4. Measurability

Definition 25 Let A and B be two sets, and $f : A \to B$ be a map. Given $A' \subseteq A$, we call **direct image** of A' by f the set denoted f(A'), and defined by $f(A') = \{f(x) : x \in A'\}.$

Definition 26 Let A and B be two sets, and $f : A \to B$ be a map. Given $B' \subseteq B$, we call **inverse image** of B' by f the set denoted $f^{-1}(B')$, and defined by $f^{-1}(B') = \{x : x \in A, f(x) \in B'\}.$

EXERCISE 1. Let A and B be two sets, and $f : A \to B$ be a bijection from A to B. Let $A' \subseteq A$ and $B' \subseteq B$.

- 1. Explain why the notation $f^{-1}(B')$ is potentially ambiguous.
- 2. Show that the inverse image of B' by f is in fact equal to the direct image of B' by f^{-1} .
- 3. Show that the direct image of A' by f is in fact equal to the inverse image of A' by f^{-1} .

Definition 27 Let (Ω, \mathcal{T}) and (S, \mathcal{T}_S) be two topological spaces. A map $f : \Omega \to S$ is said to be continuous if and only if:

$$\forall B \in \mathcal{T}_S \ , \ f^{-1}(B) \in \mathcal{T}$$

In other words, if and only if the inverse image of any open set in S is an open set in Ω .

We Write $f : (\Omega, \mathcal{T}) \to (S, \mathcal{T}_S)$ is continuous, as a way of emphasizing the two topologies \mathcal{T} and \mathcal{T}_S with respect to which f is continuous.

Definition 28 Let E be a set. A map $d : E \times E \rightarrow [0, +\infty[$ is said to be a **metric** on E, if and only if:

- (i) $\forall x, y \in E$, $d(x, y) = 0 \Leftrightarrow x = y$
- (*ii*) $\forall x, y \in E$, d(x, y) = d(y, x)
- (*iii*) $\forall x, y, z \in E$, $d(x, y) \le d(x, z) + d(z, y)$

Definition 29 A metric space is an ordered pair (E, d) where E is a set, and d is a metric on E.

Definition 30 Let (E, d) be a metric space. For all $x \in E$ and $\epsilon > 0$, we define the so-called **open ball** in E:

$$B(x,\epsilon) \stackrel{\bigtriangleup}{=} \{ y : y \in E , d(x,y) < \epsilon \}$$

We call **metric topology** on E, associated with d, the topology \mathcal{T}_E^d defined by:

$$\mathcal{T}^d_E \stackrel{\bigtriangleup}{=} \{ U \subseteq E \ , \forall x \in U, \exists \epsilon > 0, B(x, \epsilon) \subseteq U \}$$

EXERCISE 2. Let \mathcal{T}_E^d be the metric topology associated with d, where (E, d) is a metric space.

- 1. Show that \mathcal{T}_{E}^{d} is indeed a topology on E.
- 2. Given $x \in E$ and $\epsilon > 0$, show that $B(x, \epsilon)$ is an open set in E.

EXERCISE 3. Show that the usual topology on **R** is nothing but the metric topology associated with d(x, y) = |x - y|.

EXERCISE 4. Let (E, d) and (F, δ) be two metric spaces. Show that a map $f : E \to F$ is continuous, if and only if for all $x \in E$ and $\epsilon > 0$, there exists $\eta > 0$ such that for all $y \in E$:

$$d(x,y) < \eta \quad \Rightarrow \quad \delta(f(x),f(y)) < \epsilon$$

Definition 31 Let (Ω, \mathcal{T}) and (S, \mathcal{T}_S) be two topological spaces. A map $f : \Omega \to S$ is said to be a homeomorphism, if and only if f is a continuous bijection, such that f^{-1} is also continuous.

Definition 32 A topological space (Ω, \mathcal{T}) is said to be **metrizable**, if and only if there exists a metric d on Ω , such that the associated metric topology coincides with \mathcal{T} , i.e. $\mathcal{T}_{\Omega}^{d} = \mathcal{T}$.

Definition 33 Let (E, d) be a metric space and $F \subseteq E$. We call induced metric on F, denoted $d_{|F}$, the restriction of the metric d to $F \times F$, i.e. $d_{|F} = d_{|F \times F}$.

EXERCISE 5. Let (E, d) be a metric space and $F \subseteq E$. We define $\mathcal{T}_F = (\mathcal{T}_E^d)_{|F}$ as the topology on F induced by the metric topology on E. Let $\mathcal{T}'_F = \mathcal{T}_F^{d|F}$ be the metric topology on F associated with the induced metric $d|_F$ on F.

- 1. Show that $\mathcal{T}_F \subseteq \mathcal{T}'_F$.
- 2. Given $A \in \mathcal{T}'_F$, show that $A = (\bigcup_{x \in A} B(x, \epsilon_x)) \cap F$ for some $\epsilon_x > 0, x \in A$, where $B(x, \epsilon_x)$ denotes the open ball in E.
- 3. Show that $\mathcal{T}'_F \subseteq \mathcal{T}_F$.

Theorem 12 Let (E, d) be a metric space and $F \subseteq E$. Then, the topology on F induced by the metric topology, is equal to the metric topology on F associated with the induced metric, i.e. $(\mathcal{T}_E^d)_{|F} = \mathcal{T}_F^{d_{|F}}$.

EXERCISE 6. Let $\phi : \mathbf{R} \to]-1, 1[$ be the map defined by:

$$\forall x \in \mathbf{R} \quad , \quad \phi(x) \stackrel{\Delta}{=} \frac{x}{|x|+1}$$

- 1. Show that [-1, 0] is not open in **R**.
- 2. Show that [-1, 0] is open in [-1, 1].
- 3. Show that ϕ is a homeomorphism between **R** and]-1,1[.
- 4. Show that $\lim_{x\to+\infty} \phi(x) = 1$ and $\lim_{x\to-\infty} \phi(x) = -1$.

EXERCISE 7. Let $\bar{\mathbf{R}} = [-\infty, +\infty] = \mathbf{R} \cup \{-\infty, +\infty\}$. Let ϕ be defined as in exercise (6), and $\bar{\phi} : \bar{\mathbf{R}} \to [-1, 1]$ be the map defined by:

$$\bar{\phi}(x) = \begin{cases} \phi(x) & \text{if } x \in \mathbf{R} \\ 1 & \text{if } x = +\infty \\ -1 & \text{if } x = -\infty \end{cases}$$

Define:

$$\mathcal{T}_{\bar{\mathbf{R}}} \stackrel{\triangle}{=} \{ U \subseteq \bar{\mathbf{R}} \ , \ \bar{\phi}(U) \text{ is open in } [-1,1] \}$$

- 1. Show that $\bar{\phi}$ is a bijection from $\bar{\mathbf{R}}$ to [-1,1], and let $\bar{\psi} = \bar{\phi}^{-1}$.
- 2. Show that $\mathcal{T}_{\bar{\mathbf{R}}}$ is a topology on $\bar{\mathbf{R}}$.
- 3. Show that $\bar{\phi}$ is a homeomorphism between $\bar{\mathbf{R}}$ and [-1, 1].
- 4. Show that $[-\infty, 2[,]3, +\infty]$, $[3, +\infty]$ are open in $\overline{\mathbf{R}}$.
- 5. Show that if $\phi' : \bar{\mathbf{R}} \to [-1, 1]$ is an arbitrary homeomorphism, then $U \subseteq \bar{\mathbf{R}}$ is open, if and only if $\phi'(U)$ is open in [-1, 1].

Definition 34 The usual topology on $\overline{\mathbf{R}}$ is defined as:

 $\mathcal{T}_{\bar{\mathbf{R}}} \stackrel{\triangle}{=} \{ U \subseteq \bar{\mathbf{R}} \ , \ \bar{\phi}(U) \ is \ open \ in \ [-1,1] \}$

where $\bar{\phi}: \bar{\mathbf{R}} \to [-1, 1]$ is defined by $\bar{\phi}(-\infty) = -1$, $\bar{\phi}(+\infty) = 1$ and:

$$\forall x \in \mathbf{R} \quad , \quad \bar{\phi}(x) \stackrel{\triangle}{=} \frac{x}{|x|+1}$$

EXERCISE 8. Let ϕ and $\overline{\phi}$ be as in exercise (7). Define:

$$\mathcal{T}' \stackrel{\triangle}{=} (\mathcal{T}_{\bar{\mathbf{R}}})_{|\mathbf{R}} \stackrel{\triangle}{=} \{ U \cap \mathbf{R} \ , \ U \in \mathcal{T}_{\bar{\mathbf{R}}} \}$$

- 1. Recall why \mathcal{T}' is a topology on **R**.
- 2. Show that for all $U \subseteq \overline{\mathbf{R}}, \phi(U \cap \mathbf{R}) = \overline{\phi}(U) \cap [-1, 1[.$
- 3. Explain why if $U \in \mathcal{T}_{\bar{\mathbf{R}}}$, $\phi(U \cap \mathbf{R})$ is open in]-1, 1[.
- 4. Show that $\mathcal{T}' \subseteq \mathcal{T}_{\mathbf{R}}$, (the usual topology on \mathbf{R}).
- 5. Let $U \in \mathcal{T}_{\mathbf{R}}$. Show that $\overline{\phi}(U)$ is open in]-1,1[and [-1,1].
- 6. Show that $\mathcal{T}_{\mathbf{R}} \subseteq \mathcal{T}_{\bar{\mathbf{R}}}$

- 7. Show that $\mathcal{T}_{\mathbf{R}} = \mathcal{T}'$, i.e. that the usual topology on $\mathbf{\bar{R}}$ induces the usual topology on \mathbf{R} .
- 8. Show that $\mathcal{B}(\mathbf{R}) = \mathcal{B}(\bar{\mathbf{R}})_{|\mathbf{R}|} = \{B \cap \mathbf{R}, B \in \mathcal{B}(\bar{\mathbf{R}})\}$

EXERCISE 9. Let $d: \overline{\mathbf{R}} \times \overline{\mathbf{R}} \to [0, +\infty]$ be defined by:

$$\forall (x,y) \in \overline{\mathbf{R}} \times \overline{\mathbf{R}} \quad , \quad d(x,y) = |\phi(x) - \phi(y)|$$

where ϕ is an arbitrary homeomorphism from $\overline{\mathbf{R}}$ to [-1, 1].

- 1. Show that d is a metric on \mathbf{R} .
- 2. Show that if $U \in \mathcal{T}_{\bar{\mathbf{R}}}$, then $\phi(U)$ is open in [-1,1]
- 3. Show that for all $U \in \mathcal{T}_{\bar{\mathbf{R}}}$ and $y \in \phi(U)$, there exists $\epsilon > 0$ such that:

 $\forall z \in [-1,1] \ , \ |z-y| < \epsilon \ \Rightarrow \ z \in \phi(U)$

- 4. Show that $\mathcal{T}_{\bar{\mathbf{R}}} \subseteq \mathcal{T}_{\bar{\mathbf{R}}}^d$.
- 5. Show that for all $U \in \mathcal{T}^d_{\bar{\mathbf{R}}}$ and $x \in U$, there is $\epsilon > 0$ such that:

$$\forall y \in \mathbf{\bar{R}} , |\phi(x) - \phi(y)| < \epsilon \Rightarrow y \in U$$

- 6. Show that for all $U \in \mathcal{T}^d_{\bar{\mathbf{R}}}$, $\phi(U)$ is open in [-1, 1].
- 7. Show that $\mathcal{T}_{\bar{\mathbf{R}}}^d \subseteq \mathcal{T}_{\bar{\mathbf{R}}}$
- 8. Prove the following theorem.

Theorem 13 The topological space $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is metrizable.

Definition 35 Let (Ω, \mathcal{F}) and (S, Σ) be two measurable spaces. A map $f : \Omega \to S$ is said to be **measurable** with respect to \mathcal{F} and Σ , if and only if:

$$\forall B \in \Sigma \ , \ f^{-1}(B) \in \mathcal{F}$$

We Write $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ is measurable, as a way of emphasizing the two σ -algebras \mathcal{F} and Σ with respect to which f is measurable.

EXERCISE 10. Let (Ω, \mathcal{F}) and (S, Σ) be two measurable spaces. Let S' be a set and $f: \Omega \to S$ be a map such that $f(\Omega) \subseteq S' \subseteq S$. We define Σ' as the trace of Σ on S', i.e. $\Sigma' = \Sigma_{|S'}$.

- 1. Show that for all $B \in \Sigma$, we have $f^{-1}(B) = f^{-1}(B \cap S')$
- 2. Show that $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ is measurable, if and only if $f : (\Omega, \mathcal{F}) \to (S', \Sigma')$ is itself measurable.

3. Let $f: \Omega \to \mathbf{R}^+$. Show that the following are equivalent:

(i) $f: (\Omega, \mathcal{F}) \to (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$ is measurable (ii) $f: (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is measurable (iii) $f: (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable

EXERCISE 11. Let (Ω, \mathcal{F}) , (S, Σ) , (S_1, Σ_1) be three measurable spaces. let $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ and $g : (S, \Sigma) \to (S_1, \Sigma_1)$ be two measurable maps.

- 1. For all $B \subseteq S_1$, show that $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$
- 2. Show that $g \circ f : (\Omega, \mathcal{F}) \to (S_1, \Sigma_1)$ is measurable.

EXERCISE 12. Let (Ω, \mathcal{F}) and (S, Σ) be two measurable spaces. Let $f : \Omega \to S$ be a map. We define:

$$\Gamma \stackrel{\triangle}{=} \{ B \in \Sigma \ , \ f^{-1}(B) \in \mathcal{F} \}$$

- 1. Show that $f^{-1}(S) = \Omega$.
- 2. Show that for all $B \subseteq S$, $f^{-1}(B^c) = (f^{-1}(B))^c$.
- 3. Show that if $B_n \subseteq S, n \ge 1$, then $f^{-1}(\bigcup_{n=1}^{+\infty} B_n) = \bigcup_{n=1}^{+\infty} f^{-1}(B_n)$
- 4. Show that Γ is a σ -algebra on S.
- 5. Prove the following theorem.

Theorem 14 Let (Ω, \mathcal{F}) and (S, Σ) be two measurable spaces, and \mathcal{A} be a set of subsets of S generating Σ , i.e. such that $\Sigma = \sigma(\mathcal{A})$. Then $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ is measurable, if and only if:

$$\forall B \in \mathcal{A} \quad , \quad f^{-1}(B) \in \mathcal{F}$$

EXERCISE 13. Let (Ω, \mathcal{T}) and (S, \mathcal{T}_S) be two topological spaces. Let $f : \Omega \to S$ be a map. Show that if $f : (\Omega, \mathcal{T}) \to (S, \mathcal{T}_S)$ is continuous, then $f : (\Omega, \mathcal{B}(\Omega)) \to (S, \mathcal{B}(S))$ is measurable.

EXERCISE 14. We define the following subsets of the power set $\mathcal{P}(\mathbf{R})$:

$$\begin{array}{rcl} \mathcal{C}_1 & \stackrel{\bigtriangleup}{=} & \{[-\infty,c] \;,\; c \in \mathbf{R}\} \\ \mathcal{C}_2 & \stackrel{\bigtriangleup}{=} & \{[-\infty,c[\;,\; c \in \mathbf{R}\} \\ \mathcal{C}_3 & \stackrel{\bigtriangleup}{=} & \{[c,+\infty] \;,\; c \in \mathbf{R}\} \\ \mathcal{C}_4 & \stackrel{\bigtriangleup}{=} & \{]c,+\infty] \;,\; c \in \mathbf{R}\} \end{array}$$

- 1. Show that C_2 and C_4 are subsets of $\mathcal{T}_{\bar{\mathbf{R}}}$.
- 2. Show that the elements of C_1 and C_3 are closed in $\overline{\mathbf{R}}$.

- 3. Show that for all $i = 1, 2, 3, 4, \sigma(\mathcal{C}_i) \subseteq \mathcal{B}(\bar{\mathbf{R}})$.
- 4. Let U be open in $\overline{\mathbf{R}}$. Explain why $U \cap \mathbf{R}$ is open in \mathbf{R} .
- 5. Show that any open subset of \mathbf{R} is a countable union of open bounded intervals in \mathbf{R} .
- 6. Let $a < b, a, b \in \mathbf{R}$. Show that we have:

$$]a,b[=\bigcup_{n=1}^{+\infty}]a,b-1/n]=\bigcup_{n=1}^{+\infty}[a+1/n,b[$$

- 7. Show that for all $i = 1, 2, 3, 4,]a, b \in \sigma(\mathcal{C}_i)$.
- 8. Show that for all $i = 1, 2, 3, 4, \{\{-\infty\}, \{+\infty\}\} \subseteq \sigma(\mathcal{C}_i)$.
- 9. Show that for all $i = 1, 2, 3, 4, \sigma(\mathcal{C}_i) = \mathcal{B}(\bar{\mathbf{R}})$
- 10. Prove the following theorem.

Theorem 15 Let (Ω, \mathcal{F}) be a measurable space, and $f : \Omega \to \overline{\mathbf{R}}$ be a map. The following are equivalent:

 $\begin{array}{ll} (i) & f: (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}})) \text{ is measurable} \\ (ii) & \forall B \in \mathcal{B}(\bar{\mathbf{R}}) \ , \ \{f \in B\} \in \mathcal{F} \\ (iii) & \forall c \in \mathbf{R} \ , \ \{f \leq c\} \in \mathcal{F} \\ (iv) & \forall c \in \mathbf{R} \ , \ \{f < c\} \in \mathcal{F} \\ (v) & \forall c \in \mathbf{R} \ , \ \{c \leq f\} \in \mathcal{F} \\ (vi) & \forall c \in \mathbf{R} \ , \ \{c < f\} \in \mathcal{F} \end{array}$

EXERCISE 15. Let (Ω, \mathcal{F}) be a measurable space. Let $(f_n)_{n\geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$. Let g and h be the maps defined by $g(\omega) = \inf_{n\geq 1} f_n(\omega)$ and $h(\omega) = \sup_{n\geq 1} f_n(\omega)$, for all $\omega \in \Omega$.

- 1. Let $c \in \mathbf{R}$. Show that $\{c \leq g\} = \bigcap_{n=1}^{+\infty} \{c \leq f_n\}.$
- 2. Let $c \in \mathbf{R}$. Show that $\{h \leq c\} = \bigcap_{n=1}^{+\infty} \{f_n \leq c\}.$
- 3. Show that $g, h: (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are measurable.

Definition 36 Let $(v_n)_{n>1}$ be a sequence in $\overline{\mathbf{R}}$. We define:

$$u \stackrel{\triangle}{=} \liminf_{n \to +\infty} v_n \stackrel{\triangle}{=} \sup_{n \ge 1} \left(\inf_{k \ge n} v_k \right)$$

and:

$$w \stackrel{\triangle}{=} \limsup_{n \to +\infty} v_n \stackrel{\triangle}{=} \inf_{n \ge 1} \left(\sup_{k \ge n} v_k \right)$$

Then, $u, w \in \mathbf{R}$ are respectively called lower limit and upper limit of the sequence $(v_n)_{n\geq 1}$.

EXERCISE 16. Let $(v_n)_{n\geq 1}$ be a sequence in $\mathbf{\bar{R}}$. for $n \geq 1$ we define $u_n = \inf_{k\geq n} v_k$ and $w_n = \sup_{k\geq n} v_k$. Let u and w be the lower limit and upper limit of $(v_n)_{n\geq 1}$, respectively.

- 1. Show that $u_n \leq u_{n+1} \leq u$, for all $n \geq 1$.
- 2. Show that $w \leq w_{n+1} \leq w_n$, for all $n \geq 1$.
- 3. Show that $u_n \to u$ and $w_n \to w$ as $n \to +\infty$.
- 4. Show that $u_n \leq v_n \leq w_n$, for all $n \geq 1$.
- 5. Show that $u \leq w$.
- 6. Show that if u = w then $(v_n)_{n \ge 1}$ converges to a limit $v \in \mathbf{R}$, with u = v = w.
- 7. Show that if $a, b \in \mathbf{R}$ are such that u < a < b < w then for all $n \ge 1$, there exist $N_1, N_2 \ge n$ such that $v_{N_1} < a < b < v_{N_2}$.
- 8. Show that if $a, b \in \mathbf{R}$ are such that u < a < b < w then there exist two strictly increasing sequences of integers $(n_k)_{k\geq 1}$ and $(m_k)_{k\geq 1}$ such that for all $k \geq 1$, we have $v_{n_k} < a < b < v_{m_k}$.
- 9. Show that if $(v_n)_{n\geq 1}$ converges to some $v \in \overline{\mathbf{R}}$, then u = w.

Theorem 16 Let $(v_n)_{n\geq 1}$ be a sequence in **R**. Then, the following are equivalent:

(i)
$$\liminf_{n \to +\infty} v_n = \limsup_{n \to +\infty} v_n$$

(ii)
$$\lim_{n \to +\infty} v_n \text{ exists in } \bar{\mathbf{R}}.$$

in which case:

$$\lim_{n \to +\infty} v_n = \liminf_{n \to +\infty} v_n = \limsup_{n \to +\infty} v_n$$

EXERCISE 17. Let $f, g: (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ be two measurable maps, where (Ω, \mathcal{F}) is a measurable space.

- 1. Show that $\{f < g\} = \bigcup_{r \in \mathbf{Q}} (\{f < r\} \cap \{r < g\}).$
- 2. Show that the sets $\{f < g\}, \{f > g\}, \{f = g\}, \{f \le g\}, \{f \ge g\}$ belong to the σ -algebra \mathcal{F} .

EXERCISE 18. Let (Ω, \mathcal{F}) be a measurable space. Let $(f_n)_{n\geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$. We define $g = \liminf f_n$ and $h = \limsup f_n$ in the obvious way:

$$\forall \omega \in \Omega , g(\omega) \stackrel{\triangle}{=} \liminf_{n \to +\infty} f_n(\omega)$$
$$\forall \omega \in \Omega , h(\omega) \stackrel{\triangle}{=} \limsup_{n \to +\infty} f_n(\omega)$$

- 1. Show that $g, h: (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are measurable.
- 2. Show that $g \leq h$, i.e. $\forall \omega \in \Omega$, $g(\omega) \leq h(\omega)$.
- 3. Show that $\{g = h\} \in \mathcal{F}$.
- 4. Show that $\{\omega : \omega \in \Omega, \lim_{n \to +\infty} f_n(\omega) \text{ exists in } \bar{\mathbf{R}}\} \in \mathcal{F}.$
- 5. Suppose $\Omega = \{g = h\}$, and let $f(\omega) = \lim_{n \to +\infty} f_n(\omega)$, for all $\omega \in \Omega$. Show that $f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

EXERCISE 19. Let $f, g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ be two measurable maps, where (Ω, \mathcal{F}) is a measurable space.

- 1. Show that $-f, |f|, f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ are measurable with respect to \mathcal{F} and $\mathcal{B}(\mathbf{\bar{R}})$.
- 2. Let $a \in \overline{\mathbf{R}}$. Explain why the map a + f may not be well defined.
- 3. Show that $(a + f) : (\Omega, \mathcal{F}) \to (\overline{\mathbf{R}}, \mathcal{B}(\overline{\mathbf{R}}))$ is measurable, whenever $a \in \mathbf{R}$.
- 4. Show that $(a.f): (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, for all $a \in \bar{\mathbf{R}}$. (Recall the convention $0.\infty = 0$).
- 5. Explain why the map f + g may not be well defined.
- 6. Suppose that $f \geq 0$ and $g \geq 0$, i.e. $f(\Omega) \subseteq [0, +\infty]$ and also $g(\Omega) \subseteq [0, +\infty]$. Show that $\{f + g < c\} = \{f < c g\}$, for all $c \in \mathbf{R}$. Show that $f + g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.
- 7. Show that $f + g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable whenever f + g is well-defined, i.e. when the following condition holds:

$$(\{f=+\infty\}\cap\{g=-\infty\})\cup(\{f=-\infty\}\cap\{g=+\infty\})=\emptyset$$

- 8. Show that $1/f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, in the case when $f(\Omega) \subseteq \mathbf{R} \setminus \{0\}.$
- 9. Suppose that f is **R**-valued. Show that f defined by $f(\omega) = f(\omega)$ if $f(\omega) \neq 0$ and $\bar{f}(\omega) = 1$ if $f(\omega) = 0$, is measurable with respect to \mathcal{F} and $\mathcal{B}(\bar{\mathbf{R}})$.
- 10. Suppose f and g take values in **R**. Let \overline{f} be defined as in 9. Show that for all $c \in \mathbf{R}$, the set $\{fg < c\}$ can be expressed as:

 $(\{f > 0\} \cap \{g < c/\bar{f}\}) \uplus (\{f < 0\} \cap \{g > c/\bar{f}\}) \uplus (\{f = 0\} \cap \{f < c\})$

11. Show that $fg: (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, in the case when f and g take values in \mathbf{R} .

EXERCISE 20. Let $f, g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ be two measurable maps, where (Ω, \mathcal{F}) is a measurable space. Let \bar{f}, \bar{g} , be defined by:

$$\bar{f}(\omega) \stackrel{\triangle}{=} \begin{cases} f(\omega) & \text{if} \quad f(\omega) \notin \{-\infty, +\infty\} \\ 1 & \text{if} \quad f(\omega) \in \{-\infty, +\infty\} \end{cases}$$

 $\bar{g}(\omega)$ being defined in a similar way. Consider the partitions of Ω , $\Omega = A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus A_5$ and $\Omega = B_1 \oplus B_2 \oplus B_3 \oplus B_4 \oplus B_5$, where $A_1 = \{f \in]0, +\infty[\}, A_2 = \{f \in]-\infty, 0[\}, A_3 = \{f = 0\}, A_4 = \{f = -\infty\}, A_5 = \{f = +\infty\}$ and B_1, B_2, B_3, B_4, B_5 being defined in a similar way with g. Recall the conventions $0 \times (+\infty) = 0, (-\infty) \times (+\infty) = (-\infty)$, etc...

- 1. Show that \overline{f} and \overline{g} are measurable with respect to \mathcal{F} and $\mathcal{B}(\overline{\mathbf{R}})$.
- 2. Show that all A_i 's and B_j 's are elements of \mathcal{F} .
- 3. Show that for all $B \in \mathcal{B}(\bar{\mathbf{R}})$:

$$\{fg \in B\} = \biguplus_{i,j=1}^{5} (A_i \cap B_j \cap \{fg \in B\})$$

- 4. Show that $A_i \cap B_j \cap \{fg \in B\} = A_i \cap B_j \cap \{\overline{f}\overline{g} \in B\}$, in the case when $1 \leq i \leq 3$ and $1 \leq j \leq 3$.
- 5. Show that $A_i \cap B_j \cap \{fg \in B\}$ is either equal to \emptyset or $A_i \cap B_j$, in the case when $i \ge 4$ or $j \ge 4$.
- 6. Show that $fg: (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

Definition 37 Let (Ω, \mathcal{T}) be a topological space, and $A \subseteq \Omega$. We call closure of A in Ω , denoted \overline{A} , the set defined by:

$$\bar{A} \stackrel{\triangle}{=} \{ x \in \Omega : x \in U \in \mathcal{T} \Rightarrow U \cap A \neq \emptyset \}$$

EXERCISE 21. Let (E, \mathcal{T}) be a topological space, and $A \subseteq E$. Let \overline{A} be the closure of A.

- 1. Show that $A \subseteq \overline{A}$ and that \overline{A} is closed.
- 2. Show that if B is closed and $A \subseteq B$, then $\overline{A} \subseteq B$.
- 3. Show that \overline{A} is the smallest closed set in E containing A.
- 4. Show that A is closed if and only if $A = \overline{A}$.
- 5. Show that if (E, \mathcal{T}) is metrizable, then:

$$A = \{ x \in E : \forall \epsilon > 0 , B(x, \epsilon) \cap A \neq \emptyset \}$$

where $B(x, \epsilon)$ is relative to any metric d such that $\mathcal{T}_E^d = \mathcal{T}$.

EXERCISE 22. Let (E, d) be a metric space. Let $A \subseteq E$. For all $x \in E$, we define:

$$d(x,A) \stackrel{\Delta}{=} \inf\{d(x,y) : y \in A\} \stackrel{\Delta}{=} \Phi_A(x)$$

where it is understood that $\inf \emptyset = +\infty$.

- 1. Show that for all $x \in E$, $d(x, A) = d(x, \overline{A})$.
- 2. Show that d(x, A) = 0, if and only if $x \in \overline{A}$.
- 3. Show that for all $x, y \in E$, $d(x, A) \leq d(x, y) + d(y, A)$.
- 4. Show that if $A \neq \emptyset$, $|d(x, A) d(y, A)| \le d(x, y)$.
- 5. Show that $\Phi_A : (E, \mathcal{T}_E^d) \to (\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is continuous.
- 6. Show that if A is closed, then $A = \Phi_A^{-1}(\{0\})$

EXERCISE 23. Let (Ω, \mathcal{F}) be a measurable space. Let $(f_n)_{n\geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$, where (E, d) is a metric space. We assume that for all $\omega \in \Omega$, the sequence $(f_n(\omega))_{n\geq 1}$ converges to some $f(\omega) \in E$.

- 1. Explain why $\liminf f_n$ and $\limsup f_n$ may not be defined in an arbitrary metric space E.
- 2. Show that $f : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is measurable, if and only if $f^{-1}(A) \in \mathcal{F}$ for all closed subsets A of E.
- 3. Show that for all A closed in E, $f^{-1}(A) = (\Phi_A \circ f)^{-1}(\{0\})$, where the map $\Phi_A : E \to \overline{\mathbf{R}}$ is defined as in exercise (22).
- 4. Show that $\Phi_A \circ f_n : (\Omega, \mathcal{F}) \to (\overline{\mathbf{R}}, \mathcal{B}(\overline{\mathbf{R}}))$ is measurable.
- 5. Show that $f: (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is measurable.

Theorem 17 Let (Ω, \mathcal{F}) be a measurable space. Let $(f_n)_{n\geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$, where (E, d) is a metric space. Then, if the limit $f = \lim f_n$ exists on Ω , the map $f : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is itself measurable.

Definition 38 The usual topology on C, the set of complex numbers, is defined as the metric topology associated with d(z, z') = |z - z'|.

EXERCISE 24. Let $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ be a measurable map, where (Ω, \mathcal{F}) is a measurable space. Let u = Re(f) and v = Im(f). Show that $u, v, |f| : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are all measurable.

EXERCISE 25. Define the subset of the power set $\mathcal{P}(\mathbf{C})$:

$$\mathcal{C} \stackrel{\triangle}{=} \{]a, b[\times]c, d[, a, b, c, d \in \mathbf{R} \}$$

where it is understood that:

$$]a,b[\times]c,d[\stackrel{\bigtriangleup}{=} \{z=x+iy\in {\bf C}\ ,\ (x,y)\in]a,b[\times]c,d[\}$$

- 1. Show that any element of C is open in **C**.
- 2. Show that $\sigma(\mathcal{C}) \subseteq \mathcal{B}(\mathbf{C})$.
- 3. Let $z = x + iy \in \mathbb{C}$. Show that if $|x| < \eta$ and $|y| < \eta$ then we have $|z| < \sqrt{2\eta}$.
- 4. Let U be open in C. Show that for all $z \in U$, there are rational numbers a_z, b_z, c_z, d_z such that $z \in]a_z, b_z[\times]c_z, d_z[\subseteq U$.
- 5. Show that U can be written as $U = \bigcup_{n=1}^{+\infty} A_n$ where $A_n \in \mathcal{C}$.
- 6. Show that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbf{C})$.
- 7. Let (Ω, \mathcal{F}) be a measurable space, and $u, v : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ be two measurable maps. Show that $u + iv : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is measurable.

Solutions to Exercises

Exercise 1.

- 1. $f : A \to B$ being a bijection, the notation f^{-1} by itself is meaningful. From definition (26), $f^{-1}(B')$ denotes the inverse image of B' by f. However, from definition (25), $f^{-1}(B')$ also denotes the direct image of B' by f^{-1} . So $f^{-1}(B')$ is ambiguous.
- 2. Let $f^{-1}(B')$ denote the inverse image of B' by f. Let $g = f^{-1}$ and g(B')be the direct image of B' by g. Let $x \in f^{-1}(B')$. Then $x \in A$ and $f(x) \in B'$. Let y = f(x). Then x = g(y) with $y \in B'$. It follows that $x \in g(B')$, and $f^{-1}(B') \subseteq g(B')$. Conversely, let $x \in g(B')$. There exists $y \in B'$ such that $x = g(y) \in A$. Hence, $f(x) = y \in B'$, and we see that $x \in f^{-1}(B')$. It follows that $g(B') \subseteq f^{-1}(B')$. We have proved that $f^{-1}(B') = g(B')$.
- 3. Let $g = f^{-1}$. Then $f = g^{-1}$, and applying 2. to g, we have $g^{-1}(A') = f(A')$, where $g^{-1}(A')$ denotes an inverse image.

Exercise 1

Exercise 2.

- 1. Any statement of the form $\forall x \in \emptyset, \ldots$, is true. Hence, $\emptyset \in \mathcal{T}_E^d$. It is clear that $E \in \mathcal{T}_E^d$, and (i) of definition (13) is satisfied for \mathcal{T}_E^d . Let $A, B \in \mathcal{T}_E^d$, and $x \in A \cap B$. Since $x \in A \in \mathcal{T}_E^d$, there exists $\epsilon_1 > 0$ such that $B(x, \epsilon_1) \subseteq A$. Similarly, there exist $\epsilon_2 > 0$ such that $B(x, \epsilon_2) \subseteq B$. Let $\epsilon = \min(\epsilon_1, \epsilon_2)$. Then $\epsilon > 0$ and $B(x, \epsilon) \subseteq A \cap B$. It follows that $A \cap B \in \mathcal{T}_E^d$ and (ii) of definition (13) is satisfied for \mathcal{T}_E^d . Let $(A_i)_{i \in I}$ be a family of elements of \mathcal{T}_E^d , there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq A_i$. In particular, $B(x, \epsilon) \subseteq \cup_{i \in I} A_i$. It follows that $\cup_{i \in I} A_i \in \mathcal{T}_E^d$, and (iii) of definition (13) is satisfied (i), (ii) and (iii) of definition (13), we conclude that \mathcal{T}_E^d is indeed a topology on E.
- 2. Let $y \in B(x, \epsilon)$. Then $d(x, y) < \epsilon$. Let $\eta = \epsilon d(x, y)$. Then $\eta > 0$, and for all $z \in B(y, \eta)$, from *(iii)* of definition (28):

$$d(x,z) \le d(x,y) + d(y,z) < d(x,y) + \eta = \epsilon$$

It follows that $B(y,\eta) \subseteq B(x,\epsilon)$, and we have proved that $B(x,\epsilon) \in \mathcal{T}_E^d$. In other words, the *open ball* $B(x,\epsilon)$ is an open set in E, with respect to the metric topology on E.

Exercise 2

Exercise 3. If $E = \mathbf{R}$ and d(x, y) = |x - y|, then for all $x \in \mathbf{R}$ and $\epsilon > 0$, we have $B(x, \epsilon) = |x - \epsilon, x + \epsilon|$. Comparing definition (17) for the usual topology on \mathbf{R} , with definition (30), it appears that the usual topology on \mathbf{R} , $\mathcal{T}_{\mathbf{R}}$, is nothing but the metric topology $\mathcal{T}_{\mathbf{R}}^d$.

Exercise 3

Exercise 4. Let \mathcal{P} be the property that for all $x \in E$ and $\epsilon > 0$, there exists $\eta > 0$ such that for all $y \in E$:

$$d(x,y) < \eta \quad \Rightarrow \quad \delta(f(x),f(y)) < \epsilon$$

Suppose that property \mathcal{P} is true. Let $B \in \mathcal{T}_F^{\delta}$ be an open set in F, and $x \in f^{-1}(B)$. Then $f(x) \in B$. Since $B \in \mathcal{T}_F^{\delta}$, from definition (30) there exists $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq B$. However, from property \mathcal{P} , there exists $\eta > 0$, such that:

$$y \in B(x,\eta) \Rightarrow f(y) \in B(f(x),\epsilon)$$

It follows that if $y \in B(x,\eta)$, then $f(y) \in B$, i.e. $y \in f^{-1}(B)$. Hence, $B(x,\eta) \subseteq f^{-1}(B)$. We have proved that $f^{-1}(B)$ is an open set in E, i.e. $f^{-1}(B) \in \mathcal{T}_E^d$. This being true for all $B \in \mathcal{T}_F^{\delta}$, from definition (27) we conclude that $f : E \to F$ is continuous.

Conversely, suppose that f is continuous. Let $x \in E$ and $\epsilon > 0$. From exercise (2), the open ball $B(f(x), \epsilon)$ is an open set in F. Since f is continuous, it follows that $f^{-1}(B(f(x), \epsilon))$ is an open set in E, which furthermore contains x. There exists $\eta > 0$, such that $B(x, \eta) \subseteq f^{-1}(B(f(x), \epsilon))$. In other words, if $y \in B(x, \eta)$, then $f(y) \in B(f(x), \epsilon)$, or equivalently:

$$d(x,y) < \eta \quad \Rightarrow \quad \delta(f(x),f(y)) < \epsilon$$

It follows that property \mathcal{P} is true. We have proved that property \mathcal{P} is equivalent to $f: E \to F$ being continuous.

Exercise 4

Exercise 5.

1. Let $A \in \mathcal{T}_F$. From definition (23) of an induced topology, there exists $B \in \mathcal{T}_E^d$, such that $A = B \cap F$. Let $x \in A$. Then in particular $x \in B$ and from definition (30), there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq B$, where $B(x, \epsilon)$ is the open ball in E:

$$B(x,\epsilon) \stackrel{\triangle}{=} \{ y \in E : d(x,y) < \epsilon \}$$

If $B'(x,\epsilon)$ denotes the open ball in F:

$$B'(x,\epsilon) \stackrel{\bigtriangleup}{=} \{ y \in F : d_{|F}(x,y) < \epsilon \}$$

then from $d_{|F}(x,y) = d(x,y)$ for all $(x,y) \in F^2$, we conclude that $B'(x,\epsilon) = B(x,\epsilon) \cap F$, for all $x \in F$. Hence, we see that $B'(x,\epsilon) \subseteq B \cap F = A$. It follows that $A \in \mathcal{T}_F^{d_{|F}} = \mathcal{T}'_F$. We have proved that $\mathcal{T}_F \subseteq \mathcal{T}'_F$.

2. Let $A \in \mathcal{T}'_{F}$. By definition (30), for all $x \in A$, there exists $\epsilon_{x} > 0$ such that $B'(x, \epsilon_{x}) \subseteq A$, where $B'(x, \epsilon_{x})$ is the open ball in F. However, for all $x \in F$, $B'(x, \epsilon_{x}) = B(x, \epsilon_{x}) \cap F$, where $B(x, \epsilon_{x})$ is the open ball in E. It follows that $x \in B(x, \epsilon_{x}) \cap F \subseteq A$ for all $x \in A$. Finally, $A = (\bigcup_{x \in A} B(x, \epsilon_{x})) \cap F$.

3. A topology being closed under arbitrary union, and an open ball being open for the metric topology, it follows from 2. that any $A \in \mathcal{T}'_F$ can be expressed as $A = B \cap F$, where B is open for the metric topology on E, i.e. $B \in \mathcal{T}^d_E$. Hence, any $A \in \mathcal{T}'_F$ belongs to $(\mathcal{T}^d_E)_{|F} = \mathcal{T}_F$. We have proved that $\mathcal{T}'_F \subseteq \mathcal{T}_F$. The purpose of this exercise is to prove theorem (12). Given any subset F of a metric space (E, d), the topology on F induced by the metric topology on E is a very *natural* topology for F. However, $(F, d_{|F})$ being itself a metric space, the corresponding metric topology is also a very *natural* topology for F. Fortunately, theorem (12) states that these two topologies do in fact coincide.

Exercise 5

Exercise 6.

- 1. If [-1, 0[was open in **R**, there would exist $\epsilon > 0$ such that $]-1-\epsilon, -1+\epsilon[\subseteq [-1, 0]]$. This is obviously not the case.
- 2. $[-1,0[=]-2,0[\cap[-1,1]]$. Since]-2,0[is open in \mathbf{R} , [-1,0[is of the form $[-1,0[=A\cap[-1,1]]$ with $A \in \mathcal{T}_{\mathbf{R}}$. [-1,0[is therefore an element of the induced topology on [-1,1]. In other words, [-1,0[is an open set in [-1,1].
- 3. Let $\psi : |-1, 1| \to \mathbf{R}$ be defined by $\psi(y) = y/(1-|y|)$. It is easy to check that $\psi \circ \phi(x) = x$ for all $x \in \mathbf{R}$, and $\phi \circ \psi(y) = y$ for all $y \in]-1, 1[$. It follows that ϕ is a bijection and $\phi^{-1} = \psi$. The fact that ϕ and ψ are continuous, may be regarded as an obvious point. However, if one wants to prove it from principles contained in these tutorials, the following argument can be used: from exercise (3), the usual topology on \mathbf{R} is in fact the metric topology associated with d(x, y) = |x y|. From theorem (12), the induced topology on] 1, 1[is also the metric topology associated with d(x, y) = |x y|. From theorem (12), we can prove the continuity of ϕ and ψ using exercise (4). For $x \ge 0$ and $y \ge 0$, we have:

$$|\phi(x) - \phi(y)| = \frac{|x - y|}{(1 + x)(1 + y)} \le |x - y| \tag{1}$$

and:

$$|\phi(x) + \phi(y)| = \frac{x}{1+x} + \frac{y}{1+y} \le |x+y|$$

and since $\phi(-x) = -\phi(x)$ for all $x \in \mathbf{R}$, it is easy to check that equation (1) actually holds for all $x, y \in \mathbf{R}$. The continuity of ϕ is therefore an immediate consequence of exercise (4). Let $x \in]-1, 1[$ and $\epsilon > 0$ be given. For all $y \in]-1, 1[$, we have:

$$|\psi(x) - \psi(y)| = \left|\frac{x - y}{1 - |y|} + \frac{x(|x| - |y|)}{(1 - |x|)(1 - |y|)}\right|$$

Using the fact that $||x| - |y|| \le |x - y|$ and |x| < 1, we obtain:

$$|\psi(x) - \psi(y)| \le \frac{|x - y|}{1 - |y|} + \frac{|x - y|}{(1 - |x|)(1 - |y|)}$$

$$\tag{2}$$

Let $\eta_1 > 0$ be such that $-1 < x - \eta_1 < x + \eta_1 < 1$. Then, the map $y \to 1/(1 - |y|)$ is bounded on $]x - \eta_1, x + \eta_1[$. It follows from (2) that there exists $M \in \mathbf{R}^+$ such that for all $y \in]x - \eta_1, x + \eta_1[$:

$$|\psi(x) - \psi(y)| \le M|x - y| + \frac{M}{(1 - |x|)}|x - y|$$

Consequently, choosing $\eta > 0$ sufficiently small, it is possible to ensure that $|\psi(x) - \psi(y)| < \epsilon$, for all $y \in]x - \eta, x + \eta[$. We conclude from exercise (4) that ψ is continuous. Since ϕ and ψ are continuous, ϕ is a homeomorphism from **R** to] - 1, 1[.

4. Given $\epsilon > 0$ and $x \ge \max(1/\epsilon - 1, 0)$, we have:

$$|\phi(x) - 1| = \frac{1}{1+x} \le \epsilon$$

It follows that $\phi(x) \to 1$ as $x \to +\infty$. Since $\phi(-x) = -\phi(x)$ for all $x \in \mathbf{R}$, we conclude that $\phi(x) \to -1$ as $x \to -\infty$.

Exercise 6

Exercise 7.

- 1. Let $y \in [-1, 1]$. If y = 1, then $y = \overline{\phi}(+\infty)$. If y = -1, then $y = \overline{\phi}(-\infty)$. If $y \in]-1, 1[$, ϕ being onto, there exists $x \in \mathbf{R}$ such that $y = \phi(x) = \overline{\phi}(x)$. In any case, there exists $x \in \overline{\mathbf{R}}$ such that $y = \overline{\phi}(x)$. So $\overline{\phi}$ is onto. Suppose $x_1, x_2 \in \overline{\mathbf{R}}$ are such that $\overline{\phi}(x_1) = \overline{\phi}(x_2)$. If $\overline{\phi}(x_1) \in]-1, 1[$, then $\phi(x_1) = \phi(x_2)$, and ϕ being injective, $x_1 = x_2$. If $\overline{\phi}(x_1) = 1$, then $x_1 = x_2 = +\infty$. If $\overline{\phi}(x_1) = -1$, then $x_1 = x_2 = -\infty$. In any case, $x_1 = x_2$. It follows that $\overline{\phi}$ is injective. Finally, $\overline{\phi}$ is a bijection.
- 2. $\bar{\phi}(\emptyset) = \emptyset$ is open in [-1, 1]. So $\emptyset \in \mathcal{T}_{\mathbf{\bar{R}}}$. $\bar{\phi}(\mathbf{\bar{R}}) = [-1, 1]$ is open in [-1, 1], so $\mathbf{\bar{R}} \in \mathcal{T}_{\mathbf{\bar{R}}}$. Let $A, B \in \mathcal{T}_{\mathbf{\bar{R}}}$. Using exercise (1), any direct image by $\bar{\phi}$ can also be viewed as an inverse image by $\bar{\psi}$. Hence, we have:

$$\bar{\phi}(A \cap B) = \bar{\psi}^{-1}(A \cap B) = \bar{\psi}^{-1}(A) \cap \bar{\psi}^{-1}(B) = \bar{\phi}(A) \cap \bar{\phi}(B)$$

Since A and B lie in $\mathcal{T}_{\bar{\mathbf{R}}}$, both $\bar{\phi}(A)$ and $\bar{\phi}(B)$ are open in [-1, 1]. It follows that $\bar{\phi}(A \cap B)$ is open in [-1, 1], so $A \cap B \in \mathcal{T}_{\bar{\mathbf{R}}}$. Hence, we see that $\mathcal{T}_{\bar{\mathbf{R}}}$ is closed under finite intersection. Let $(A_i)_{i \in I}$ be a family of elements of $\mathcal{T}_{\bar{\mathbf{R}}}$. We have:

$$\bar{\phi}(\bigcup_{i\in I}A_i) = \bar{\psi}^{-1}(\bigcup_{i\in I}A_i) = \bigcup_{i\in I}\bar{\psi}^{-1}(A_i) = \bigcup_{i\in I}\bar{\phi}(A_i)$$

Each $\bar{\phi}(A_i)$ being open in [-1,1], $\bar{\phi}(\bigcup_{i\in I}A_i)$ is also open in [-1,1]. It follows that $\bigcup_{i\in I}A_i \in \mathcal{T}_{\bar{\mathbf{R}}}$. Hence, we see that $\mathcal{T}_{\bar{\mathbf{R}}}$ is closed under arbitrary union. we have proved that $\mathcal{T}_{\bar{\mathbf{R}}}$ is indeed a topology on $\bar{\mathbf{R}}$.

3. From 1. we know that $\overline{\phi}$ is a bijection from $\overline{\mathbf{R}}$ to [-1,1]. Let B be open in [-1,1]. We have:

$$B = (\bar{\phi} \circ \bar{\psi})^{-1}(B) = \bar{\psi}^{-1}(\bar{\phi}^{-1}(B))$$

Using exercise (1), we see that $B = \bar{\phi}(\bar{\phi}^{-1}(B))$. So $\bar{\phi}(\bar{\phi}^{-1}(B))$ is open in [-1, 1]. From the very definition of $\mathcal{T}_{\mathbf{\bar{R}}}$, it follows that $\bar{\phi}^{-1}(B) \in \mathcal{T}_{\mathbf{\bar{R}}}$. From definition (27) we conclude that $\bar{\phi}$ is continuous. Let A be open in $\mathbf{\bar{R}}$, i.e. $A \in \mathcal{T}_{\mathbf{\bar{R}}}$. By definition, $\bar{\phi}(A)$ is open in [-1, 1]. Using exercise (1), $\bar{\phi}(A) = \bar{\psi}^{-1}(A)$. Hence, $\bar{\psi}^{-1}(A)$ is open in [-1, 1]. From definition (27) we conclude that $\bar{\psi}$ is continuous. Finally, $\bar{\phi}$ is a homeomorphism from $\mathbf{\bar{R}}$ to [-1, 1].

4. We have:

$$\begin{split} \bar{\phi}([-\infty,2[) &= [-1,2/3[=] - \infty,2/3[\cap[-1,1]] \\ \bar{\phi}(]3,+\infty]) &=]3/4,1] &=]3/4,+\infty[\cap[-1,1]] \\ \bar{\phi}(]3,+\infty[) &=]3/4,1[=]3/4,1[\cap[-1,1]] \end{split}$$

It follows that $\bar{\phi}([-\infty, 2[), \bar{\phi}(]3, +\infty])$ and $\bar{\phi}(]3, +\infty[)$ are all open sets in [-1, 1]. Consequently, $[-\infty, 2[,]3, +\infty]$ and $]3, +\infty[$ are open in $\bar{\mathbf{R}}$.

5. Let $\phi' : \mathbf{R} \to [-1,1]$ be an arbitrary homeomorphism, and $\psi' = (\phi')^{-1}$. Suppose $U \subseteq \bar{\mathbf{R}}$ is open in $\bar{\mathbf{R}}$, i.e. $U \in \mathcal{T}_{\bar{\mathbf{R}}}$. Since ψ' is continuous, $(\psi')^{-1}(U)$ is open in [-1,1]. Using exercise (1), $(\psi')^{-1}(U) = \phi'(U)$. So $\phi'(U)$ is open in [-1,1]. Conversely, suppose $\phi'(U)$ is open in [-1,1] for $U \subseteq \bar{\mathbf{R}}$. Since ϕ' is continuous, $(\phi')^{-1}(\phi'(U))$ is open in $\bar{\mathbf{R}}$. However, using exercise (1):

$$(\phi')^{-1}(\phi'(U)) = (\phi')^{-1}((\psi')^{-1}(U)) = (\psi' \circ \phi')^{-1}(U) = U$$

Hence, U is open in $\overline{\mathbf{R}}$. The purpose of this exercise is to give a formal description of the *usual topology* on $\overline{\mathbf{R}}$, leading to definition (34).

Exercise 7

Exercise 8.

- 1. From definition (23), \mathcal{T}' is the topology on **R** induced by $\mathcal{T}_{\bar{\mathbf{R}}}$.
- 2. Let $U \subseteq \overline{\mathbf{R}}$. Let $y \in \phi(U \cap \mathbf{R})$. There exists $x \in U \cap \mathbf{R}$ such that $y = \phi(x)$. In particular, $y \in]-1, 1[$ and $y = \overline{\phi}(x)$ with $x \in U$. So $y \in \overline{\phi}(U) \cap]-1, 1[$. Conversely, suppose that $y \in \overline{\phi}(U) \cap]-1, 1[$. There exists $x \in U$ such that $y = \overline{\phi}(x)$. But $\overline{\phi}(x) \in]-1, 1[$ implies that that $x \in \mathbf{R}$, and therefore $\overline{\phi}(x) = \phi(x) = y$. So $x \in U \cap \mathbf{R}$ and $\phi(x) = y$. It follows that $y \in \phi(U \cap \mathbf{R})$. We have proved that $\phi(U \cap \mathbf{R}) = \overline{\phi}(U) \cap]-1, 1[$.
- 3. Let $U \in \mathcal{T}_{\bar{\mathbf{R}}}$. By definition, $\bar{\phi}(U)$ is open in [-1,1]. There exists B open in \mathbf{R} , such that $\bar{\phi}(U) = B \cap [-1,1]$. Hence, $\bar{\phi}(U) \cap] - 1, 1[=B \cap] - 1, 1[$. From 2., $\phi(U \cap \mathbf{R}) = B \cap] - 1, 1[$. We conclude that $\phi(U \cap \mathbf{R})$ is open in] - 1, 1[.
- 4. Let $V \in \mathcal{T}'$. By definition, there exists $U \in \mathcal{T}_{\bar{\mathbf{R}}}$ such that $V = U \cap \mathbf{R}$. From 3., we see that $\phi(V)$ is open in]-1,1[. ϕ being continuous, $\phi^{-1}(\phi(V))$ is therefore open in \mathbf{R} . However, using exercise (1):

$$\phi^{-1}(\phi(V)) = \phi^{-1}(\psi^{-1}(V)) = (\psi \circ \phi)^{-1}(V) = V$$

It follows that V is open in \mathbf{R} , i.e. $V \in \mathcal{T}_{\mathbf{R}}$. We have proved that $\mathcal{T}' \subseteq \mathcal{T}_{\mathbf{R}}$

- 5. Let $U \in \mathcal{T}_{\mathbf{R}}$. Since $U \subseteq \mathbf{R}$, it is easy to check that $\bar{\phi}(U) = \phi(U)$. Using exercise (1), $\phi(U) = \psi^{-1}(U)$, and ψ being continuous, $\psi^{-1}(U)$ is open in]-1,1[. It follows that $\bar{\phi}(U)$ is open in]-1,1[. There exists B open in \mathbf{R} , such that $\bar{\phi}(U) = B \cap]-1,1[$. In particular $\bar{\phi}(U)$ is also open in \mathbf{R} , with $\bar{\phi}(U) = \bar{\phi}(U) \cap [-1,1]$. We conclude that $\bar{\phi}(U)$ is open in [-1,1].
- 6. For all $U \in \mathcal{T}_{\mathbf{R}}$, from 5., $\bar{\phi}(U)$ is open in [-1,1]. It follows that $U \in \mathcal{T}_{\bar{\mathbf{R}}}$. We have proved that $\mathcal{T}_{\mathbf{R}} \subseteq \mathcal{T}_{\bar{\mathbf{R}}}$.
- 7. Let $U \in \mathcal{T}_{\mathbf{R}}$. From 6., $U \in \mathcal{T}_{\bar{\mathbf{R}}}$. However, since $U \subseteq \mathbf{R}$, we have $U = U \cap \mathbf{R}$. From $U \in \mathcal{T}_{\bar{\mathbf{R}}}$ we conclude that $U \in \mathcal{T}'$. We have proved that $\mathcal{T}_{\mathbf{R}} \subseteq \mathcal{T}'$. From 4., $\mathcal{T}' \subseteq \mathcal{T}_{\mathbf{R}}$. It follows that $\mathcal{T}_{\mathbf{R}} = \mathcal{T}'$. In other words, the topology on \mathbf{R} induced by the usual topology on $\bar{\mathbf{R}}$, is nothing but the usual topology on \mathbf{R} .
- 8. Using the trace theorem (10), we have:

$$\mathcal{B}(\mathbf{R})_{|\mathbf{R}|} = \sigma(\mathcal{T}_{\bar{\mathbf{R}}})_{|\mathbf{R}|} = \sigma((\mathcal{T}_{\bar{\mathbf{R}}})_{|\mathbf{R}|}) = \sigma(\mathcal{T}_{\mathbf{R}}) = \mathcal{B}(\mathbf{R})$$

Exercise 8

Exercise 9.

1. d(x,y) = 0 is equivalent to $\phi(x) = \phi(y)$, which is in turn equivalent to x = y. So (i) of definition (28) is satisfied for d. The fact that (ii) is also satisfied is completely obvious. Given $x, y, z \in \overline{\mathbf{R}}$, we have:

 $|\phi(x) - \phi(y)| \le |\phi(x) - \phi(z)| + |\phi(z) - \phi(y)|$

It follows that (iii) of definition (28) is also satisfied for d. We have proved that d is indeed a metric on $\overline{\mathbf{R}}$.

- 2. Let $U \in \mathcal{T}_{\bar{\mathbf{R}}}$ and $\psi = \phi^{-1}$. Since, $\phi(U) = \psi^{-1}(U)$, ψ being continuous, $\phi(U)$ is open in [-1, 1].
- 3. Let $U \in \mathcal{T}_{\overline{\mathbf{R}}}$ and $y \in \phi(U)$. From 2., $\phi(U)$ is open in [-1, 1]. From theorem (12), the induced topology on [-1, 1] is also the metric topology associated with d(x, y) = |x y| on $[-1, 1]^2$. Hence, there exists $\epsilon > 0$ such that $B'(y, \epsilon) \subseteq \phi(U)$, where $B'(y, \epsilon)$ is the open ball in [-1, 1]. Equivalently, there exists $\epsilon > 0$, such that:

$$\forall z \in [-1,1] , \ |z-y| < \epsilon \ \Rightarrow \ z \in \phi(U) \tag{3}$$

4. Let $U \in \mathcal{T}_{\mathbf{\bar{R}}}$. Let $x \in U$ and $y = \phi(x)$. Then $y \in \phi(U)$. From 3., there exists $\epsilon > 0$ such that property (3) holds. Let $x' \in B(x, \epsilon)$ where $B(x, \epsilon)$ is the open ball in $\mathbf{\bar{R}}$. Then $d(x, x') < \epsilon$, i.e. $|\phi(x') - y| < \epsilon$. Since $\phi(x') \in [-1, 1]$, from property (3), we see that $\phi(x') \in \phi(U)$. There exists $x'' \in U$ such that $\phi(x') = \phi(x'')$. ϕ being injective, x' = x'' and in particular $x' \in U$. We have proved that $B(x, \epsilon) \subseteq U$. It follows that $U \in \mathcal{T}_{\mathbf{\bar{R}}}^d$. This being true for all $U \in \mathcal{T}_{\mathbf{\bar{R}}}$, we conclude that $\mathcal{T}_{\mathbf{\bar{R}}} \subseteq \mathcal{T}_{\mathbf{\bar{R}}}^d$.

5. Let $U \in \mathcal{T}^d_{\mathbf{R}}$ and $x \in U$. From definition (30), there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. In other words, there exists $\epsilon > 0$ such that:

$$\forall y \in \mathbf{R} , \ |\phi(x) - \phi(y)| < \epsilon \ \Rightarrow \ y \in U \tag{4}$$

- 6. Let $U \in \mathcal{T}_{\mathbf{R}}^d$ and $z \in \phi(U)$. There exists $x \in U$ such that $z = \phi(x)$. Let $\epsilon > 0$ be such that property (4) holds. Let $z' \in B'(z, \epsilon)$, where $B'(z, \epsilon)$ is the open ball in [-1, 1]. ϕ being onto, there exists $y \in \mathbf{R}$ such that $z' = \phi(y)$. Since $|z z'| < \epsilon$, we have $|\phi(x) \phi(y)| < \epsilon$. Using property (4), $y \in U$. It follows that $z' \in \phi(U)$. We have proved that $B'(z, \epsilon) \subseteq \phi(U)$. So $\phi(U)$ is open in [-1, 1] with respect to the metric topology on [-1, 1]. From theorem (12), this topology coincide with the induced topology on [-1, 1]. Finally, $\phi(U)$ is open in [-1, 1].
- 7. Let $U \in \mathcal{T}_{\overline{\mathbf{R}}}^d$, and $\psi = \phi^{-1}$. From 6., $\phi(U) = \psi^{-1}(U)$ is open in [-1, 1]. ϕ being continuous $\phi^{-1}(\psi^{-1}(U)) = (\psi \circ \phi)^{-1}(U) = U$ is open in $\overline{\mathbf{R}}$. We have proved that $\mathcal{T}_{\overline{\mathbf{R}}}^d \subseteq \mathcal{T}_{\overline{\mathbf{R}}}$.
- 8. We have $\mathcal{T}_{\bar{\mathbf{R}}}^d = \mathcal{T}_{\bar{\mathbf{R}}}$. *d* is a metric on $\bar{\mathbf{R}}$, for which the associated metric topology coincide with the usual topology on $\bar{\mathbf{R}}$. From definition (32), $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is metrizable. This proves theorem (13).

Exercise 9

Exercise 10.

- 1. Let $B \subseteq S$. For all $x \in \Omega$, since $f(\Omega) \subseteq S'$, $f(x) \in B$ is equivalent to $f(x) \in B \cap S'$. Hence, $f^{-1}(B) = f^{-1}(B \cap S')$.
- 2. From definition (35), $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ is measurable, if and only if $f^{-1}(B) \in \mathcal{F}$, for all $B \in \Sigma$. From 1., this is equivalent to $f^{-1}(B \cap S') \in \mathcal{F}$, for all $B \in \Sigma$, or in other words, $f^{-1}(B') \in \mathcal{F}$, for all $B' \in \Sigma_{|S'} = \Sigma'$. It follows that the measurability of f viewed as a function with values in (S, Σ) , is equivalent to the measurability of f viewed as a function with values in (S', Σ') .
- 3. From the trace theorem (10) and the fact that the topologies on \mathbf{R} and \mathbf{R}^+ are induced from the topology on $\mathbf{\bar{R}}$, $\mathcal{B}(\mathbf{R}) = \mathcal{B}(\mathbf{\bar{R}})_{|\mathbf{R}|}$ and $\mathcal{B}(\mathbf{R}^+) = \mathcal{B}(\mathbf{\bar{R}})_{|\mathbf{R}^+}$. So the equivalence between (i), (ii) and (iii) is a direct application of 2.

Exercise 10

Exercise 11.

1. Let $B \subseteq S_1$. For all $x \in \Omega$, $g \circ f(x) \in B$ is equivalent to $f(x) \in g^{-1}(B)$, which is in turn equivalent to $x \in f^{-1}(g^{-1}(B))$. It follows that $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$. Note that we have used this property on several occasions in the solutions of exercises (7) and (8).

2. Let $B \in \Sigma_1$. Since $g: (S, \Sigma) \to (S_1, \Sigma_1)$ is measurable, we have $g^{-1}(B) \in \Sigma$. Since $f: (\Omega, \mathcal{F}) \to (S, \Sigma)$ is measurable, we have $f^{-1}(g^{-1}(B)) \in \mathcal{F}$. Using 1., we see that $(f \circ g)^{-1}(B) \in \mathcal{F}$. It follows that $f \circ g: (\Omega, \mathcal{F}) \to (S_1, \Sigma_1)$ is measurable.

Exercise 11

Exercise 12.

- 1. f being defined on Ω , any inverse image by f is by definition (26) a subset of Ω . Moreover, for all $x \in \Omega$, $f(x) \in S$. So $x \in f^{-1}(S)$ and $\Omega \subseteq f^{-1}(S)$. We have proved that $\Omega = f^{-1}(S)$.
- 2. For all $x \in \Omega$, $f(x) \in B^c$ is equivalent to $x \notin f^{-1}(B)$. So $f^{-1}(B^c) = (f^{-1}(B))^c$.
- 3. Let $(B_i)_{i\in I}$ be a family of subsets of S. $f(x) \in \bigcup_{i\in I} B_i$ is equivalent to $f(x) \in B_i$ for some $i \in I$, which is in turn equivalent to $x \in \bigcup_{i\in I} f^{-1}(B_i)$. So $f^{-1}(\bigcup_{i\in I} B_i) = \bigcup_{i\in I} f^{-1}(B_i)$. Note that we have used this property in the solution of exercise (7).
- 4. Σ being a σ -algebra on $S, S \in \Sigma$. From 1., $f^{-1}(S) = \Omega$, and \mathcal{F} being a σ -algebra on $\Omega, \Omega \in \mathcal{F}$. So $f^{-1}(S) \in \mathcal{F}$, and $S \in \Gamma$. Let $B \in \Gamma$. In particular $B \in \Sigma$ and therefore $B^c \in \Sigma$. Moreover from 2., $f^{-1}(B^c) = (f^{-1}(B))^c$. Since $B \in \Gamma, f^{-1}(B) \in \mathcal{F}$ and therefore $(f^{-1}(B))^c \in \mathcal{F}$. It follows that $f^{-1}(B^c) \in \mathcal{F}$ and we see that $B^c \in \Gamma$. We have proved that Γ is closed under complementation. Let $(B_n)_{n\geq 1}$ be a sequence of elements of Γ . In particular $(B_n)_{n\geq 1}$ is a sequence of elements of Σ and therefore $\cup_{n=1}^{+\infty} B_n \in \Sigma$. Moreover, $f^{-1}(\bigcup_{n=1}^{+\infty} B_n) = \bigcup_{n=1}^{+\infty} f^{-1}(B_n)$. Since $B_n \in \Gamma$, for all $n \geq 1$, $f^{-1}(B_n) \in \mathcal{F}$ for all $n \geq 1$ and therefore $\bigcup_{n=1}^{+\infty} f^{-1}(B_n) \in \mathcal{F}$. It follows that $f^{-1}(\bigcup_{n=1}^{+\infty} B_n) \in \mathcal{F}$ and we see that $\bigcup_{n=1}^{+\infty} B_n \in \Gamma$. We have proved that Γ is closed under countable union. Finally, Γ is a σ -algebra on S.
- 5. Suppose $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ is measurable. Since $\mathcal{A} \subseteq \Sigma$, for all $B \in \mathcal{A}$, $f^{-1}(B) \in \mathcal{F}$. Conversely, suppose that the weaker condition of $f^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{A}$, is satisfied. Then, $\mathcal{A} \subseteq \Gamma$. From 4., Γ is a σ -algebra on S. Since the σ -algebra $\sigma(\mathcal{A})$ generated by \mathcal{A} is the smallest σ -algebra on S containing \mathcal{A} , we obtain that $\sigma(\mathcal{A}) \subseteq \Gamma$. However $\sigma(\mathcal{A}) = \Sigma$. It follows that $\Sigma \subseteq \Gamma$, and in particular, $f^{-1}(B) \in \mathcal{F}$ for all $B \in \Sigma$. So $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ is measurable. This proves theorem (14).

Exercise 12

Exercise 13. Let $f : (\Omega, \mathcal{T}) \to (S, \mathcal{T}_S)$ be continuous. By definition (16), the Borel σ -algebra $\mathcal{B}(S)$ is generated by the set of all open sets, i.e. $\mathcal{B}(S) = \sigma(\mathcal{T}_S)$. Since f is continuous, for all $B \in \mathcal{T}_S$, we have $f^{-1}(B) \in \mathcal{T}$. In particular, for all $B \in \mathcal{T}_S$, $f^{-1}(B) \in \mathcal{B}(\Omega)$. Using theorem (14), we conclude that $f : (\Omega, \mathcal{B}(\Omega)) \to (S, \mathcal{B}(S))$ is measurable.

Exercise 14.

- 1. Let $\phi : \mathbf{R} \to [-1, 1]$ be defined as in definition (34). Then, for all $c \in \mathbf{R}$, $\bar{\phi}([-\infty, c]) = [-1, \bar{\phi}(c)[\text{ and } \bar{\phi}(]c, +\infty]) =]\bar{\phi}(c), 1]$. Both sets being open in [-1, 1], we conclude that $\mathcal{C}_2 \subseteq \mathcal{T}_{\bar{\mathbf{R}}}$ and $\mathcal{C}_4 \subseteq \mathcal{T}_{\bar{\mathbf{R}}}$.
- 2. Using 1., for all $c \in \mathbf{R}$, we have $[-\infty, c]^c =]c, +\infty] \in \mathcal{T}_{\bar{\mathbf{R}}}$ and $[c, +\infty]^c = [-\infty, c] \in \mathcal{T}_{\bar{\mathbf{R}}}$. Hence, the complements of any element of \mathcal{C}_1 or \mathcal{C}_3 is open in $\bar{\mathbf{R}}$. It follows that any element of \mathcal{C}_1 or \mathcal{C}_3 is closed in $\bar{\mathbf{R}}$.
- 3. Let i = 1, ..., 4. From 1. and 2., any element of C_i is either closed or open in $\overline{\mathbf{R}}$. In any case, it is a Borel set in $\overline{\mathbf{R}}$. Hence, $C_i \subseteq \mathcal{B}(\overline{\mathbf{R}})$. Since $\sigma(C_i)$ is the smallest σ -algebra on $\overline{\mathbf{R}}$ containing C_i , we conclude that $\sigma(C_i) \subseteq \mathcal{B}(\overline{\mathbf{R}})$.
- 4. From exercise (8), the usual topology on $\overline{\mathbf{R}}$ induces the usual topology on \mathbf{R} . Hence, for all $U \in \mathcal{T}_{\overline{\mathbf{R}}}, U \cap \mathbf{R} \in (\mathcal{T}_{\overline{\mathbf{R}}})_{|\mathbf{R}} = \mathcal{T}_{\mathbf{R}}$, i.e. $U \cap \mathbf{R}$ is open in \mathbf{R} .
- 5. Let U be open in **R**. For all $x \in U$, there exists $\epsilon_x > 0$ such that $]x \epsilon_x, x + \epsilon_x \subseteq U$. Let $p_x \in]x \epsilon_x, x \cap \mathbf{Q}$ and $q_x \in]x, x + \epsilon_x \cap \mathbf{Q}$. Then, $x \in]p_x, q_x \subseteq U$. It follows that $U = \bigcup_{i \in I} A_i$, where I is the countable set $I = \{]p_x, q_x [: x \in U\}$ and $A_i = i$ for all $i \in I$. We have proved that U can be expressed as a countable union of open bounded intervals in \mathbf{R}^1 .
- 6. For all $n \ge 1$, $]a, b 1/n] \subseteq]a, b[$ and $[a + 1/n, b] \subseteq]a, b[$. Moreover, for all $x \in]a, b[$, there exists $n \ge 1$ with $a + 1/n \le x \le b 1/n$. It follows that:

$$]a, b[=\bigcup_{n=1}^{+\infty}]a, b-1/n] = \bigcup_{n=1}^{+\infty}[a+1/n, b[$$

- 7. For all $a, b \in \mathbf{R}$, $]a, b] =]a, +\infty] \setminus [b, +\infty] = [-\infty, b] \setminus [-\infty, a]$. So $]a, b] \in \sigma(\mathcal{C}_4) \cap \sigma(\mathcal{C}_1)$. Similarly $[a, b] \in \sigma(\mathcal{C}_2) \cap \sigma(\mathcal{C}_3)$. Using 6., we conclude that $]a, b] \in \sigma(\mathcal{C}_i)$, for all $i = 1, \ldots, 4$.
- 8. $\{+\infty\} = \bigcap_n [n, +\infty] = \bigcap_n [n, +\infty] = \bigcap_n [-\infty, n]^c = \bigcap_n [-\infty, n]^c$. We conclude that $\{+\infty\} \in \sigma(\mathcal{C}_i)$, and similarly $\{-\infty\} \in \sigma(\mathcal{C}_i)$, for all $i = 1, \ldots, 4$.
- 9. Let i = 1, ..., 4. Let $U \in \mathcal{T}_{\mathbf{\bar{R}}}$. From 4., $U \cap \mathbf{R} \in \mathcal{T}_{\mathbf{R}}$. From 5., $U \cap \mathbf{R}$ can be expressed as a countable union of open bounded intervals in \mathbf{R} . From 7., any such interval is an element of $\sigma(\mathcal{C}_i)$. It follows that $U \cap \mathbf{R} \in \sigma(\mathcal{C}_i)$. However, $U = (U \cap \mathbf{R}) \uplus A$, where A is either \emptyset , $\{-\infty\}$, $\{+\infty\}$ or $\{-\infty, +\infty\}$. We conclude from 8. that in any case, $U \in \sigma(\mathcal{C}_i)$. We have proved that $\mathcal{T}_{\mathbf{\bar{R}}} \subseteq \sigma(\mathcal{C}_i)$, and therefore $\mathcal{B}(\mathbf{\bar{R}}) \subseteq \sigma(\mathcal{C}_i)$. From 3., $\sigma(\mathcal{C}_i) \subseteq \mathcal{B}(\mathbf{\bar{R}})$. Finally $\sigma(\mathcal{C}_i) = \mathcal{B}(\mathbf{\bar{R}})$.

¹If you think this proof was a bit quick, see Exercise (7) of the previous tutorial.

10. Given $B \subseteq \overline{\mathbf{R}}$, $\{f \in B\}$ denotes $f^{-1}(B)$. $(i) \Leftrightarrow (ii)$ is just definition (35). Similarly, $\{f \leq c\} = f^{-1}([-\infty, c])$, etc... and the equivalence between (i), and (iii), (iv), (v) and (vi), stems from a direct application of theorem (14), using $\sigma(\mathcal{C}_i) = \mathcal{B}(\overline{\mathbf{R}})$.

Exercise 14

Exercise 15.

- 1. Let $\omega \in \{c \leq g\} = g^{-1}([c, +\infty])$. Then $c \leq g(\omega) = \inf_{n\geq 1} f_n(\omega)$. In particular, for all $n \geq 1$, $c \leq f_n(\omega)$. So $\omega \in \bigcap_{n=1}^{+\infty} \{c \leq f_n\}$. Conversely, suppose that $c \leq f_n(\omega)$ for all $n \geq 1$. Then c is a lower-bound of all $f_n(\omega)$'s for $n \geq 1$. $g(\omega)$ being the greatest of such lower-bound, we have $c \leq g(\omega)$. We have proved that $\{c \leq g\} = \bigcap_{n=1}^{+\infty} \{c \leq f_n\}$.
- 2. Let $\omega \in \{h \leq c\} = h^{-1}([-\infty, c])$. Then $\sup_{n\geq 1} f_n(\omega) = h(\omega) \leq c$. In particular, for all $n \geq 1$, $f_n(\omega) \leq c$. So $\omega \in \bigcap_{n=1}^{+\infty} \{f_n \leq c\}$. Conversely, suppose that $f_n(\omega) \leq c$ for all $n \geq 1$. Then c is an upper-bound of all $f_n(\omega)$'s for $n \geq 1$. $h(\omega)$ being the smallest of such upper-bound, we have $h(\omega) \leq c$. We have proved that $\{h \leq c\} = \bigcap_{n=1}^{+\infty} \{f_n \leq c\}$.
- 3. All f_n 's being measurable, using theorem (15), we conclude from 1. and 2. that $g, h: (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are measurable.

Exercise 15

Exercise 16.

- 1. Let $n \geq 1$. For all $k \geq n$, $u_n = \inf_{k \geq n} v_k \leq v_k$. In particular, u_n is a lower-bound of all v_k 's for $k \geq n+1$. u_{n+1} being the greatest of such lower-bound, we see that $u_n \leq u_{n+1}$. From definition (36), we have $u = \sup_{n \geq 1} u_n$. In particular, u is an upper-bound of all u_n 's. We have proved that $u_n \leq u_{n+1} \leq u$.
- 2. Let $n \ge 1$. For all $k \ge n$, $v_k \le \sup_{k\ge n} v_k = w_n$. In particular, w_n is an upper-bound of all v_k 's for $k \ge n+1$. w_{n+1} being the smallest of such upper-bound, we see that $w_{n+1} \le w_n$. From definition (36), we have $w = \inf_{n\ge 1} w_n$. In particular, w is a lower-bound of all w_n 's. We have proved that $w \le w_{n+1} \le w_n$.
- 3. From 1., $(u_n)_{n\geq 1}$ is a non-decreasing sequence in **R**. It therefore converges to $\sup_{n\geq 1} u_n = u$. Indeed, suppose $u = +\infty$. Then, u being the smallest of all u_n 's upper-bounds, for all $A \in \mathbf{R}^+$, there exists $N \geq 1$ such that $A < u_N$. Since $(u_n)_{n\geq 1}$ is non-decreasing, we have $A < u_n$ for all $n \geq N$. It follows that $u_n \uparrow +\infty$. If $u = -\infty$, then $u_n = -\infty$ for all $n \geq 1$ and $u_n \uparrow -\infty$. If $u \in \mathbf{R}$, then given $\epsilon > 0$, $u - \epsilon < u$. So $u - \epsilon$ cannot be an upper-bound of all u_n 's. There exists $N \geq 1$ such that $u - \epsilon < u_N \leq u$. Since $(u_n)_{n\geq 1}$ is non-decreasing, we have $u - \epsilon < u_n \leq u$ for all $n \geq N$. It follows that $u_n \uparrow u$. Similarly, $(w_n)_{n\geq 1}$ being a non-increasing sequence in $\mathbf{\bar{R}}$, it converges to $\inf_{n\geq 1} w_n = w$. So $w_n \downarrow w$.

- 4. For all $n \ge 1$, $u_n = \inf_{k \ge n} v_k \le v_n \le \sup_{k > n} v_k = w_n$.
- 5. From $u_n \leq w_n$, taking the limit as $n \to +\infty$, we obtain $u \leq w$.
- 6. From 5., for all $n \ge 1$, $u_n \le v_n \le w_n$. If u = w, then $(u_n)_{n\ge 1}$ and $(w_n)_{n\ge 1}$ converge to the same limit $u \in \overline{\mathbf{R}}$. It follows that $(v_n)_{n\ge 1}$ also converges to $u \in \overline{\mathbf{R}}$.
- 7. Let $a, b \in \mathbf{R}$, with u < a < b < w. Let $n \ge 1$. In particular, we have $u_n < a < b < w_n$. Since $u_n = \inf_{k\ge n} v_k$, u_n is the greatest lower-bound of all v_k 's for $k \ge n$. It follows that a cannot be such lower-bound. There exists $N_1 \ge n$ such that $v_{N_1} < a$. Similarly, b cannot be an upper-bound of all v_k 's for $k \ge n$. There exists $N_2 \ge n$ such that $b < v_{N_2}$.
- 8. From 7., there exist $n_1, m_1 \geq 1$, such that $v_{n_1} < a < b < v_{m_1}$. Let $n = \max(n_1 + 1, m_1 + 1)$. Using 7. once more, there exist $n_2, m_2 \geq n$ such that $v_{n_2} < a < b < v_{m_2}$. In particular, we have $n_1 < n_2$ and $m_1 < m_2$. By induction, we can therefore construct two strictly increasing sequences of integers $(n_k)_{k\geq 1}$ and $(m_k)_{k\geq 1}$ such that $v_{n_k} < a < b < v_{m_k}$ for all $k \geq 1$.
- 9. Suppose that $(v_n)_{n\geq 1}$ converges to some $v \in \mathbf{R}$. From 5., $u \leq w$. Suppose u < w, and let $a, b \in \mathbf{R}$, u < a < b < w. Using 8., let $(n_k)_{k\geq 1}$ and $(m_k)_{k\geq 1}$ be two strictly increasing sequences of integers such that $v_{n_k} < a < b < v_{m_k}$. Taking the limit as $k \to +\infty$, we obtain $v \leq a < b \leq v$ which is a contradiction. It follows that if $(v_n)_{n\geq 1}$ converges to some $v \in \mathbf{R}$, then u = w.

Exercise 16

Exercise 17.

- 1. Let $\omega \in \{f < g\}$. Then $f(\omega) < g(\omega)$. There exists a rational number $r \in \mathbf{Q}$ such that $f(\omega) < r < g(\omega)$. It follows that $\omega \in \{f < r\} \cap \{r < g\}$. So $\{f < g\} \subseteq \bigcup_{r \in \mathbf{Q}} \{f < r\} \cap \{r < g\}$. The reverse inclusion is clear.
- 2. Since f and g are measurable, $\{f < r\} = f^{-1}([-\infty, r[) \text{ and } \{r < g\} = g^{-1}(]r, +\infty])$ are both elements of \mathcal{F} , for all $r \in \mathbf{Q}$. Using 1., and the fact that \mathbf{Q} is a countable set, it follows that $\{f < g\} \in \mathcal{F}$. Similarly, $\{g < f\} \in \mathcal{F}$. Moreover, we have $\{f \leq g\} = \{g < f\}^c \in \mathcal{F}$ and $\{g \leq f\} = \{f < g\}^c \in \mathcal{F}$. Finally, $\{f = g\} = \{f \leq g\} \cap \{g \leq f\} \in \mathcal{F}$.

Exercise 17

Exercise 18.

1. Let $g_n = \inf_{k \ge n} f_k$ and $h_n = \sup_{k \ge n} f_k$, for all $n \ge 1$. Being a countable infimum and supremum of measurable maps, using exercise (15), we see that g_n and h_n are measurable for all $n \ge 1$. Since $g = \sup_{n \ge 1} g_n$ and $h = \inf_{n \ge 1} h_n$, we conclude also from exercise (15), that $g, h : (\Omega, \mathcal{F}) \to$ $(\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are measurable.

- 2. Using 5. of exercise (16), $g(\omega) \leq h(\omega)$, for all $\omega \in \Omega$. So $g \leq h$.
- 3. Since $f, g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are measurable, using exercise (17), we conclude that $\{g = h\} \in \mathcal{F}$.
- 4. The set $\{\omega : \omega \in \Omega, \lim_{n \to +\infty} f_n(\omega) \text{ exists in } \mathbf{R}\}$ is by virtue of theorem (16), equal to $\{g = h\}$. From 3., it is therefore an element of \mathcal{F} .
- 5. If $f_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$, using theorem (16), f = g = h. From 1., $f: (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is itself measurable.

Exercise 18

Exercise 19.

- 1. For all $c \in \mathbf{R}$, $\{-f < c\} = \{-c < f\}$. From theorem (15), we see that -f is measurable. From $\{|f| < c\} = \{-c < f\} \cap \{f < c\}, |f|$ is measurable. If $c \leq 0$, then $\{f^+ < c\} = \emptyset$. If c > 0, then $\{f^+ < c\} = \{f < c\}$. In any case $\{f^+ < c\} \in \mathcal{F}$ and it follows that f^+ is measurable. Similarly, f^- is measurable.
- 2. An expression of the form $(+\infty) + (-\infty)$ is meaningless. Since f takes values in $\bar{\mathbf{R}}$, given $a \in \bar{\mathbf{R}}$ and $\omega \in \Omega$, the sum $a + f(\omega)$ may not be meaningful.
- 3. Let $a \in \mathbf{R}$. Then a + f is meaningful as a map defined on Ω . Given $c \in \mathbf{R}$, we have $\{a + f < c\} = \{f < c a\}$. We conclude from theorem (15) that a + f is measurable.
- 4. Let $a \in \mathbf{R}$. From 1., -f is measurable whenever f is measurable. Without loss of generality, we can therefore assume that $a \ge 0$. If $0 < a < +\infty$, then for all $c \in \mathbf{R}$, $\{a.f < c\} = \{f < c/a\}$. It follows from theorem (15) that a.f is measurable. If a = 0, since by convention $0.(+\infty) = 0.(-\infty) = 0$, we have a.f = 0. Given $c \in \mathbf{R}$, $\{a.f < c\}$ is either \emptyset or Ω . In any case $\{a.f < c\} \in \mathcal{F}$, and a.f is measurable. If $a = +\infty$, then for all $c \in \mathbf{R}$, we have $\{a.f < c\} = \{f < 0\}$ if $c \le 0$, and $\{a.f < c\} = \{f < 0\} \ \mbox{ ff } = 0\}$ if c > 0. In any case, $\{a.f < c\} \in \mathcal{F}$ and a.f is measurable.
- 5. Given $\omega \in \Omega$, the sum $f(\omega) + g(\omega)$ may not be meaningful.
- 6. If $f \ge 0$ and $g \ge 0$, the sum f + g is meaningful as a map defined on Ω . Let $\omega \in \{f + g < c\}$ where $c \in \mathbf{R}$. In particular, $g(\omega) < +\infty$. Subtracting $g(\omega)$ from both side of the inequality, we obtain $f(\omega) < c - g(\omega)$, i.e. $\omega \in \{f < c - g\}$. Conversely, if $f(\omega) < c - g(\omega)$, then $g(\omega)$ is again finite, and $f(\omega) + g(\omega) < c$. So $\{f + g < c\} = \{f < c - g\}$. This equality may have looked obvious in the first place. However, it is easy to make mistake with algebra and inequalities involving $+\infty$ and $-\infty$... From 1., -g is a measurable map. Using 3., for all $c \in \mathbf{R}$, c - g is also measurable. From exercise (17), $\{f < c - g\} \in \mathcal{F}$. Finally, using theorem (15), we conclude

that $f + g : (\Omega, \mathcal{F}) \to (\mathbf{\bar{R}}, \mathcal{B}(\mathbf{\bar{R}}))$ is measurable. The sum of two non-negative and measurable maps, is itself a non-negative and measurable map.

7. Suppose we have:

$$(\{f=+\infty\}\cap\{g=-\infty\})\cup(\{f=-\infty\}\cap\{g=+\infty\})=\emptyset$$

Then f + g is meaningful as a map defined on Ω . As in 6., given $c \in \mathbf{R}$ we wish to argue that $\{f + g < c\} = \{f < c - g\}$. Given $\omega \in \Omega$, this amounts to checking the equivalence between the two inequalities $f(\omega) + g(\omega) < c$ and $f(\omega) < c - g(\omega)$, which is obviously true in the case when $f(\omega), g(\omega) \in \mathbf{R}$. Since the only other possible case is $f(\omega) = g(\omega) = +\infty$ or $f(\omega) = g(\omega) = -\infty$, such equivalence is clear and we have proved that the equality $\{f + g < c\} = \{f < c - g\}$ holds. As in 6. we conclude that $f + g : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is measurable. The sum of two \mathbf{R} -valued measurable maps is itself measurable, provided it is well-defined.

- 8. If $f(\Omega) \subseteq \mathbf{R} \setminus \{0\}$, then 1/f is meaningful as a map defined on Ω . Let $c \in \mathbf{R}$. If c > 0, then $\{1/f < c\} = \{f < 0\} \uplus \{f > 1/c\}$. If c = 0, then $\{1/f < c\} = \{f < 0\}$. In the final case when c < 0, we have $\{1/f < c\} = \{1/c < f\} \cap \{f < 0\}$. In any case, $\{1/f < c\} \in \mathcal{F}$, and we conclude from theorem (15) that $1/f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.
- 9. Let $B \in \mathcal{B}(\bar{\mathbf{R}})$. Then $\{\bar{f} \in B\} = (\{f \in B\} \cap \{f = 0\}^c) \uplus \{f = 0\}$, if $1 \in B$. Otherwise, $\{\bar{f} \in B\} = \{f \in B\} \cap \{f = 0\}^c$. In any case, $\{\bar{f} \in B\} \in \mathcal{F}$ and $\bar{f} : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.
- 10. We have $\Omega = \{f > 0\} \uplus \{f < 0\} \uplus \{f = 0\}$. If $f(\omega) > 0$, then $f(\omega)g(\omega) < c$ is equivalent to $g(\omega) < c/\overline{f}(\omega)$. If $f(\omega) < 0$, then $f(\omega)g(\omega) < c$ is equivalent to $g(\omega) > c/\overline{f}(\omega)$. Finally, if $f(\omega) = 0$, then $f(\omega)g(\omega) < c$ is equivalent to $f(\omega) < c$. It follows that $\{fg < c\}$ can be expressed as:

$$(\{f > 0\} \cap \{g < c/\bar{f}\}) \uplus (\{f < 0\} \cap \{g > c/\bar{f}\}) \uplus (\{f = 0\} \cap \{f < c\})$$

11. Whether or not f and g take values in \mathbf{R} , the product fg is meaningful as a map defined on Ω . In the case when $f(\Omega) \subseteq \mathbf{R}$ and $g(\Omega) \subseteq \mathbf{R}$, given $c \in \mathbf{R}$, we can use the decomposition of $\{fg < c\}$ obtained in 10. Furthermore, from 9., \bar{f} is a measurable map with values in $\mathbf{R} \setminus \{0\}$. Using 8., $1/\bar{f}$ is measurable. From 4., c/\bar{f} is also measurable. It follows from exercise (17), that $\{g < c/\bar{f}\} \in \mathcal{F}$ and $\{g > c/\bar{f}\} \in \mathcal{F}$. Hence, all sets involved in 10. are elements of \mathcal{F} . So $\{fg < c\} \in \mathcal{F}$. We conclude from theorem (15) that $fg: (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable. In the following exercise, we shall extend this result to the more general case when f and g have arbitrary values in $\bar{\mathbf{R}}$.

Exercise 19

Exercise 20.

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1. For all $B \in \mathcal{B}(\bar{\mathbf{R}})$, the inverse image $\bar{f}^{-1}(B)$ can be written as:

$$\bar{f}^{-1}(B) = (f^{-1}(B) \cap f^{-1}(\mathbf{R})) \uplus (A \cap (\{f = +\infty\} \uplus \{f = -\infty\}))$$

where $A = \Omega$ if $1 \in B$, and $A = \emptyset$ otherwise. It follows that $\bar{f}^{-1}(B) \in \mathcal{F}$, and \bar{f} is measurable. Similarly, \bar{g} is measurable.

2. All A_i 's and B_j 's are inverse images of Borel sets in $\mathbf{\bar{R}}$, by measurable maps. They are therefore elements of \mathcal{F} .

3. Since $\Omega = \bigcup_{i,j=1}^{5} A_i \cap B_j$, for all $B \in \mathcal{B}(\bar{\mathbf{R}})$, we have:

$$\{fg \in B\} = \biguplus_{i,j=1}^{5} (A_i \cap B_j \cap \{fg \in B\})$$

- 4. For all $1 \leq i, j \leq 3$ and $\omega \in A_i \cap B_j$, $f(\omega) \in \mathbf{R}$ and $g(\omega) \in \mathbf{R}$. In particular, $f(\omega) = \overline{f}(\omega)$, and $g(\omega) = \overline{g}(\omega)$. Hence, we conclude that $A_i \cap B_j \cap \{fg \in B\} = A_i \cap B_j \cap \{\overline{fg} \in B\}$.
- 5. Suppose $i \ge 4$ or $j \ge 4$. Then, for all $\omega \in A_i \cap B_j$, $f(\omega)g(\omega)$ is either $-\infty$, 0 or $+\infty$. More specifically, $f(\omega)g(\omega) = a$, with:

$$a = \begin{cases} -\infty & \text{if} \quad (i,j) \in \{(1,4), (2,5), (4,5), (5,4), (5,2), (4,1)\} \\ 0 & \text{if} \quad (i,j) \in \{(3,4), (3,5), (4,3), (5,3)\} \\ +\infty & \text{if} \quad (i,j) \in \{(1,5), (2,4), (4,4), (5,5), (5,1), (4,2)\} \end{cases}$$

Hence, given $B \in \mathcal{B}(\mathbf{\bar{R}})$, $A_i \cap B_j \cap \{fg \in B\} = \emptyset$ if $a \notin B$, and $A_i \cap B_j \cap \{fg \in B\} = A_i \cap B_j$ if $a \in B$.

6. Let $B \in \mathcal{B}(\bar{\mathbf{R}})$. From 1., \bar{f} and \bar{g} are measurable. Moreover, by construction, both \bar{f} and \bar{g} take values in \mathbf{R} . From exercise (19), it follows that $\bar{f}\bar{g}$ is measurable. Hence, $\{\bar{f}\bar{g} \in B\} \in \mathcal{F}$. From 2., all A_i 's and B_j 's are elements of \mathcal{F} . Using 4., whenever $1 \leq i, j \leq 3, A_i \cap B_j \cap \{fg \in B\} \in \mathcal{F}$. However, from 5., we also have $A_i \cap B_j \cap \{fg \in B\} \in \mathcal{F}$, for all $i \geq 4$ or $j \geq 4$. We conclude from 3. that $\{fg \in B\} \in \mathcal{F}$. We have proved that $fg: (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

Exercise 20

Exercise 21.

1. Let $x \in A$. Suppose $U \in \mathcal{T}$ is such that $x \in U$. Then $x \in U \cap A$. In particular, $U \cap A \neq \emptyset$. So $x \in \overline{A}$. We have proved that $A \subseteq \overline{A}$. Suppose $x \notin \overline{A}$. From definition (37), there exists an open set $U_x \in \mathcal{T}$ such that $x \in U_x$ and $U_x \cap A = \emptyset$. Moreover, for all $y \in U_x$, from $U_x \in \mathcal{T}$, $U_x \cap A = \emptyset$ and definition (37), we see that $y \notin \overline{A}$. Hence, for all $x \in \overline{A^c}$, there exists $U_x \in \mathcal{T}$, such that $x \in U_x \subseteq \overline{A^c}$. It follows that $\overline{A^c} = \bigcup_{x \notin \overline{A}} U_x$, and $\overline{A^c}$ is therefore an open set in E. Hence, \overline{A} is closed in E.

- 2. Suppose that B is closed and $A \subseteq B$. Then $B^c \in \mathcal{T}$. Suppose that $\bar{A} \subseteq B$ is false. There exists $x \in \bar{A} \cap B^c$. From $x \in B^c \in \mathcal{T}$ and definition (37), we see that $B^c \cap A \neq \emptyset$. This contradicts the assumption that $A \subseteq B$. It follows that $\bar{A} \subseteq B$.
- 3. From 1., \overline{A} is indeed a closed set containing A. From 2., \overline{A} is the smallest closed set containing A.
- 4. Suppose $A = \overline{A}$. Then from 1., A is closed. Conversely, suppose that A is closed. Since $A \subseteq A$, using 2., $\overline{A} \subseteq A$. However from 1., $A \subseteq \overline{A}$. So $A = \overline{A}$. We have proved that A is closed, if and only if $A = \overline{A}$.
- 5. Suppose \mathcal{T} is the metric topology associated with some metric d on E. Let A' be defined by:

$$A' = \{ x \in E : \forall \epsilon > 0 , B(x, \epsilon) \cap A \neq \emptyset \}$$

Let $x \in \overline{A}$. For all $\epsilon > 0$, from exercise (2), $B(x, \epsilon)$ is an open set in E, which furthermore contains x. Hence, from definition (37), $B(x, \epsilon) \cap A \neq \emptyset$ and we see that $x \in A'$. So $\overline{A} \subseteq A'$. Conversely, suppose $x \in A'$. Let $U \in \mathcal{T}$ be such that $x \in U$. \mathcal{T} being the metric topology, from definition (30), there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. However, since $x \in A', B(x, \epsilon) \cap A \neq \emptyset$. In particular, $U \cap A \neq \emptyset$. It follows that $x \in \overline{A}$, and $A' \subseteq \overline{A}$. We have proved that $\overline{A} = A'$.

Exercise 21

Exercise 22.

1. By definition, for all $y \in \overline{A}$, $d(x, \overline{A}) \leq d(x, y)$. From exercise (21), $A \subseteq \overline{A}$. It follows that $d(x, \overline{A})$ is a lower-bound of all d(x, y) for $y \in A$. d(x, A) being the greatest of such lower-bound, we have $d(x, \overline{A}) \leq d(x, A)$. Suppose $d(x, \overline{A}) < d(x, A)$. Let $\alpha \in \mathbf{R}$ be such that $d(x, \overline{A}) < \alpha < d(x, A)$. It follows from $d(x, \overline{A}) < \alpha$, that α cannot be a lower-bound of all d(x, y) for $y \in \overline{A}$. There exists $y \in \overline{A}$ such that $d(x, y) < \alpha$. Since $y \in \overline{A}$, from exercise (21), for all $\epsilon > 0$, $B(y, \epsilon) \cap A \neq \emptyset$. There exists $z \in A$ such that $d(y, z) < \epsilon$. In particular:

$$d(x,A) \le d(x,z) \le d(x,y) + d(y,z) < \alpha + \epsilon$$

 $\epsilon > 0$ being arbitrary, it follows that $d(x, A) \leq \alpha$. This is a contradiction. We conclude that $d(x, \overline{A}) = d(x, A)$.

Suppose that d(x, A) = 0. For all ε > 0, ε cannot be a lower-bound of all d(x, y) for y ∈ A. There exists y ∈ A, such that d(x, y) < ε. In other words, B(x, ε) ∩ A ≠ Ø. Hence, from exercise (21), x ∈ Ā. Conversely, suppose x ∈ Ā. Then for all ε > 0, B(x, ε) ∩ A ≠ Ø. Let y ∈ B(x, ε) ∩ A. We have d(x, A) ≤ d(x, y) < ε. ε > 0 being arbitrary, it follows that d(x, A) ≤ 0. However, 0 is a lower-bound of all d(x, y) for y ∈ A. So 0 ≤ d(x, A). Hence d(x, A) = 0. We have proved that d(x, A) = 0, if and only if x ∈ Ā.

3. Let $x, y \in E$. For all $z \in A$, we have:

$$d(x, A) \le d(x, z) \le d(x, y) + d(y, z)$$

Subtracting $d(x, y) \in \mathbf{R}^+$ from both side of the inequality, the difference d(x, A) - d(x, y) appears as a lower-bound of all d(y, z) for $z \in A$. d(y, A) being the greatest of such lower-bound, $d(x, A) - d(x, y) \leq d(y, A)$. Hence, $d(x, A) \leq d(x, y) + d(y, A)$.

- 4. Let $x, y \in E$. If $A \neq \emptyset$, there exists $z \in A$. From the inequality $d(x, A) \leq d(x, z)$, we have in particular $d(x, A) < +\infty$ and similarly $d(y, A) < +\infty$. The difference d(x, A) - d(y, A) is therefore meaningful. $d(x, A) \leq d(x, y) + d(y, A)$ is obtained from 3. Similarly, $d(y, A) \leq d(y, x) + d(x, A)$. It follows that $|d(x, A) - d(y, A)| \leq d(x, y)$.
- 5. If $A = \emptyset$, then for all $x \in E$, $\Phi_A(x) = +\infty$. The map Φ_A is therefore continuous. If $A \neq \emptyset$, then from 4., for all $x, y \in E$, $|\Phi_A(x) - \Phi_A(y)| \leq d(x, y)$. From theorem (12), the induced topology on \mathbf{R}^+ coincide with the metric topology. Using exercise (4), it follows that $\Phi_A : (E, \mathcal{T}_E^d) \to (\mathbf{R}^+, \mathcal{T}_{\mathbf{R}^+})$ is continuous. However, for all $U \in \mathcal{T}_{\mathbf{\bar{R}}}, U \cap \mathbf{R}^+ \in \mathcal{T}_{\mathbf{R}^+}$ and therefore, $\Phi_A^{-1}(U) = \Phi_A^{-1}(U \cap \mathbf{R}^+) \in \mathcal{T}_E^d$. So $\Phi_A : (E, \mathcal{T}_E^d) \to (\mathbf{\bar{R}}, \mathcal{T}_{\mathbf{\bar{R}}})$ is also continuous. Note that $\delta(u, v) = |u - v|$ is not a metric on $\mathbf{\bar{R}}$. Hence, we could not use exercise (4) to prove directly the continuity of Φ_A , viewed as a map with values in $\mathbf{\bar{R}}$.
- 6. Suppose that A is closed. From exercise (21), $A = \overline{A}$. Hence, from 2., d(x, A) = 0 is equivalent to $x \in A$. So $A = \Phi_A^{-1}(\{0\})$.

Exercise 22

Exercise 23.

- 1. The upper and lower limits as defined in definition (36), require the notions of infimums and supremums. Such notions may not be meaningful on an arbitrary metric space (E, d).
- 2. Let \mathcal{A} be the set of all closed sets in E. \mathcal{T}_E^d being the metric topology on E, the Borel σ -algebra on E is generated by \mathcal{T}_E^d , i.e. $\mathcal{B}(E) = \sigma(\mathcal{T}_E^d)$. In fact, $\mathcal{B}(E)$ is also generated by \mathcal{A} . Indeed, for all $A \in \mathcal{A}$, $A^c \in \mathcal{T}_E^d$. In particular $A^c \in \mathcal{B}(E)$, and therefore we have $A \in \mathcal{B}(E)$. So $\mathcal{A} \subseteq \mathcal{B}(E)$ and consequently, $\sigma(\mathcal{A}) \subseteq \mathcal{B}(E)$. However, for all $U \in \mathcal{T}_E^d$, $U^c \in \mathcal{A}$. In particular, $U^c \in \sigma(\mathcal{A})$, and therefore $U \in \sigma(\mathcal{A})$. So $\mathcal{T}_E^d \subseteq \sigma(\mathcal{A})$ and consequently, we have $\mathcal{B}(E) \subseteq \sigma(\mathcal{A})$. We have proved that $\mathcal{B}(E) = \sigma(\mathcal{A})$. From theorem (14), we conclude that a map $f : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is measurable, if and only if $f^{-1}(\mathcal{A}) \in \mathcal{F}$, for all $\mathcal{A} \in \mathcal{A}$.
- 3. Let A be closed in E. From exercise (22), $A = \Phi_A^{-1}(\{0\})$. Hence, $f^{-1}(A) = f^{-1}(\Phi_A^{-1}(\{0\})) = (\Phi_A \circ f)^{-1}(\{0\})$.

- 4. Let $n \geq 1$. By assumption, $f_n : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is measurable. From exercise (22), $\Phi_A : (E, \mathcal{T}_E^d) \to (\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is continuous. Using exercise (13), it follows that $\Phi_A : (E, \mathcal{B}(E)) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable. We conclude from exercise (11) that the map $\Phi_A \circ f_n : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable. Note that this is true for all $A \subseteq E$, irrespective of whether or not A is closed.
- 5. Let $A \subseteq E$. By assumption, for all $\omega \in \Omega$, $f_n(\omega) \to f(\omega)$. Since Φ_A is continuous, it follows that $\Phi_A \circ f_n(\omega) \to \Phi_A \circ f(\omega)$. A more direct justification of this fact is as follows: $f_n(\omega) \to f(\omega)$ is a short way of saying that given $\epsilon > 0$, there exists $N \ge 1$, such that $n \ge N$ implies that $d(f_n(\omega), f(\omega)) < \epsilon$. In the case when $A \neq \emptyset$, from exercise (22), we see that $n \ge N$ also implies that $|\Phi_A(f_n(\omega)) - \Phi_A(f(\omega))| \le d(f_n(\omega), f(\omega)) < \epsilon$. Hence, $\Phi_A \circ f_n(\omega) \to \Phi_A \circ f(\omega)$. The fact that this is still true when $A = \emptyset$ is clear. Since $\Phi_A \circ f_n$ is a measurable map for all $n \ge 1$, we see from exercise (18) that $\Phi_A \circ f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable. In particular, $(\Phi_A \circ f)^{-1}(\{0\}) \in \mathcal{F}$. However, from 3., $(\Phi_A \circ f)^{-1}(\{0\}) =$ $f^{-1}(A)$, whenever A is closed in E. We have proved that $f^{-1}(A) \in \mathcal{F}$, for all A closed in E. From 2., we conclude that $f : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is measurable. The purpose of this exercise is to prove theorem (17).

Exercise 23

Exercise 24. For all $z, z' \in \mathbf{C}$, we have $|Re(z) - Re(z')| \leq |z - z'|$, $|Im(z) - Im(z')| \leq |z - z'|$ and $||z| - |z'|| \leq |z - z'|$. From exercise (4), it follows that Re, Im, |.|: $(\mathbf{C}, \mathcal{T}_{\mathbf{C}}) \to (\mathbf{R}, \mathcal{T}_{\mathbf{R}})$ are all continuous maps. From exercise (13), Re, Im, |.|: $(\mathbf{C}, \mathcal{B}(\mathbf{C})) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ are therefore measurable. Since $f: (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is measurable, using exercise (11), we conclude that $u = Re \circ f, v = Im \circ f$ and $|f| = |.| \circ f$ are all measurable with respect to \mathcal{F} and $\mathcal{B}(\mathbf{R})$. In fact, using exercise (10), they are also measurable with respect to \mathcal{F} and $\mathcal{B}(\mathbf{R})$. Essentially, this last point is a direct consequence of the fact that given $B \in \mathcal{B}(\mathbf{R}), B \cap \mathbf{R} \in \mathcal{B}(\mathbf{R})$.

Exercise 24

Exercise 25.

- 1. Let $A =]a, b[\times]c, d[\in \mathcal{C}, \text{ and } z = x + iy \in A$. Then $x \in]a, b[$ and $y \in]c, d[$. Let $\epsilon > 0$ be such that $|x - x'| < \epsilon \Rightarrow x' \in]a, b[$, and $|y - y'| < \epsilon \Rightarrow y' \in]c, d[$. Then $|z - z'| < \epsilon \Rightarrow z' \in A$, for all $z' \in \mathbf{C}$. Hence, there exists $\epsilon > 0$ such that $B(z, \epsilon) \subseteq A$. We have proved that A is open in \mathbf{C} .
- 2. From 1., $C \subseteq T_{\mathbf{C}}$. In particular, $C \subseteq \mathcal{B}(\mathbf{C})$. The σ -algebra $\sigma(\mathcal{C})$ generated by \mathcal{C} being the smallest σ -algebra on \mathbf{C} containing \mathcal{C} , we conclude that $\sigma(\mathcal{C}) \subseteq \mathcal{B}(\mathbf{C})$.
- 3. If $|x| < \eta$ and $|y| < \eta$, then $|z| \le \sqrt{x^2 + y^2} < \sqrt{2\eta}$.
- 4. Let U be open in C, and $z = x + iy \in U$. There exists $\epsilon > 0$, such that $B(z,\epsilon) \subseteq U$. Let $\eta = \epsilon/\sqrt{2}$. Using 3., we have $]x \eta, x + \eta[\times]y \eta, y + \eta[\subseteq$

U. Let $a_z \in]x - \eta, x[\cap \mathbf{Q}, \text{ and } b_z \in]x, x + \eta[\cap \mathbf{Q}.$ Let $c_z \in]y - \eta, y[\cap \mathbf{Q} \text{ and } d_z \in]y, y + \eta[\cap \mathbf{Q}.$ Then, we have $z \in]a_z, b_z[\times]c_z, d_z[\subseteq U.$

- 5. Let *I* be the set $I = \{]a_z, b_z[\times]c_z, d_z[, z \in U\}$. Then *I* is finite or countable, and $U = \bigcup_{i \in I} B_i$ where $B_i = i \in \mathcal{C}$, for all $i \in I$. In order to express *U* as a union indexed by the set of positive integers \mathbf{N}^* , the following can be done: Let $\psi : I \to \mathbf{N}^*$ be an arbitrary injection. For all $n \ge 1$, define A_n as $A_n = B_i$ if $n \in \psi(I)$ and $n = \psi(i)$, and $A_n = \emptyset$ if $n \notin \psi(I)$. Then, $A_n \in \mathcal{C}$ for all $n \ge 1$, and we have $U = \bigcup_{n=1}^{+\infty} A_n$.
- 6. It follows from 5. that $\mathcal{T}_{\mathbf{C}} \subseteq \sigma(\mathcal{C})$. The Borel σ -algebra $\mathcal{B}(\mathbf{C})$ being the smallest σ -algebra on \mathbf{C} containing all open sets, we see that $\mathcal{B}(\mathbf{C}) \subseteq \sigma(\mathcal{C})$. Hence, from 2., $\sigma(\mathcal{C}) = \mathcal{B}(\mathbf{C})$.
- 7. Let f = u + iv. Then, $f^{-1}(A) = u^{-1}(]a, b[) \cap v^{-1}(]c, d[)$, for all $A =]a, b[\times]c, d[\in \mathcal{C}$. Since u and v are assumed to be measurable, $u^{-1}(]a, b[) \in \mathcal{F}$ and $v^{-1}(]c, d[) \in \mathcal{F}$. It follows that $f^{-1}(A) \in \mathcal{F}$. Using 6., we conclude from theorem (14) that f is measurable with respect to \mathcal{F} and $\mathcal{B}(\mathbf{C})$.

Exercise 25