf{R}$. Moreover, from exercise (6), K is bounded with respect to any metric inducing the usual topology on $\mathbf{R}$. In particular, it is bounded with respect to the usual metric on $\mathbf{R}$. We have proved that any compact subset of $\mathbf{R}$ is closed and bounded.

6. From 4., any subset of $\mathbf{R}$ which is closed and bounded, is compact. Conversely, from 5., any compact subset of $\mathbf{R}$ is closed and bounded. This proves theorem (36).

Exercise 7

Exercise 8.

1. Let $(W_i)_{i\in I}$ be an open covering of $f(\Omega)$. For all $i \in I$, W_i is open, and $f(\Omega) \subseteq \cup_{i\in I} W_i$. Let $x \in \Omega$. Then $f(x) \in f(\Omega)$. There exists $i \in I$, such that $f(x) \in W_i$, i.e. $x \in f^{-1}(W_i)$. It follows that $\Omega \subseteq \cup_{i\in I} f^{-1}(W_i)$. Moreover, f being continuous and W_i open, each $f^{-1}(W_i)$ is open in Ω. We have proved that $(f^{-1}(W_i))_{i\in I}$ is an open covering of Ω.

2. Let $f : (\Omega, \mathcal{T}) \to (S, \mathcal{T}_S)$ be a continuous map, where $(\Omega, \mathcal{T})$ is a compact topological space. Let $(W_i)_{i\in I}$ be a family of open sets in S, such that $f(\Omega) \subseteq \cup_{i\in I} W_i$. From 1., $(f^{-1}(W_i))_{i\in I}$ is a family of open sets in Ω, such that $\Omega \subseteq \cup_{i\in I} f^{-1}(W_i)$. $(\Omega, \mathcal{T})$ being compact, there exists $\{i_1, \dots, i_n\}$ finite subset of I, such that $\Omega \subseteq f^{-1}(W_{i_1}) \cup \dots \cup f^{-1}(W_{i_n})$. Let $y \in f(\Omega)$. There exists $x \in \Omega$, such that $y = f(x)$. There exists $k \in \{1, \dots, n\}$, such that $x \in f^{-1}(W_{i_k})$, i.e. $f(x) \in W_{i_k}$. So $y \in W_{i_k}$. We have proved that $f(\Omega) \subseteq W_{i_1} \cup \dots \cup W_{i_n}$. Given an arbitrary family $(W_i)_{i\in I}$ of open sets, such that $f(\Omega) \subseteq \cup_{i\in I} W_i$, we have found a finite subset $\{i_1, \dots, i_n\}$ of I, such that $f(\Omega) \subseteq W_{i_1} \cup \dots \cup W_{i_n}$. This shows that $f(\Omega)$ is a compact subset of $(S, \mathcal{T}_S)$.

Exercise 8

Exercise 9.

1. By construction, the topological space $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is homeomorphic to $[-1,1]$ [definition (34)]. In particular, there exists a continuous map $h : [-1,1] \to \bar{\mathbf{R}}$. From theorem (34), the topological space $[-1,1]$ is compact. From exercise (8), we conclude that $\bar{\mathbf{R}} = h([-1,1])$ is a compact subset of $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$. In other words, $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is a compact topological space.

2. Let K be a compact subset of $\mathbf{R}$. The usual topology $\mathcal{T}_{\mathbf{R}}$ on $\mathbf{R}$, is nothing but the topology induced on $\mathbf{R}$, by the usual topology on $\bar{\mathbf{R}}$, i.e. $\mathcal{T}_{\mathbf{R}} = (\mathcal{T}_{\bar{\mathbf{R}}})_{|\mathbf{R}}$. From exercise (2)[3.], we conclude that K is also a compact subset of $\bar{\mathbf{R}}$.

3. Let K be a compact subset of $\bar{\mathbf{R}}$. Since $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is metrizable, it is a Hausdorff topological space. It follows from theorem (35) that K is closed in $\bar{\mathbf{R}}$. Conversely, suppose K is a closed subset of $\bar{\mathbf{R}}$. From 1., $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is compact. We conclude from exercise (2)[6.], that K is a compact subset of $\bar{\mathbf{R}}$.

4. Let A be a non-empty subset of $\bar{\mathbf{R}}$, and $\alpha = \sup A$. We assume that $\alpha \neq -\infty$ (i.e. A is not reduced to $\{-\infty\}$). Let $U \in \mathcal{T}_{\bar{\mathbf{R}}}$ with $\alpha \in U$. Let $h : \bar{\mathbf{R}} \to [-1,1]$ be an increasing homeomorphism. Then, $h(U)$ is open in $[-1,1]$, and $h(\alpha) \in h(U)$. Since $\alpha \neq -\infty$, we have $h(\alpha) \neq -1$. There exists $\epsilon > 0$, such that we have $]h(\alpha) - \epsilon, h(\alpha)] \subseteq h(U)$, together with $-1 < h(\alpha) - \epsilon$. It follows that $]\beta, \alpha] \subseteq U$, where $\beta = h^{-1}(h(\alpha) - \epsilon) \in \mathbf{R}$. Let $\bar{A}$ be the closure of A in $\bar{\mathbf{R}}$ [definition 37]. If $\alpha = -\infty$, since $A \neq \emptyset$, we have $A = \{-\infty\}$. So $\alpha \in A \subseteq \bar{A}$. Suppose that $\alpha \neq -\infty$. We claim that $\alpha \in \bar{A}$. Let $U \in \mathcal{T}_{\bar{\mathbf{R}}}$ be such that $\alpha \in U$. As shown above, there exists $\beta < \alpha$, $\beta \in \mathbf{R}$, such that $]\beta, \alpha] \subseteq U$. α being the supremum of A, its is the smallest of all upper-bounds of A. Hence, β cannot be such upper-bound, and there exists $c \in A$ such that $c \in]\beta, \alpha] \subseteq U$. Hence, we see that $A \cap U \neq \emptyset$. This being true for all open sets U in $\bar{\mathbf{R}}$ containing α, we have proved that $\alpha \in \bar{A}$. We conclude that for any non-empty subset A of $\bar{\mathbf{R}}$, we have $\alpha = \sup A \in \bar{A}$.

5. Let A be a non-empty closed subset of $\bar{\mathbf{R}}$. From 4., we have $\sup A \in \bar{A}$, and similarly $\inf A \in \bar{A}$. A being closed in $\bar{\mathbf{R}}$, it coincides with its closure in $\bar{\mathbf{R}}$, i.e. $A = \bar{A}$. So $\sup A \in A$ and $\inf A \in A$. Any non-empty closed subset of $\bar{\mathbf{R}}$ contains its supremum and infimum.

6. Let $A = \{x \in \mathbf{R} : \sin x = 0\}$. The map 'sin' being continuous, $A = \sin^{-1}(\{0\})$ is a closed subset of $\mathbf{R}$. However, $\inf A = -\infty$ and $\sup A = +\infty$, and consequently, A does not contain its supremum or infimum. In 5., we showed that any non-empty closed subset of $\bar{\mathbf{R}}$ contains its supremum and infimum. This property does not hold for non-empty closed subset of $\mathbf{R}$. Indeed, $\mathbf{R}$ itself is a closed subset of itself, and does not contain its supremum or infimum. [Note that $\mathbf{R}$ is not closed in $\bar{\mathbf{R}}$].

7. Let A be a non-empty closed and bounded subset of $\mathbf{R}$. From theorem (36), A is a non-empty compact subset of $\mathbf{R}$. It follows that it is also a non-empty compact of subset of $\bar{\mathbf{R}}$, and consequently from theorem (35), it is a non-empty closed subset of $\bar{\mathbf{R}}$. We conclude from 5. that A contains its supremum and infimum, i.e. $\sup A \in A$ and $\inf A \in A$.

Exercise 9

Exercise 10.

1. Let $f : (\Omega, \mathcal{T}) \to (\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ be a map with $f(\Omega) \subseteq \mathbf{R}$. Suppose f is continuous with respect to $\mathcal{T}_{\mathbf{R}}$. Let U be open in $\bar{\mathbf{R}}$. Then $U \cap \mathbf{R}$ is open in $\mathbf{R}$, and therefore $f^{-1}(U) = f^{-1}(U \cap \mathbf{R}) \in \mathcal{T}$. So f is continuous with respect to $\mathcal{T}_{\bar{\mathbf{R}}}$. Conversely, suppose f is continuous with respect to $\mathcal{T}_{\bar{\mathbf{R}}}$. Let $V \in \mathcal{T}_{\mathbf{R}}$. There exists $U \in \mathcal{T}_{\bar{\mathbf{R}}}$, such that $V = U \cap \mathbf{R}$. So $f^{-1}(V) = f^{-1}(U) \in \mathcal{T}$. So f is continuous with respect to $\mathcal{T}_{\mathbf{R}}$. We have proved that whenever $f(\Omega) \subseteq \mathbf{R}$, the continuity with respect to $\mathcal{T}_{\mathbf{R}}$ and $\mathcal{T}_{\bar{\mathbf{R}}}$ are equivalent.

2. Let $f : (\Omega, \mathcal{T}) \to (\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ be a continuous map, where $(\Omega, \mathcal{T})$ is a non-empty compact topological space. From exercise (8), $f(\Omega)$ is a non-empty compact subset of $\bar{\mathbf{R}}$. In particular, from theorem (35), it is a non-empty closed subset of $\bar{\mathbf{R}}$. From exercise (9)[5.], we conclude that $f(\Omega)$ contains its supremum and infimum, i.e. $\sup f(\Omega) \in f(\Omega)$ and $\inf f(\Omega) \in f(\Omega)$. In other words, there exist x_m and x_M in Ω, such that;
$$f(x_m) = \inf_{x\in\Omega} f(x) \ , \ f(x_M) = \sup_{x\in\Omega} f(x)$$
This proves theorem (37).

Exercise 10

Exercise 11.

1. Suppose $c \in]a, b[$ and $f(c) = \sup f([a, b])$. By assumption, $f'(x)$ exists for all $x \in]a, b[$. So in particular, $f'(c)$ is well defined. For all $x \in [a, b]$, we have $f(x) \leq f(c)$. Hence, for all $x \in]c, b]$, we have $(f(x)-f(c))/(x-c) \leq 0$. Taking the limit as $x \to c$, $c < x$, we obtain $f'(c) \leq 0$. Moreover, for all $x \in [a, c[$, we have $(f(c) - f(x))/(c - x) \geq 0$. Taking the limit as $x \to c$, $x < c$, we obtain $f'(c) \geq 0$. We conclude that $f'(c) = 0$.

2. Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$, differentiable on $]a, b[$, with $f(a) = f(b)$. From theorem (34), $[a, b]$ is a compact subset of $\mathbf{R}$. f being continuous, from theorem (37), it attains its maximum and minimum on $[a, b]$. Suppose $\sup f([a, b]) = \inf f([a, b])$. Then f is constant on $[a, b]$, and $f'(c) = 0$ for all $c \in]a, b[$. Suppose that we have $\sup f([a, b]) \neq \inf f([a, b])$. Then $\sup f([a, b])$ and $\inf f([a, b])$ cannot both be equal to $f(a) = f(b)$. Changing f into $-f$ if necessary, without loss of generality we can assume that $\sup f([a, b]) \neq f(a)$. Let $c \in [a, b]$ be such that $f(c) = \sup f([a, b])$. Then $f(c) \neq f(a)$ and $f(c) \neq f(b)$. So in fact, we have $c \in]a, b[$. Since $f(c) = \sup_{x\in[a,b]} f(x)$, from 1., we conclude that

$f'(c) = 0$. We have proved the existence of $c \in]a, b[$, such that $f'(c) = 0$. This proves theorem (38).

Exercise 11

Exercise 12.

1. h is of the form $h = f + \alpha p$, where $\alpha \in \mathbf{R}$, and p is a polynomial. Since f is continuous on $[a, b]$ and differentiable on $]a, b[$, the same is true of h.

2. We have $h(a) = f(a)$ and $h(b) = f(a)$. So $h(a) = h(b)$, and we can apply Rolle's theorem (38). There exists $c \in]a, b[$ such that $h'(c) = 0$. Since for all $x \in [a, b]$,we have:

$$h(x) = f(x) - (x - a)\frac{f(b) - f(a)}{b - a}$$

we have found $c \in]a, b[$, such that:

$$f(b) - f(a) = (b - a)f'(c)$$

Exercise 12

Exercise 13.

1. f is continuous on $[a, b]$, and f' exists on $]a, b[$. Since f is of class C^n, each $f^{(k)}$ is well defined and continuous on $[a, b]$, for all $k \in \{1, \ldots, n\}$. Moreover, each $f^{(k)}$ is differentiable on $[a, b]$, and in particular on $]a, b[$, for all $k \in \{1, \ldots, n-1\}$. In fact, since $f^{(n+1)}$ exist on $]a, b[$, each $f^{(k)}$ is differentiable on $]a, b[$ for all $k \in \{1, \ldots, n\}$. We conclude that h is continuous on $[a, b]$, and differentiable on $]a, b[$.

2. For all $k \in \{1, \ldots, n\}$, we have:

$$[(b - x)^k f^{(k)}]' = -k(b - x)^{k-1} f^{(k)} + (b - x)^k f^{(k+1)}$$

Therefore, if we define:

$$g(x) = \sum_{k=1}^{n} \frac{(b - x)^k}{k!} f^{(k)}(x)$$

we have:

$$\begin{aligned} g'(x) &= -\sum_{k=1}^{n} \frac{(b - x)^{k-1}}{(k-1)!} f^{(k)}(x) + \sum_{k=1}^{n} \frac{(b - x)^k}{k!} f^{(k+1)}(x) \\ &= -\sum_{k=0}^{n-1} \frac{(b - x)^k}{k!} f^{(k+1)}(x) + \sum_{k=1}^{n} \frac{(b - x)^k}{k!} f^{(k+1)}(x) \\ &= -f'(x) + \frac{(b - x)^n}{n!} f^{(n+1)}(x) \end{aligned}$$

and from:

$$h(x) = f(b) - f(x) - g(x) - \alpha\frac{(b - x)^{n+1}}{(n + 1)!}$$

we conclude that:

$$\begin{aligned} h'(x) &= -f'(x) + f'(x) - \frac{(b-x)^n}{n!} f^{(n+1)}(x) + \alpha \frac{(b-x)^n}{n!} \\ &= \frac{(b-x)^n}{n!}(\alpha - f^{(n+1)}(x)) \end{aligned}$$

3. h is continuous on $[a,b]$, and differentiable on $]a,b[$. Moreover, $h(b) = 0 = h(a)$. From theorem (38), there exists $c \in]a,b[$, such that $h'(c) = 0$. Hence, from 2., there exists $c \in]a,b[$ such that $f^{(n+1)}(c) = \alpha$. From $h(a) = 0$, we have:

$$f(b) - f(a) = \sum_{k=1}^{n} \frac{(b-a)^k}{k!} f^{(k)}(a) + \frac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(c) \tag{5}$$

Given $a, b \in \mathbf{R}$, $a < b$ and $n \geq 0$, given $f : [a,b] \to \mathbf{R}$ of class C^n on $[a,b]$, such that $f^{(n+1)}$ exists on $]a,b[$, we have found $c \in]a,b[$ such that equation (5) holds. This proves theorem (39).

Exercise 13

Exercise 14.

1. Let $\phi :]a,b[\to \mathbf{R}$ be convex and differentiable. Let $x, y \in]a,b[$, $x < y$. For all $z, z' \in]x,y[$ such that $z < z'$, from exercise (1), we have:

$$\frac{\phi(z) - \phi(x)}{z - x} \leq \frac{\phi(z') - \phi(z)}{z' - z} \leq \frac{\phi(y) - \phi(z')}{y - z'}$$

z' being fixed, taking the limit as $z \downarrow\downarrow x$, we obtain:

$$\phi'(x) \leq \frac{\phi(y) - \phi(z')}{y - z'}$$

and finally, taking the limit as $z' \uparrow\uparrow y$, $\phi'(x) \leq \phi'(y)$. We have proved that if a convex function is differentiable, its derivative is non-decreasing.

2. Let $x, y, z \in]a,b[$ with $x < y < z$. Since f is differentiable on $]a,b[$, in particular, it is continuous on $[x,y]$ and differentiable on $]x,y[$. From exercise (12), there exists $c_1 \in]x,y[$ such that;

$$\phi(y) - \phi(x) = \phi'(c_1)(y - x) \tag{6}$$

Similarly, there exists $c_2 \in]y,z[$, such that:

$$\phi(z) - \phi(y) = \phi'(c_2)(z - y) \tag{7}$$

From $x < y < x$, we conclude that $c_1 < c_2$.

3. Let $\phi :]a,b[\to \mathbf{R}$ be differentiable, and such that ϕ' is non-decreasing. Let $x, y, z \in]a,b[$ be such that $x < y < z$. From 2., there exist $c_1, c_2 \in]a,b[$,

$c_1 < c_2$, such that equations (6) and (7) are satisfied. ϕ' being non-decreasing, we have $\phi'(c_1) \leq \phi'(c_2)$. We conclude from (6) and (7) that:

$$\frac{\phi(y) - \phi(x)}{y - x} \leq \frac{\phi(z) - \phi(y)}{z - y}$$

From exercise (1), it follows that ϕ is convex. We have proved that a differentiable map on $]a,b[$, with non-decreasing derivative is convex.

4. $x \to e^x$ is differentiable on $\mathbf{R}$, with non-decreasing derivative. It is therefore convex.

5. $x \to -\ln(x)$ is differentiable on $]0,+\infty[$, with non-decreasing derivative. It is therefore convex.

Exercise 14

Exercise 15.

1. Since $\phi :]a,b[\to \mathbf{R}$ is convex, from exercise (1), it is continuous. It follows that $\phi : (]a,b[, \mathcal{B}(]a,b[)) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is measurable. Since $X \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$, the map $X : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is measurable. In fact, since $X(\Omega) \subseteq]a,b[$, it is also true that $X : (\Omega, \mathcal{F}) \to (]a,b[, \mathcal{B}(]a,b[))$ is measurable. We conclude that $\phi \circ X : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is measurable.

2. Since from 1., $\phi \circ X$ is measurable and $\mathbf{R}$-valued, it is an element of $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$, if and only if:

$$E[|\phi \circ X|] \stackrel{\triangle}{=} \int |\phi \circ X| dP < +\infty$$

3. Suppose $E[X] = a$. Since by assumption, $X \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$, $E[X] \in \mathbf{R}$. So $a \in \mathbf{R}$. Since $X(\Omega) \subseteq]a,b[$, in particular $X \geq a$. So $X - a \geq 0$ and $\int (X - a) dP = 0$. From exercise (7) [6.] of Tutorial 5, we conclude that $X = a$ P-a.s., which contradicts $X(\Omega) \subseteq]a,b[$.

4. Suppose $E[X] = b$. Since by assumption, $X \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$, $E[X] \in \mathbf{R}$. So $b \in \mathbf{R}$. Since $X(\Omega) \subseteq]a,b[$, in particular $X \leq b$. So $b - X \geq 0$ and $\int (b - X) dP = 0$. From exercise (7) [6.] of Tutorial 5, we conclude that $X = b$ P-a.s., which contradicts $X(\Omega) \subseteq]a,b[$.

5. Let $m = E[X]$. Since $X(\Omega) \subseteq]a,b[$, we have $a < X < b$. It follows that $a \leq m \leq b$. From 3. and 4., $m = a$ or $m = b$ leads to a contradiction. We conclude that $m \in]a,b[$.

6. We define:

$$\beta \stackrel{\triangle}{=} \sup_{x \in]a,m[} \frac{\phi(m) - \phi(x)}{m - x}$$

Since $a < m$, $]a,m[\neq \emptyset$ and $\beta \neq -\infty$. Let $z \in]m,b[$. Since ϕ is convex, from exercise (1), for all $x \in]a,m[$, we have:

$$\frac{\phi(m) - \phi(x)}{m - x} \leq \frac{\phi(z) - \phi(m)}{z - m}$$

It follows that:

$$\beta \leq \frac{\phi(z) - \phi(m)}{z - m}$$

In particular, $\beta < +\infty$ and finally $\beta \in \mathbf{R}$.

7. Let $x \in]a, b[$. If $x \in]a, m[$, then by definition of β, we have:

$$\frac{\phi(m) - \phi(x)}{m - x} \leq \beta$$

and consequently:

$$\phi(m) + \beta(x - m) \leq \phi(x) \tag{8}$$

If $x \in]m, b[$, then from 6., we have:

$$\beta \leq \frac{\phi(x) - \phi(m)}{x - m}$$

and consequently, inequality (8) still holds. We conclude that inequality (8) holds for all $x \in]a, b[$.

8. For all $\omega \in \Omega$, $X(\omega) \in]a, b[$. From 7., we obtain:

$$\phi(m) + \beta(X(\omega) - m) \leq \phi(X(\omega)) \tag{9}$$

9. If $\phi \circ X \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$, then $E[\phi \circ X]$ is meaningful. Taking expectations on both sides of (9), we obtain:

$$\phi(m) + \beta(E[X] - m) \leq E[\phi \circ X]$$

and since $m = E[X]$, we conclude that $\phi(m) \leq E[\phi \circ X]$. This proves theorem (40).

Exercise 15