12. Radon-Nikodym Theorem

In the following, $(\Omega, \mathcal{F})$ is an arbitrary measurable space.

Definition 96 *Let μ and ν be two (possibly complex) measures on $(\Omega, \mathcal{F})$. We say that ν is* **absolutely continuous** *with respect to μ, and we write $\nu << \mu$, if and only if, for all $E \in \mathcal{F}$:*

$$\mu(E) = 0 \ \Rightarrow \ \nu(E) = 0$$

Exercise 1. Let μ be a measure on $(\Omega, \mathcal{F})$ and $\nu \in M^1(\Omega, \mathcal{F})$. Show that $\nu << \mu$ is equivalent to $|\nu| << \mu$.

Exercise 2. Let μ be a measure on $(\Omega, \mathcal{F})$ and $\nu \in M^1(\Omega, \mathcal{F})$. Let $\epsilon > 0$. Suppose there exists a sequence $(E_n)_{n\geq 1}$ in $\mathcal{F}$ such that:

$$\forall n \geq 1 \ , \ \mu(E_n) \leq \frac{1}{2^n} \ , \ |\nu(E_n)| \geq \epsilon$$

Define:

$$E \stackrel{\triangle}{=} \limsup_{n\geq 1} E_n \stackrel{\triangle}{=} \bigcap_{n\geq 1} \bigcup_{k\geq n} E_k$$

1. Show that:

$$\mu(E) = \lim_{n\to+\infty} \mu\left(\bigcup_{k\geq n} E_k\right) = 0$$

2. Show that:

$$|\nu|(E) = \lim_{n\to+\infty} |\nu|\left(\bigcup_{k\geq n} E_k\right) \geq \epsilon$$

3. Let λ be a measure on $(\Omega, \mathcal{F})$. Can we conclude in general that:

$$\lambda(E) = \lim_{n\to+\infty} \lambda\left(\bigcup_{k\geq n} E_k\right)$$

4. Prove the following:

Theorem 58 *Let μ be a measure on $(\Omega, \mathcal{F})$ and ν be a complex measure on $(\Omega, \mathcal{F})$. The following are equivalent:*

$$\begin{array}{ll} (i) & \nu << \mu \\ (ii) & |\nu| << \mu \\ (iii) & \forall \epsilon > 0, \exists \delta > 0, \forall E \in \mathcal{F}, \mu(E) \leq \delta \Rightarrow |\nu(E)| < \epsilon \end{array}$$

Exercise 3. Let μ be a measure on $(\Omega, \mathcal{F})$ and $\nu \in M^1(\Omega, \mathcal{F})$ such that $\nu << \mu$. Let $\nu_1 = Re(\nu)$ and $\nu_2 = Im(\nu)$.

1. Show that $\nu_1 << \mu$ and $\nu_2 << \mu$.

2. Show that $\nu_1^+, \nu_1^-, \nu_2^+, \nu_2^-$ are absolutely continuous w.r. to μ.

EXERCISE 4. Let μ be a finite measure on $(\Omega, \mathcal{F})$ and $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Let S be a closed proper subset of $\mathbf{C}$. We assume that for all $E \in \mathcal{F}$ such that $\mu(E) > 0$, we have:

$$\frac{1}{\mu(E)} \int_E f d\mu \ \in \ S$$

1. Show there is a sequence $(D_n)_{n\geq 1}$ of closed discs in $\mathbf{C}$, with:

$$S^c = \bigcup_{n=1}^{+\infty} D_n$$

   Let $\alpha_n \in \mathbf{C}$, $r_n > 0$ be such that $D_n = \{z \in \mathbf{C} : \ |z - \alpha_n| \leq r_n\}$.

2. Suppose $\mu(E_n) > 0$ for some $n \geq 1$, where $E_n = \{f \in D_n\}$. Show that:

$$\left| \frac{1}{\mu(E_n)} \int_{E_n} f d\mu - \alpha_n \right| \leq \frac{1}{\mu(E_n)} \int_{E_n} |f - \alpha_n| d\mu \leq r_n$$

3. Show that for all $n \geq 1$, $\mu(\{f \in D_n\}) = 0$.

4. Prove the following:

Theorem 59 *Let μ be a finite measure on $(\Omega, \mathcal{F})$, $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Let S be a closed subset of $\mathbf{C}$ such that for all $E \in \mathcal{F}$ with $\mu(E) > 0$, we have:*

$$\frac{1}{\mu(E)} \int_E f d\mu \ \in \ S$$

Then, $f \in S$ μ-a.s.

EXERCISE 5. Let μ be a σ-finite measure on $(\Omega, \mathcal{F})$. Let $(E_n)_{n\geq 1}$ be a sequence in $\mathcal{F}$ such that $E_n \uparrow \Omega$ and $\mu(E_n) < +\infty$ for all $n \geq 1$. Define $w : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ as:

$$w \stackrel{\triangle}{=} \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{1}{1 + \mu(E_n)} 1_{E_n}$$

1. Show that for all $\omega \in \Omega$, $0 < w(\omega) \leq 1$.

2. Show that $w \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

EXERCISE 6. Let μ be a σ-finite measure on $(\Omega, \mathcal{F})$ and ν be a finite measure on $(\Omega, \mathcal{F})$, such that $\nu << \mu$. Let $w \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ be such that $0 < w \leq 1$. We define $\bar{\mu} = \int w d\mu$, i.e.

$$\forall E \in \mathcal{F} \ , \ \bar{\mu}(E) \stackrel{\triangle}{=} \int_E w d\mu$$

1. Show that $\bar{\mu}$ is a finite measure on $(\Omega, \mathcal{F})$.

2. Show that $\phi = \nu + \bar{\mu}$ is also a finite measure on $(\Omega, \mathcal{F})$.

3. Show that for all $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$, we have $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu)$, $fw \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, and:
$$\int f d\phi = \int f d\nu + \int f w d\mu$$

4. Show that for all $f \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$, we have:
$$\int |f| d\nu \leq \int |f| d\phi \leq \left(\int |f|^2 d\phi\right)^{\frac{1}{2}} (\phi(\Omega))^{\frac{1}{2}}$$

5. Show that $L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi) \subseteq L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu)$, and for $f \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$:
$$\left|\int f d\nu\right| \leq \sqrt{\phi(\Omega)}.\|f\|_2$$

6. Show the existence of $g \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$ such that:
$$\forall f \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi) \ , \ \int f d\nu = \int f g d\phi \tag{1}$$

7. Show that for all $E \in \mathcal{F}$ such that $\phi(E) > 0$, we have:
$$\frac{1}{\phi(E)} \int_E g d\phi \ \in \ [0, 1]$$

8. Show the existence of $g \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$ such that $g(\omega) \in [0, 1]$ for all $\omega \in \Omega$, and (1) still holds.

9. Show that for all $f \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$, we have:
$$\int f(1 - g) d\nu = \int f g w d\mu$$

10. Show that for all $n \geq 1$ and $E \in \mathcal{F}$,
$$f \stackrel{\triangle}{=} (1 + g + \ldots + g^n) 1_E \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$$

11. Show that for all $n \geq 1$ and $E \in \mathcal{F}$,
$$\int_E (1 - g^{n+1}) d\nu = \int_E g(1 + g + \ldots + g^n) w d\mu$$

12. Define:
$$h \stackrel{\triangle}{=} g w \left(\sum_{n=0}^{+\infty} g^n\right)$$
Show that if $A = \{0 \leq g < 1\}$, then for all $E \in \mathcal{F}$:
$$\nu(E \cap A) = \int_E h d\mu$$

13. Show that $\{h = +\infty\} = A^c$ and conclude that $\mu(A^c) = 0$.

14. Show that for all $E \in \mathcal{F}$, we have $\nu(E) = \int_E h d\mu$.

15. Show that if μ is σ-finite on $(\Omega, \mathcal{F})$, and ν is a finite measure on $(\Omega, \mathcal{F})$ such that $\nu << \mu$, there exists $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, such that $h \geq 0$ and:

$$\forall E \in \mathcal{F} \ , \ \nu(E) = \int_E h d\mu$$

16. Prove the following:

Theorem 60 (Radon-Nikodym:1) *Let μ be a σ-finite measure on $(\Omega, \mathcal{F})$. Let ν be a complex measure on $(\Omega, \mathcal{F})$ such that $\nu << \mu$. Then, there exists some $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that:*

$$\forall E \in \mathcal{F} \ , \ \nu(E) = \int_E h d\mu$$

If ν is a signed measure on $(\Omega, \mathcal{F})$, we can assume $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.
If ν is a finite measure on $(\Omega, \mathcal{F})$, we can assume $h \geq 0$.

Exercise 7. Let $f = u + iv \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, such that:

$$\forall E \in \mathcal{F} \ , \ \int_E f d\mu = 0$$

where μ is a measure on $(\Omega, \mathcal{F})$.

1. Show that:

$$\int u^+ d\mu = \int_{\{u \geq 0\}} u d\mu$$

2. Show that $f = 0$ μ-a.s.

3. State and prove some uniqueness property in theorem (60).

Exercise 8. Let μ and ν be two σ-finite measures on $(\Omega, \mathcal{F})$ such that $\nu << \mu$. Let $(E_n)_{n \geq 1}$ be a sequence in $\mathcal{F}$ such that $E_n \uparrow \Omega$ and $\nu(E_n) < +\infty$ for all $n \geq 1$. We define:

$$\forall n \geq 1 \ , \ \nu_n \stackrel{\triangle}{=} \nu^{E_n} \stackrel{\triangle}{=} \nu(E_n \cap \cdot)$$

1. Show that there exists $h_n \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ with $h_n \geq 0$ and:

$$\forall E \in \mathcal{F} \ , \ \nu_n(E) = \int_E h_n d\mu \qquad (2)$$

for all $n \geq 1$.

2. Show that for all $E \in \mathcal{F}$,

$$\int_E h_n d\mu \leq \int_E h_{n+1} d\mu$$

3. Show that for all $n, p \geq 1$,

$$\mu(\{h_n - h_{n+1} > \frac{1}{p}\}) = 0$$

4. Show that $h_n \leq h_{n+1}$ μ-a.s.

5. Show the existence of a sequence $(h_n)_{n\geq 1}$ in $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ such that $0 \leq h_n \leq h_{n+1}$ for all $n \geq 1$ and with (2) still holding.

6. Let $h = \sup_{n\geq 1} h_n$. Show that:

$$\forall E \in \mathcal{F} \ , \ \nu(E) = \int_E h d\mu \tag{3}$$

7. Show that for all $n \geq 1$, $\int_{E_n} h d\mu < +\infty$.

8. Show that $h < +\infty$ μ-a.s.

9. Show there exists $h : (\Omega, \mathcal{F}) \to \mathbf{R}^+$ measurable, while (3) holds.

10. Show that for all $n \geq 1$, $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu^{E_n})$.

Theorem 61 (Radon-Nikodym:2) *Let μ and ν be two σ-finite measures on $(\Omega, \mathcal{F})$ such that $\nu << \mu$. There exists a measurable map $h : (\Omega, \mathcal{F}) \to (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$ such that:*

$$\forall E \in \mathcal{F} \ , \ \nu(E) = \int_E h d\mu$$

Exercise 9. Let $h, h' : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Let μ be a σ-finite measure on $(\Omega, \mathcal{F})$. We assume:

$$\forall E \in \mathcal{F} \ , \ \int_E h d\mu = \int_E h' d\mu$$

Let $(E_n)_{n\geq 1}$ be a sequence in $\mathcal{F}$ with $E_n \uparrow \Omega$ and $\mu(E_n) < +\infty$ for all $n \geq 1$. We define $F_n = E_n \cap \{h \leq n\}$ for all $n \geq 1$.

1. Show that for all n and $E \in \mathcal{F}$, $\int_E h d\mu^{F_n} = \int_E h' d\mu^{F_n} < +\infty$.

2. Show that for all $n, p \geq 1$, $\mu(F_n \cap \{h > h' + 1/p\}) = 0$.

3. Show that for all $n \geq 1$, $\mu(\{F_n \cap \{h \neq h'\}) = 0$.

4. Show that $\mu(\{h \neq h'\} \cap \{h < +\infty\}) = 0$.

5. Show that $h = h'$ μ-a.s.

6. State and prove some uniqueness property in theorem (61).

EXERCISE 10. Take $\Omega = \{*\}$ and $\mathcal{F} = \mathcal{P}(\Omega) = \{\emptyset, \{*\}\}$. Let μ be the measure on $(\Omega, \mathcal{F})$ defined by $\mu(\emptyset) = 0$ and $\mu(\{*\}) = +\infty$. Let $h, h' : (\Omega, \mathcal{F}) \to [0, +\infty]$ be defined by $h(*) = 1 \neq 2 = h'(*)$. Show that we have:

$$\forall E \in \mathcal{F}\ ,\ \int_E h d\mu = \int_E h' d\mu$$

Explain why this does not contradict the previous exercise.

EXERCISE 11. Let μ be a complex measure on $(\Omega, \mathcal{F})$.

1. Show that $\mu << |\mu|$.
2. Show the existence of some $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ such that:

$$\forall E \in \mathcal{F}\ ,\ \mu(E) = \int_E h d|\mu|$$

3. If μ is a signed measure, can we assume $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, |\mu|)$?

EXERCISE 12. Further to ex. (11), define $A_r = \{|h| < r\}$ for all $r > 0$.

1. Show that for all measurable partition $(E_n)_{n\geq 1}$ of A_r:

$$\sum_{n=1}^{+\infty} |\mu(E_n)| \leq r|\mu|(A_r)$$

2. Show that $|\mu|(A_r) = 0$ for all $0 < r < 1$.
3. Show that $|h| \geq 1$ $|\mu|$-a.s.
4. Suppose that $E \in \mathcal{F}$ is such that $|\mu|(E) > 0$. Show that:

$$\left| \frac{1}{|\mu|(E)} \int_E h d|\mu| \right| \leq 1$$

5. Show that $|h| \leq 1$ $|\mu|$-a.s.
6. Prove the following:

Theorem 62 *For all complex measure μ on $(\Omega, \mathcal{F})$, there exists h belonging to $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ such that $|h| = 1$ and:*

$$\forall E \in \mathcal{F}\ ,\ \mu(E) = \int_E h d|\mu|$$

If μ is a signed measure on $(\Omega, \mathcal{F})$, we can assume $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, |\mu|)$.

EXERCISE 13. Let $A \in \mathcal{F}$, and $(A_n)_{n\geq 1}$ be a sequence in $\mathcal{F}$.

1. Show that if $A_n \uparrow A$ then $1_{A_n} \uparrow 1_A$.

2. Show that if $A_n \downarrow A$ then $1_{A_n} \downarrow 1_A$.
3. Show that if $1_{A_n} \to 1_A$, then for all $\mu \in M^1(\Omega, \mathcal{F})$:
$$\mu(A) = \lim_{n\to+\infty} \mu(A_n)$$

EXERCISE 14. Let μ be a measure on $(\Omega, \mathcal{F})$ and $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

1. Show that $\nu = \int f d\mu \in M^1(\Omega, \mathcal{F})$.
2. Let $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\nu|)$ be such that $|h| = 1$ and $\nu = \int h d|\nu|$. Show that for all $E, F \in \mathcal{F}$:
$$\int_E f 1_F d\mu = \int_E h 1_F d|\nu|$$
3. Show that if $g : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is bounded and measurable:
$$\forall E \in \mathcal{F} \ , \ \int_E fg d\mu = \int_E hg d|\nu|$$
4. Show that:
$$\forall E \in \mathcal{F} \ , \ |\nu|(E) = \int_E f\bar{h} d\mu$$
5. Show that for all $E \in \mathcal{F}$,
$$\int_E Re(f\bar{h}) d\mu \geq 0 \quad , \quad \int_E Im(f\bar{h}) d\mu = 0$$
6. Show that $f\bar{h} \in \mathbf{R}^+$ μ-a.s.
7. Show that $f\bar{h} = |f|$ μ-a.s.
8. Prove the following:

Theorem 63 *Let μ be a measure on $(\Omega, \mathcal{F})$ and $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Then, $\nu = \int f d\mu$ defined by:*
$$\forall E \in \mathcal{F} \ , \ \nu(E) \stackrel{\triangle}{=} \int_E f d\mu$$
is a complex measure on $(\Omega, \mathcal{F})$ with total variation:
$$\forall E \in \mathcal{F} \ , \ |\nu|(E) = \int_E |f| d\mu$$

EXERCISE 15. Let $\mu \in M^1(\Omega, \mathcal{F})$ be a signed measure. Suppose that $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, |\mu|)$ is such that $|h| = 1$ and $\mu = \int h d|\mu|$. Define $A = \{h = 1\}$ and $B = \{h = -1\}$.

1. Show that for all $E \in \mathcal{F}$, $\mu^+(E) = \int_E \frac{1}{2}(1 + h) d|\mu|$.
2. Show that for all $E \in \mathcal{F}$, $\mu^-(E) = \int_E \frac{1}{2}(1 - h) d|\mu|$.

3. Show that $\mu^+ = \mu^A = \mu(A \cap \cdot)$.
4. Show that $\mu^- = -\mu^B = -\mu(B \cap \cdot)$.

Theorem 64 (Hahn Decomposition) *Let μ be a signed measure on $(\Omega, \mathcal{F})$. There exist $A, B \in \mathcal{F}$, such that $A \cap B = \emptyset$, $\Omega = A \uplus B$ and for all $E \in \mathcal{F}$, $\mu^+(E) = \mu(A \cap E)$ and $\mu^-(E) = -\mu(B \cap E)$.*

Definition 97 *Let μ be a complex measure on $(\Omega, \mathcal{F})$. We define:*

$$L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$$

and for all $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, the **Lebesgue integral** *of f with respect to μ, is defined as:*

$$\int f d\mu \stackrel{\triangle}{=} \int f h d|\mu|$$

where $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ is such that $|h| = 1$ and $\mu = \int h d|\mu|$.

Exercise 16. Let μ be a complex measure on $(\Omega, \mathcal{F})$.

1. Show that for all $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ measurable:
$$f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \ \Leftrightarrow \ \int |f| d|\mu| < +\infty$$
2. Show that for $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, $\int f d\mu$ is unambiguously defined.
3. Show that for all $E \in \mathcal{F}$, $1_E \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $\int 1_E d\mu = \mu(E)$.
4. Show that if μ is a finite measure, then $|\mu| = \mu$.
5. Show that if μ is a finite measure, definition (97) of integral and space $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is consistent with that already known for measures.
6. Show that $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is a $\mathbf{C}$-vector space and that:
$$\int (f + \alpha g) d\mu = \int f d\mu + \alpha \int g d\mu$$
for all $f, g \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $\alpha \in \mathbf{C}$.
7. Show that for all $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we have:
$$\left| \int f d\mu \right| \leq \int |f| d|\mu|$$

Exercise 17. Let $\mu, \nu \in M^1(\Omega, \mathcal{F})$, let $\alpha \in \mathbf{C}$.

1. Show that $|\alpha \nu| = |\alpha| . |\nu|$

2. Show that $|\mu+\nu| \leq |\mu|+|\nu|$
3. Show that $L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu)\cap L^1_{\mathbf{C}}(\Omega,\mathcal{F},\nu)\subseteq L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu+\alpha\nu)$
4. Show that for all $E\in\mathcal{F}$:
$$\int 1_E d(\mu+\alpha\nu)=\int 1_E d\mu+\alpha\int 1_E d\nu$$
5. Show that for all $f\in L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu)\cap L^1_{\mathbf{C}}(\Omega,\mathcal{F},\nu)$:
$$\int f d(\mu+\alpha\nu)=\int f d\mu+\alpha\int f d\nu$$

EXERCISE 18. Let $f:(\Omega,\mathcal{F})\to[0,+\infty]$ be non-negative and measurable. Let μ and ν be measures on $(\Omega,\mathcal{F})$, and $\alpha\in[0,+\infty]$:

1. Show that $\mu+\alpha\nu$ is a measure on $(\Omega,\mathcal{F})$ and:
$$\int f d(\mu+\alpha\nu)=\int f d\mu+\alpha\int f d\nu$$
2. Show that if $\mu\leq\nu$, then:
$$\int f d\mu\leq\int f d\nu$$

EXERCISE 19. Let $\mu\in M^1(\Omega,\mathcal{F})$, $\mu_1=Re(\mu)$ and $\mu_2=Im(\mu)$.

1. Show that $|\mu_1|\leq|\mu|$ and $|\mu_2|\leq|\mu|$.
2. Show that $|\mu|\leq|\mu_1|+|\mu_2|$.
3. Show that $L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu)=L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu_1)\cap L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu_2)$.
4. Show that:
$$\begin{array}{rcl} L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu_1) &=& L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu_1^+)\cap L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu_1^-)\\ L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu_2) &=& L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu_2^+)\cap L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu_2^-)\end{array}$$
5. Show that for all $f\in L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu)$:
$$\int f d\mu=\int f d\mu_1^+-\int f d\mu_1^-+i\left(\int f d\mu_2^+-\int f d\mu_2^-\right)$$

EXERCISE 20. Let $\mu\in M^1(\Omega,\mathcal{F})$. Let $A\in\mathcal{F}$. Let $h\in L^1_{\mathbf{C}}(\Omega,\mathcal{F},|\mu|)$ be such that $|h|=1$ and $\mu=\int hd|\mu|$. Recall that $\mu^A=\mu(A\cap\cdot)$ and $\mu_{|A}=\mu_{|(\mathcal{F}_{|A})}$ where $\mathcal{F}_{|A}=\{A\cap E\ ,\ E\in\mathcal{F}\}\subseteq\mathcal{F}$.

1. Show that we also have $\mathcal{F}_{|A}=\{E:\ E\in\mathcal{F}\ ,\ E\subseteq A\}$.
2. Show that $\mu^A\in M^1(\Omega,\mathcal{F})$ and $\mu_{|A}\in M^1(A,\mathcal{F}_{|A})$.

3. Let $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E. Show:

$$\sum_{n=1}^{+\infty} |\mu^A(E_n)| \leq |\mu|^A(E)$$

4. Show that we have $|\mu^A| \leq |\mu|^A$.

5. Let $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of $A \cap E$. Show that:

$$\sum_{n=1}^{+\infty} |\mu(E_n)| \leq |\mu^A|(A \cap E)$$

6. Show that $|\mu^A|(A^c) = 0$.

7. Show that $|\mu^A| = |\mu|^A$.

8. Let $E \in \mathcal{F}_{|A}$ and $(E_n)_{n\geq 1}$ be an $\mathcal{F}_{|A}$-measurable partition of E. Show that:

$$\sum_{n=1}^{+\infty} |\mu_{|A}(E_n)| \leq |\mu|_{|A}(E)$$

9. Show that $|\mu_{|A}| \leq |\mu|_{|A}$.

10. Let $E \in \mathcal{F}_{|A} \subseteq \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E. Show that $(E_n)_{n\geq 1}$ is also an $\mathcal{F}_{|A}$-measurable partition of E, and conclude:

$$\sum_{n=1}^{+\infty} |\mu(E_n)| \leq |\mu_{|A}|(E)$$

11. Show that $|\mu_{|A}| = |\mu|_{|A}$.

12. Show that $\mu^A = \int h d|\mu^A|$.

13. Show that $h_{|A} \in L^1_{\mathbf{C}}(A, \mathcal{F}_{|A}, |\mu_{|A}|)$ and $\mu_{|A} = \int h_{|A} d|\mu_{|A}|$.

14. Show that for all $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we have:

$$f 1_A \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)\ ,\ f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu^A)\ ,\ f_{|A} \in L^1_{\mathbf{C}}(A, \mathcal{F}_{|A}, \mu_{|A})$$

and:

$$\int f 1_A d\mu = \int f d\mu^A = \int f_{|A} d\mu_{|A}$$

Definition 98 *Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, where μ is a complex measure on $(\Omega, \mathcal{F})$. let $A \in \mathcal{F}$. We call* **partial Lebesgue integral** *of f with respect to μ over A, the integral denoted $\int_A f d\mu$, defined as:*

$$\int_A f d\mu \stackrel{\triangle}{=} \int (f 1_A) d\mu = \int f d\mu^A = \int (f_{|A}) d\mu_{|A}$$

where μ^A is the complex measure on $(\Omega, \mathcal{F})$, $\mu^A = \mu(A \cap \cdot\,)$, $f_{|A}$ is the restriction of f to A and $\mu_{|A}$ is the restriction of μ to $\mathcal{F}_{|A}$, the trace of $\mathcal{F}$ on A.

Exercise 21. Prove the following:

Theorem 65 *Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, where μ is a complex measure on $(\Omega, \mathcal{F})$. Then, $\nu = \int f d\mu$ defined as:*

$$\forall E \in \mathcal{F} \ , \ \nu(E) \stackrel{\triangle}{=} \int_E f d\mu$$

is a complex measure on $(\Omega, \mathcal{F})$, with total variation:

$$\forall E \in \mathcal{F} \ , \ |\nu|(E) = \int_E |f| d|\mu|$$

Moreover, for all measurable map $g : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$, we have:

$$g \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu) \ \Leftrightarrow \ gf \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$$

and when such condition is satisfied:

$$\int g d\nu = \int g f d\mu$$

Exercise 22. Let $(\Omega_1, \mathcal{F}_1), \ldots, (\Omega_n, \mathcal{F}_n)$ be measurable spaces, where $n \geq 2$. Let $\mu_1 \in M^1(\Omega_1, \mathcal{F}_1)$, ..., $\mu_n \in M^1(\Omega_n, \mathcal{F}_n)$. For all $i \in \mathbf{N}_n$, let h_i belonging to $L^1_{\mathbf{C}}(\Omega_i, \mathcal{F}_i, |\mu_i|)$ be such that $|h_i| = 1$ and $\mu_i = \int h_i d|\mu_i|$. For all $E \in \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$, we define:

$$\mu(E) \stackrel{\triangle}{=} \int_E h_1 \ldots h_n d|\mu_1| \otimes \ldots \otimes |\mu_n|$$

1. Show that $\mu \in M^1(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)$
2. Show that for all measurable rectangle $A_1 \times \ldots \times A_n$:

$$\mu(A_1 \times \ldots \times A_n) = \mu_1(A_1) \ldots \mu_n(A_n)$$

3. Prove the following:

Theorem 66 *Let $\mu_1, \ldots, \mu_n$ be n complex measures on measurable spaces $(\Omega_1, \mathcal{F}_1), \ldots, (\Omega_n, \mathcal{F}_n)$ respectively, where $n \geq 2$. There exists a unique complex measure $\mu_1 \otimes \ldots \otimes \mu_n$ on $(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)$ such that for all measurable rectangle $A_1 \times \ldots \times A_n$, we have:*

$$\mu_1 \otimes \ldots \otimes \mu_n(A_1 \times \ldots \times A_n) = \mu_1(A_1) \ldots \mu_n(A_n)$$

Exercise 23. Further to theorem (66) and exercise (22):

1. Show that $|\mu_1 \otimes \ldots \otimes \mu_n| = |\mu_1| \otimes \ldots \otimes |\mu_n|$.
2. Show that $\|\mu_1 \otimes \ldots \otimes \mu_n\| = \|\mu_1\| \ldots \|\mu_n\|$.
3. Show that for all $E \in \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$:

$$\mu_1 \otimes \ldots \otimes \mu_n(E) = \int_E h_1 \ldots h_n d|\mu_1 \otimes \ldots \otimes \mu_n|$$

4. Let $f \in L^1_{\mathbf{C}}(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n, \mu_1 \otimes \ldots \otimes \mu_n)$. Show:
$$\int f d\mu_1 \otimes \ldots \otimes \mu_n = \int f h_1 \ldots h_n d|\mu_1| \otimes \ldots \otimes |\mu_n|$$

5. let σ be a permutation of $\{1, \ldots, n\}$. Show that:
$$\int f d\mu_1 \otimes \ldots \otimes \mu_n = \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$

Solutions to Exercises

Exercise 1. Let μ be a measure on $(\Omega, \mathcal{F})$ and $\nu \in M^1(\Omega, \mathcal{F})$. Suppose that $\nu << \mu$. Let $E \in \mathcal{F}$ be such that $\mu(E) = 0$. Let $(E_n)_{n\geq 1}$ be a measurable partition of E. For each $n \geq 1$, we have $E_n \subseteq E$ and consequently $\mu(E_n) \leq \mu(E)$. It follows that $\mu(E_n) = 0$ for all $n \geq 1$, and from $\nu << \mu$ we obtain that $\nu(E_n) = 0$ for all $n \geq 1$. Hence:

$$\sum_{n=1}^{+\infty} |\nu(E_n)| = 0$$

This being true for all measurable partition $(E_n)_{n\geq 1}$ of E, it follows from definition (94) that $|\nu|(E) = 0$. We have proved the implication that $\mu(E) = 0 \Rightarrow |\nu|(E) = 0$ and consequently $|\nu| << \mu$. Conversely, if $|\nu| << \mu$ and $\mu(E) = 0$, then $|\nu|(E) = 0$. From $|\nu(E)| \leq |\nu|(E)$ we conclude that $\nu(E) = 0$. So $\nu << \mu$. We have proved the equivalence between $\nu << \mu$ and $|\nu| << \mu$. Note that μ is assumed to be a measure, and not a complex measure.

Exercise 1

Exercise 2.

1. Define $B_n = \cup_{k\geq n} E_k$ for $n \geq 1$. By assumption, $\mu(E_k) \leq 1/2^k$ for all $k \geq 1$ and consequently:

$$\mu(B_n) \leq \sum_{k=n}^{+\infty} \mu(E_k) \leq \sum_{k=n}^{+\infty} \frac{1}{2^k} = \frac{1}{2^{n-1}} < +\infty$$

   It follows that $\mu(B_n) \to 0$ as $n \to +\infty$. Furthermore, since E is defined as $E = \cap_{n\geq 1} B_n$ and $B_{n+1} \subseteq B_n$ for all $n \geq 1$, we have $B_n \downarrow E$. From $\mu(B_1) < +\infty$ and theorem (8), we obtain $\mu(B_n) \to \mu(E)$ as $n \to +\infty$. We have proved that:

$$\mu(E) = \lim_{n\to+\infty} \mu\left(\bigcup_{k\geq n} E_k\right) = 0$$

2. If $B_n = \cup_{k\geq n} E_k$, then $E = \cap_{n\geq 1} B_n$ and $B_{n+1} \subseteq B_n$ for all $n \geq 1$. From theorem (57), the total variation $|\nu|$ of the complex measure ν is a finite measure. In particular $|\nu|(B_1) < +\infty$, and applying theorem (8), it follows that $|\nu|(B_n) \to |\nu|(E)$ as $n \to +\infty$. Furthermore, since $E_n \subseteq B_n$ for all $n \geq 1$, we have:

$$\epsilon \leq |\nu(E_n)| \leq |\nu|(E_n) \leq |\nu|(B_n)$$

   and in particular $\lim |\nu|(B_n) \geq \epsilon$. We have proved that:

$$|\nu|(E) = \lim_{n\to+\infty} |\nu|\left(\bigcup_{k\geq n} E_k\right) \geq \epsilon$$

3. Let λ be a measure on $(\Omega, \mathcal{F})$ and $B_n = \cup_{k\geq n} E_k$ for $n \geq 1$. Since $E = \cap_{n\geq 1} B_n$ and $B_{n+1} \subseteq B_n$ for all $n \geq 1$, it is very tempting to conclude that $\lambda(B_n) \to \lambda(E)$ as $n \to +\infty$. However, a careful reading of theorem (8) shows that we cannot safely apply this theorem, unless $\lambda(B_1) < +\infty$ (or at least $\lambda(B_p) < +\infty$ for some $p \geq 1$), which in general is not true. So in general, we cannot conclude that:

$$\lambda(E) = \lim_{n\to+\infty} \lambda\left(\bigcup_{k\geq n} E_k\right)$$

When $\lambda = \mu$ or $\lambda = |\nu|$, we crucially used the assumption that $\mu(E_k) \leq 1/2^k$ for all $k \geq 1$, and the fact that $|\nu|$ is a finite measure, to obtain $\lambda(B_1) < +\infty$.

4. Let μ be a measure on $(\Omega, \mathcal{F})$ and ν be a complex measure on $(\Omega, \mathcal{F})$. The fact that $\nu << \mu$ is equivalent to $|\nu| << \mu$, has already been proved in exercise (1). Suppose the condition:

$$\forall \epsilon > 0, \exists \delta > 0, \forall E \in \mathcal{F}, \mu(E) \leq \delta \Rightarrow |\nu(E)| < \epsilon \tag{4}$$

holds. Let $E \in \mathcal{F}$ be such that $\mu(E) = 0$. Applying (4), for all $\epsilon > 0$, there exists $\delta > 0$ such that if $E' \in \mathcal{F}$ satisfies $\mu(E') \leq \delta$, then $|\nu(E')| < \epsilon$. Since $\mu(E) = 0$, we have $\mu(E) \leq \delta$ for all $\delta > 0$ and consequently $|\nu(E)| < \epsilon$ for all $\epsilon > 0$. So $\nu(E) = 0$. This shows that ν is absolutely continuous with respect to μ, and we have proved that (4) $\Rightarrow$ $\nu << \mu$. Conversely, suppose that $\nu << \mu$, and that condition (1) does not hold. There exists $\epsilon > 0$ such that for all $\delta > 0$ we can find some $E_\delta \in \mathcal{F}$ with the property that $\mu(E_\delta) \leq \delta$ and $|\nu(E_\delta)| \geq \epsilon$. Taking δ of the form $\delta = 1/2^n$ for $n \geq 1$, there exists a sequence $(E_n)_{n\geq 1}$ in $\mathcal{F}$, such that $\mu(E_n) \leq 1/2^n$ and $|\nu(E_n)| \geq \epsilon$ for all $n \geq 1$. Defining $E = \limsup E_n = \cap_{n\geq 1} \cup_{k\geq n} E_k$, we have $\mu(E) = 0$ from 1. and $|\nu|(E) \geq \epsilon$ from 2. This contradicts the fact that $|\nu| << \mu$, or equivalently the fact that $\nu << \mu$. We have proved that $\nu << \mu \Rightarrow$ (4), which completes the proof of theorem (58).

Exercise 2

Exercise 3.

1. Let μ be a measure on $(\Omega, \mathcal{F})$ and $\nu \in M^1(\Omega, \mathcal{F})$. Suppose that $\nu << \mu$. Let $E \in \mathcal{F}$ be such that $\mu(E) = 0$. Then $\nu(E) = 0$. In particular, $\nu_1(E) = Re(\nu(E)) = 0$ and $\nu_2(E) = Im(\nu(E)) = 0$. This shows that $\nu_1 << \mu$ and $\nu_2 << \mu$.

2. From 1. we have $\nu_1 << \mu$. From exercise (1), this is equivalent to $|\nu_1| << \mu$. Hence, if $E \in \mathcal{F}$ is such that $\mu(E) = 0$, then $\nu_1(E) = 0$ and $|\nu_1|(E) = 0$. It follows that:

$$\nu_1^+(E) = \frac{1}{2}(|\nu_1|(E) + \nu_1(E)) = 0$$

and

$$\nu_1^-(E) = \frac{1}{2}(|\nu_1|(E) - \nu_1(E)) = 0$$

We conclude that $\nu_1^+ << \mu$ and $\nu_1^- << \mu$. We prove similarly that ν_2^+ and ν_2^- are absolutely continuous with respect to μ.

Exercise 3

Exercise 4.

1. Since S is a closed proper subset of $\mathbf{C}$, its complement S^c is an open subset of $\mathbf{C}$, which is not empty. Let $z = x + iy \in S^c$. There exists $\epsilon > 0$ such that $B(z,\epsilon) \subseteq S^c$. Let $x', y' \in \mathbf{Q}$ be such that $|x - x'| < \epsilon/2\sqrt{2}$ and $|y - y'| < \epsilon/2\sqrt{2}$, and define $z' = x' + iy'$. Then:

$$|z - z'| = \sqrt{|x - x'|^2 + |y - y'|^2} < \epsilon/2$$

Let $\epsilon' \in \mathbf{Q}$ be such that $|z - z'| < \epsilon' < \epsilon/2$. Then it is clear that $z \in B(z', \epsilon')$ and furthermore, for all $z'' \in \mathbf{C}$ such that $|z' - z''| \leq \epsilon'$, we have:

$$|z - z''| \leq |z - z'| + |z' - z''| < 2\epsilon' < \epsilon$$

It follows that $z \in \bar{B}(z', \epsilon') \subseteq B(z, \epsilon) \subseteq S^c$, where $\bar{B}(z', \epsilon')$ denotes the closed disc with center z' and radius ϵ'. Hence, for all $z \in S^c$, we are able to find a closed disc D_z in $\mathbf{C}$, such that $z \in D_z \subseteq S^c$, and furthermore, such closed disc can be chosen to have a rational radius ($\epsilon' \in \mathbf{Q}$), and a center with rational coordinates ($x', y' \in \mathbf{Q}$). In particular, to each D_z where $z \in S^c$, can be associated a triple (x_z, y_z, ϵ_z) in $\mathbf{Q}^3$, defining a mapping which is injective. $\mathbf{Q}^3$ being a countable set, it follows that $\mathcal{D} = \{D_z : z \in S^c\}$ is at most countable (and non-empty), and consequently there exists a surjective map $\phi : \mathbf{N}^* \to \mathcal{D}$. Defining $D_n = \phi(n)$, from $S^c = \cup_{z \in S^c} D_z$ we obtain:

$$S^c = \bigcup_{D \in \mathcal{D}} D = \bigcup_{n=1}^{+\infty} \phi(n) = \bigcup_{n=1}^{+\infty} D_n$$

2. Since μ is a finite measure and $\mu(E_n) > 0$, it is always possible to write the complex number α_n as $\alpha_n = \mu(E_n)^{-1} \int_{E_n} \alpha_n d\mu$. Consequently, using theorem (24), we have:

$$\left| \frac{1}{\mu(E_n)} \int_{E_n} f d\mu - \alpha_n \right| \leq \frac{1}{\mu(E_n)} \int_{E_n} |f - \alpha_n| d\mu$$

Since $E_n = \{f \in D_n\} = \{|f - \alpha_n| \leq r_n\}$, we have the inequality $|f - \alpha_n| 1_{E_n} \leq r_n 1_{E_n}$, and consequently:

$$\frac{1}{\mu(E_n)} \int_{E_n} |f - \alpha_n| d\mu \leq \frac{1}{\mu(E_n)} \int r_n 1_{E_n} d\mu = r_n$$

We have proved that:

$$\left| \frac{1}{\mu(E_n)} \int_{E_n} f d\mu - \alpha_n \right| \leq \frac{1}{\mu(E_n)} \int_{E_n} |f - \alpha_n| d\mu \leq r_n$$

3. Let $n \geq 1$ and $E_n = \{f \in D_n\}$. Suppose $\mu(E_n) > 0$. Then:
$$\frac{1}{\mu(E_n)} \int_{E_n} f d\mu \ \in \ S \tag{5}$$
by assumption. However, from 2.:
$$\left| \frac{1}{\mu(E_n)} \int_{E_n} f d\mu - \alpha_n \right| \leq r_n$$
or equivalently:
$$\frac{1}{\mu(E_n)} \int_{E_n} f d\mu \ \in \ D_n \tag{6}$$
Since $D_n \subseteq S^c$, (5) and (6) form a contradiction. It follows that the assumption $\mu(E_n) > 0$ is absurd and therefore $\mu(E_n) = 0$. We have proved that $\mu(\{f \in D_n\}) = 0$ for all $n \geq 1$.

4. Let μ be a finite measure on $(\Omega, \mathcal{F})$ and $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Let S be a closed subset of $\mathbf{C}$ such that for all $E \in \mathcal{F}$ with $\mu(E) > 0$:
$$\frac{1}{\mu(E)} \int_E f d\mu \ \in \ S$$
We claim that $f \in S$ μ-a.s. If $S = \mathbf{C}$, there is nothing further to prove. We assume that S is a proper (closed) subset of $\mathbf{C}$. Let $(D_n)_{n \geq 1}$ be a sequence of closed discs in $\mathbf{C}$ as in 1. Then $S^c = \cup_{n\geq 1} D_n$ and from 3. $\mu(\{f \in D_n\}) = 0$ for all $n \geq 1$. From $\{f \in S^c\} = \cup_{n \geq 1}\{f \in D_n\}$ we obtain:
$$\mu(\{f \in S^c\}) \leq \sum_{n=1}^{+\infty} \mu(\{f \in D_n\}) = 0$$
It follows that if $N = \{f \in S^c\}$, then $N \in \mathcal{F}$, $\mu(N) = 0$ and $f(\omega) \in S$ for all $\omega \in N^c$. This shows that $f \in S$ μ-a.s. We have proved theorem (59).

Exercise 4

Exercise 5.

1. Let $\omega \in \Omega$. Since $E_n \uparrow \Omega$, in particular $\Omega = \cup_{n \geq 1} E_n$. There exists $p \geq 1$ such that $\omega \in E_p$. Hence:
$$w(\omega) = \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{1}{1 + \mu(E_n)} 1_{E_n}(\omega) \geq \frac{1}{2^p} \frac{1}{1 + \mu(E_p)} > 0$$
Furthermore:
$$w(\omega) = \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{1}{1 + \mu(E_n)} 1_{E_n}(\omega) \leq \sum_{n=1}^{+\infty} \frac{1}{2^n} = 1$$

2. w is $\mathbf{R}$-valued, measurable, and from theorem (19):

$$\int |w|d\mu = \int w d\mu = \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{1}{1+\mu(E_n)} \int 1_{E_n} d\mu < +\infty$$

So $w \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

Exercise 5

Exercise 6.

1. The fact that $\bar{\mu} = \int w d\mu$ is a measure on $(\Omega, \mathcal{F})$ stems from a direct application of theorem (21). However, the result is pretty straightforward, with or without theorem (21): it is clear that $\bar{\mu}(\emptyset) = 0$ and furthermore from the monotone convergence theorem (19):

$$\bar{\mu}(E) = \int 1_E w d\mu = \sum_{n=1}^{+\infty} \int 1_{E_n} w d\mu = \sum_{n=1}^{+\infty} \bar{\mu}(E_n)$$

for any $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ measurable partition of E. Since w is non-negative and is an element of $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, we have:

$$\bar{\mu}(\Omega) = \int w d\mu = \int |w| d\mu < +\infty$$

So $\bar{\mu}$ is a finite measure.

2. Since both ν and $\bar{\mu}$ are finite measures on $(\Omega, \mathcal{F})$, they are complex measures with values in $\mathbf{R}^+$. So $\phi = \nu + \bar{\mu}$ is a complex measure on $(\Omega, \mathcal{F})$ ($M^1(\Omega, \mathcal{F})$ is a vector space), and it has values in $\mathbf{R}^+$. It follows that ϕ is a finite measure. Alternatively, you may wish to argue that ϕ is a measure (as the sum of two measures), and that $\phi(\Omega) = \nu(\Omega) + \bar{\mu}(\Omega) < +\infty$ since both ν and $\bar{\mu}$ are finite.

3. Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be a non-negative and measurable map, and consider the equality:

$$\int f d\phi = \int f d\nu + \int f w d\mu \tag{7}$$

Since $\phi = \nu + \bar{\mu}$ and $\bar{\mu} = \int w d\mu$, this equality is true whenever f is of the form $f = 1_E$ with $E \in \mathcal{F}$. By linearity, equation (7) is also true whenever f is a simple function on $(\Omega, \mathcal{F})$. If f is an arbitrary non-negative and measurable map, from theorem (18) there exists a sequence $(s_n)_{n\geq 1}$ of simple functions on $(\Omega, \mathcal{F})$, such that $s_n \uparrow f$. Applying equation (7) for each $n \geq 1$, we obtain:

$$\int s_n d\phi = \int s_n d\nu + \int s_n w d\mu \tag{8}$$

Since w is non-negative, $(s_n w)_{n\geq 1}$ is a non-decreasing sequence of non-negative and measurable maps, converging simply (i.e. pointwise) to fw.

In short, we have $s_n w \uparrow fw$, and from the monotone convergence theorem (19), taking the limit in (8) as $n \to +\infty$, we conclude that equation (7) is also true for f. Suppose now that $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$. Applying (7) to $|f|$, we obtain:

$$\int |f| d\nu + \int |f| w d\mu = \int |f| d\phi < +\infty$$

and consequently $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu)$ and $fw \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. If f is real-valued, Applying equation (7) once more to f^+ and f^-, we obtain:

$$\int f^+ d\phi = \int f^+ d\nu + \int f^+ w d\mu \tag{9}$$

and:

$$\int f^- d\phi = \int f^- d\nu + \int f^- w d\mu \tag{10}$$

Subtracting (10) to (9) (all terms being finite, w being non-negative and f^+w, f^-w being finite), we see that equation (7) is true for f, whenever $f \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \phi)$. If $f = u + iv$ where u and v are elements of $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \phi)$, we conclude that equation (7) is true for f by the linearity of the integral, and the fact that it is true for u and v. This proves that equation (7) is in fact true for all $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$.

4. Let $f \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$. From the Cauchy-Schwarz inequality (42):

$$\int |f| d\phi \leq \left(\int |f|^2 d\phi \right)^{\frac{1}{2}} \left(\int 1^2 d\phi \right)^{\frac{1}{2}} = \left(\int |f|^2 d\phi \right)^{\frac{1}{2}} (\phi(\Omega))^{\frac{1}{2}}$$

In particular, ϕ being a finite measure, $\int |f| d\phi < +\infty$ and f is also an element of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$. [1] Applying 3. to $|f|$, we have:

$$\int |f| d\nu \leq \int |f| d\nu + \int |f| w d\mu = \int |f| d\phi$$

It follows that:

$$\int |f| d\nu \leq \int |f| d\phi \leq \left(\int |f|^2 d\phi \right)^{\frac{1}{2}} (\phi(\Omega))^{\frac{1}{2}}$$

5. ϕ being a finite measure, from 4. the inequality $\int |f|^2 d\phi < +\infty$ implies $\int |f| d\nu < +\infty$. So $L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi) \subseteq L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu)$. Furthermore, given $f \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$, from 4. and theorem (24):

$$\left| \int f d\nu \right| \leq \int |f| d\nu \leq \sqrt{\phi(\Omega)} \|f\|_2$$

6. Consider the map $\lambda : L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi) \to \mathbf{C}$ defined by:

$$\forall f \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi) \ , \ \lambda(f) = \int f d\nu$$

[1] This shows that $L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi) \subseteq L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$ whenever ϕ is a finite measure. We don't need $|f| \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$ for equation (7) to be true (see proof of 3.)

Since $L^2_{\mathbf{C}}(\Omega,\mathcal{F},\phi) \subseteq L^1_{\mathbf{C}}(\Omega,\mathcal{F},\nu)$ λ is well-defined, and it is clearly linear. Furthermore from 5., $|\lambda(f)| \leq \sqrt{\phi(\Omega)}\|f\|_2$ for all $f \in L^2_{\mathbf{C}}(\Omega,\mathcal{F},\phi)$. So λ is also continuous. Applying theorem (55), there exists $g' \in L^2_{\mathbf{C}}(\Omega,\mathcal{F},\phi)$ such that $\lambda(f) = \int f\bar{g}'d\phi$ for all f's. Taking $g = \bar{g}' \in L^2_{\mathbf{C}}(\Omega,\mathcal{F},\phi)$, we obtain:

$$\forall f \in L^2_{\mathbf{C}}(\Omega,\mathcal{F},\phi)\ ,\ \int f d\nu = \int fgd\phi$$

7. Let $E \in \mathcal{F}$ be such that $\phi(E) > 0$. ϕ being a finite measure, the map 1_E is an element of $L^2_{\mathbf{C}}(\Omega,\mathcal{F},\phi)$. From 6. we have:

$$\int_E gd\phi = \int 1_E gd\phi = \int 1_E d\nu = \nu(E)$$

Furthermore, since $0 \leq \nu(E) \leq \nu(E) + \bar{\mu}(E) = \phi(E)$, we obtain:

$$0 \leq \int_E gd\phi \leq \phi(E)$$

Finally since $\phi(E) > 0$, we see that $\phi(E)^{-1}\int_E gd\phi \in [0,1]$.

8. Since ϕ is a finite measure, we have $L^2_{\mathbf{C}}(\Omega,\mathcal{F},\phi) \subseteq L^1_{\mathbf{C}}(\Omega,\mathcal{F},\phi)$, as can be seen from the Cauchy-Schwarz inequality (42). In particular, g is an element of $L^1_{\mathbf{C}}(\Omega,\mathcal{F},\phi)$. Furthermore, the interval $[0,1]$ is a closed subset of $\mathbf{C}$, and for all $E \in \mathcal{F}$ with $\phi(E) > 0$, we have:

$$\frac{1}{\phi(E)}\int_E gd\phi\ \in\ [0,1]$$

Applying theorem (59), it follows that $g \in [0,1]$ ϕ-almost surely. There exists $N \in \mathcal{F}$ with $\phi(N) = 0$ such that $g(\omega) \in [0,1]$ for all $\omega \in N^c$. Define $h = g1_{N^c}$. Then $h \in L^2_{\mathbf{C}}(\Omega,\mathcal{F},\phi)$ and $h(\omega) \in [0,1]$ for all $\omega \in \Omega$. Furthermore, for all $f \in L^2_{\mathbf{C}}(\Omega,\mathcal{F},\phi)$ we have from 6.:

$$\int fd\nu = \int fgd\phi = \int fg1_N d\phi + \int fg1_{N^c}d\phi = \int fhd\phi$$

Renaming h by 'g', we have found $g \in L^2_{\mathbf{C}}(\Omega,\mathcal{F},\phi)$ such that $g(\omega) \in [0,1]$ for all $\omega \in \Omega$ and (1) still holds.

9. Let $f \in L^2_{\mathbf{C}}(\Omega,\mathcal{F},\phi)$. Since $g \in L^2_{\mathbf{C}}(\Omega,\mathcal{F},\phi)$, from the Cauchy-Schwarz inequality (42):

$$\int |fg|d\phi \leq \left(\int |f|^2 d\phi\right)^{\frac{1}{2}}\left(\int |g|^2 d\phi\right)^{\frac{1}{2}} < +\infty$$

It follows that $fg \in L^1_{\mathbf{C}}(\Omega,\mathcal{F},\phi)$. From 3. we have:

$$\int fgd\phi = \int fgd\nu + \int fgwd\mu \tag{11}$$

all three integrals being well-defined. From 6. we have:

$$\int fd\nu = \int fgd\phi \tag{12}$$

From (11) and (12), using the linearity of the integral, we obtain:

$$\int f(1-g)d\nu = \int fgwd\mu$$

10. Let $n \geq 1$ and $E \in \mathcal{F}$. Let $f = (1+g+\ldots+g^n)1_E$. Then f is a measurable map and furthermore, since $0 \leq g \leq 1$, f is also bounded. ϕ being a finite measure on $(\Omega, \mathcal{F})$, we conclude that $f \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$.

11. Let $n \geq 1$ and $E \in \mathcal{F}$. Let $f = (1+g+\ldots+g^n)1_E$. From 10. f is an element of $L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$. Applying 9. we obtain:

$$\int f(1-g)d\nu = \int fgwd\mu$$

or equivalently:

$$\int_E (1-g^{n+1})d\nu = \int_E g(1+g+\ldots+g^n)wd\mu \tag{13}$$

12. Let $A = \{0 \leq g < 1\}$ and define:

$$h \stackrel{\triangle}{=} gw\left(\sum_{k=0}^{+\infty} g^k\right)$$

and $h_n = gw(\sum_{k=0}^n g^k)$ for $n \geq 1$. Then, for all $E \in \mathcal{F}$, $(h_n 1_E)_{n\geq 1}$ is a non-decreasing sequence of non-negative and measurable maps, converging simply to $h1_E$. By the monotone convergence theorem (19), we have $\int h_n 1_E d\mu \to \int h1_E d\mu$, i.e.

$$\lim_{n\to+\infty} \int_E g(1+g+\ldots+g^n)wd\mu = \int_E hd\mu \tag{14}$$

Furthermore for all $\omega \in A$, $(1-g^{n+1}(\omega)) \to 1$ as $n \to +\infty$, and if $\omega \notin A$, since $0 \leq g \leq 1$ we have $1-g^{n+1}(\omega) = 0$ for all $n \geq 1$. It follows that $(1-g^{n+1})1_E \to 1_{E\cap A}$, and ν being a finite measure, the condition $|(1-g^{n+1})1_E| \leq 1$ allows us to apply to dominated convergence theorem (23) to obtain:

$$\lim_{n\to+\infty} \int_E (1-g^{n+1})d\nu = \int 1_{E\cap A}d\nu = \nu(E\cap A) \tag{15}$$

Using (14) and (15), taking the limit in (13) as $n \to +\infty$:

$$\nu(E\cap A) = \int_E hd\mu$$

13. Let $\omega \in \Omega$ with $h(\omega) = +\infty = g(\omega)w(\omega)\sum_{k=0}^{\infty} g^k(\omega)$. Since $0 \leq g \leq 1$ and $0 < w \leq 1$, the series $\sum_{k=0}^{+\infty} g^k(\omega)$ cannot be convergent, and consequently $g(\omega) = 1$. So $\omega \in A^c$ and we have proved that $\{h = +\infty\} \subseteq A^c$. Conversely, suppose that $\omega \in A^c$. Since $0 \leq g \leq 1$, we have $g(\omega) = 1$. Hence $\sum_{k=0}^{+\infty} g^k(\omega) = +\infty$, and since $w(\omega) > 0$ it follows that $h(\omega) = +\infty$. This

shows that $A^c \subseteq \{h = +\infty\}$ and finally that $\{h = +\infty\} = A^c$. Applying 12. to $E = A^c$, we obtain:

$$0 = \nu(A^c \cap A) = \int_{A^c} h d\mu = (+\infty)\mu(A^c)$$

from which we conclude that $\mu(A^c) = 0$.

14. Let $E \in \mathcal{F}$. From 12. we have:

$$\nu(E \cap A) = \int_E h d\mu$$

From 13. we have $\mu(A^c) = 0$. Since by assumption, ν is absolutely continuous with respect to μ (i.e. $\nu << \mu$), we also have $\nu(A^c) = 0$. It follows that:

$$\nu(E) = \nu(E \cap A) + \nu(E \cap A^c) = \int_E h d\mu$$

15. Let μ be a σ-finite measure on $(\Omega, \mathcal{F})$ and ν be a finite measure on $(\Omega, \mathcal{F})$ such that $\nu << \mu$. From 14. we have found a map $h : (\Omega, \mathcal{F}) \to [0, +\infty]$ non-negative and measurable, such that:

$$\forall E \in \mathcal{F}\ ,\ \nu(E) = \int_E h d\mu \tag{16}$$

Furthermore, from 13. we have $\mu(\{h = +\infty\}) = 0$. It follows that property (16) will also hold, if we replace h by $h1_{\{h<+\infty\}}$. Hence, without loss of generality, we can assume that h satisfying (16) has values in $\mathbf{R}^+$. Since ν is a finite measure, taking $E = \Omega$ in (16) we obtain:

$$\int |h| d\mu = \int h d\mu = \nu(\Omega) < +\infty$$

So $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. We have proved the existence of a map $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ such that $h \geq 0$ and property (16) holds.

16. Let μ be a σ-finite measure on $(\Omega, \mathcal{F})$, and ν be a complex measure on $(\Omega, \mathcal{F})$ such that $\nu << \mu$. If ν is in fact a finite measure, then 15. guarantees the existence of $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that:

$$\forall E \in \mathcal{F}\ ,\ \nu(E) = \int_E h d\mu \tag{17}$$

In fact, the result in 15. is slightly stronger, and allows us to choose h with values in $\mathbf{R}^+$. If ν is a signed measure (i.e. it has values in $\mathbf{R}$), then it can be written as $\nu = \nu^+ - \nu^-$, where ν^+ and ν^- are respectively the positive part and negative part of ν. Since ν^+ and ν^- are finite measures (see exercise (12) of Tutorial 11), which are absolutely continuous with respect to μ (see exercise (3)), there exist h^+, h^- elements of $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ with values in $\mathbf{R}^+$, such that $\nu^+ = \int h^+ d\mu$ and $\nu^- = \int h^- d\mu$. Defining $h = h^+ - h^-$, we obtain an element of $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ for which (by linearity of the integral) property (17) holds. In the general case when ν is an arbitrary complex measure, ν can be written as $\nu = \nu_1 + i\nu_2$ where ν_1, ν_2 are two signed measures which are absolutely continuous with respect to μ (see exercise (3)). Hence, there exist h_1, h_2 in $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ such that $\nu_1 = \int h_1 d\mu$ and $\nu_2 = \int h_2 d\mu$. Defining $h = h_1 + ih_2$, we obtain an element of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ for which (by linearity of the integral) property (17) holds. This proves the complex version of the Radon-Nikodym theorem (60).

Exercise 6

Exercise 7.

1. The positive part u^+ of u is defined as $u^+ = \max(0, u)$. It follows that $u^+ = u1_{\{u\geq 0\}}$ and consequently:

$$\int u^+ d\mu = \int u1_{\{u\geq 0\}} d\mu = \int_{\{u\geq 0\}} u d\mu$$

2. By assumption, using $E = \{u \geq 0\} \in \mathcal{F}$, we have:

$$\int_{\{u\geq 0\}} f d\mu = 0 = \int_{\{u\geq 0\}} u d\mu + i \int_{\{u\geq 0\}} v d\mu$$

   It follows in particular that $\int_{\{u\geq 0\}} u d\mu = 0$ and consequently using 1., $\int u^+ d\mu = 0$. Since u^+ is non-negative, this implies that $u^+ = 0$ μ-a.s. (See Exercise (7) of Tutorial 5.). Similarly, from $u^- = \max(-u, 0) = -u1_{\{u\leq 0\}}$ we obtain:

$$\int u^- d\mu = -\int u1_{\{u\leq 0\}} d\mu = -\int_{\{u\leq 0\}} u d\mu$$

   and from:

$$\int_{\{u\leq 0\}} f d\mu = 0 = \int_{\{u\leq 0\}} u d\mu + i \int_{\{u\leq 0\}} v d\mu$$

   we see that $\int u^- d\mu = 0$ and finally $u^- = 0$ μ-a.s. An identical proof will show that $v^+ = 0$ μ-a.s. and $v^- = 0$ μ-a.s. Having proved that u^+, u^-, v^+ and v^- are all μ-almost surely equal to zero, there exist sets N_1, N_2, N_3 and N_4, elements of $\mathcal{F}$, with $\mu(N_1) = \mu(N_2) = \mu(N_3) = \mu(N_4) = 0$ and such that $u^+(\omega) = u^-(\omega) = v^+(\omega) = v^-(\omega) = 0$ for all $\omega \in N_1^c \cap \ldots \cap N_4^c$. Taking $N = N_1 \cup \ldots \cup N_4$, we have found $N \in \mathcal{F}$ with $\mu(N) = 0$ such that $f(\omega) = 0$ for all $\omega \in N^c$. This shows that $f = 0$ μ-a.s.

3. Suppose there exist two maps $h_1, h_2 \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ which satisfy the conclusion of theorem (60), i.e. such that:

$$\forall E \in \mathcal{F}\ ,\ \nu(E) = \int_E h_1 d\mu = \int_E h_2 d\mu$$

   Defining $f = h_1 - h_2 \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we obtain:

$$\forall E \in \mathcal{F}\ ,\ \int_E f d\mu = 0$$

and from 2. we conclude that $f = 0$ μ-a.s., or equivalently $h_1 = h_2$ μ-a.s. This shows that the *Radon-Nikodym derivative of* ν *with respect to* μ (i.e. the element h of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ which satisfies the conclusion of theorem (60)), is unique up to μ-almost sure equality.

Exercise 7

Exercise 8.

1. Let $n \geq 1$. We have $\nu_n(\Omega) = \nu(E_n \cap \Omega) = \nu(E_n) < +\infty$. So ν_n is a finite measure, and in particular a complex measure on $(\Omega, \mathcal{F})$. Furthermore, if $E \in \mathcal{F}$ is such that $\mu(E) = 0$, then $\mu(E_n \cap E) = 0$ and it follows from $\nu << \mu$ that $\nu(E_n \cap E) = 0$ i.e. $\nu_n(E) = 0$. This shows that $\nu_n << \mu$, and the assumptions of theorem (60) are therefore all satisfied. There exists $h_n \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that:

$$\forall E \in \mathcal{F} \ , \ \nu_n(E) = \int_E h_n d\mu$$

Furthermore, ν_n being a finite measure, the map h_n can be chosen to lie in $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, with $h_n \geq 0$.

2. Let $E \in \mathcal{F}$ and $n \geq 1$. By assumption, $E_n \subseteq E_{n+1}$. Hence:

$$\int_E h_n d\mu = \nu(E_n \cap E) \leq \nu(E_{n+1} \cap E) = \int_E h_{n+1} d\mu$$

3. Let $n, p \geq 1$. Since h_n, h_{n+1} have values in $\mathbf{R}$ (in fact $\mathbf{R}^+$), the difference $h_n - h_{n+1}$ is meaningful, and from 2. we have:

$$\int_E (h_n - h_{n+1}) d\mu \leq 0$$

Applying this inequality to $E = \{h_n - h_{n+1} > 1/p\}$ we obtain:

$$\frac{1}{p}\mu(\{h_n - h_{n+1} > \frac{1}{p}\}) \leq \int_E (h_n - h_{n+1}) d\mu \leq 0$$

from which we conclude that $\mu(\{h_n - h_{n+1} > 1/p\}) = 0$.

4. Let $n \geq 1$. From:

$$\{h_n > h_{n+1}\} = \bigcup_{p=1}^{+\infty} \{h_n - h_{n+1} > \frac{1}{p}\}$$

and the fact that $\mu(\{h_n - h_{n+1} > 1/p\}) = 0$ for all $p \geq 1$, we conclude that $\mu(\{h_n > h_{n+1}\}) = 0$. So $h_n \leq h_{n+1}$ μ-.a.s.

5. Given $n \geq 1$, let $N_n = \{h_n > h_{n+1}\}$. Define $N = \cup_{n \geq 1} N_n$. Then, $\mu(N) = 0$ and replacing all h_n's by $h_n 1_{N^c}$, we obtain a sequence $(h_n)_{n \geq 1}$ in $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ such that $0 \leq h_n \leq h_{n+1}$ (this time *everywhere*), where the *new* h_n's are μ-almost equal to our original $h'_n s$, and therefore such that equation (2) still holds.

6. Let $h = \sup_{n\geq 1} h_n$ and $E \in \mathcal{F}$. From (2), for all $n \geq 1$ we have:
$$\nu(E_n \cap E) = \int 1_E h_n d\mu \tag{18}$$
From $(E_n \cap E) \uparrow E$ and theorem (7), $\nu(E_n \cap E) \to \nu(E)$ as $n \to +\infty$. From $1_E h_n \uparrow 1_E h$ and the monotone convergence theorem (19), $\int 1_E h_n d\mu \to \int 1_E h d\mu$ as $n \to +\infty$. Taking the limit in (18) as $n \to +\infty$, we conclude that:
$$\forall E \in \mathcal{F}\ ,\ \nu(E) = \int_E h d\mu$$

7. Let $n \geq 1$. From 6. we have:
$$\int_{E_n} h d\mu = \nu(E_n) < +\infty$$

8. From $(+\infty)1_{\{h=+\infty\}} \leq h$ and 7. we obtain:
$$(+\infty)\mu(E_n \cap \{h = +\infty\}) \leq \int_{E_n} h d\mu < +\infty$$
It follows that $\mu(E_n \cap \{h = +\infty\}) = 0$ for all $n \geq 1$. From $E_n \cap \{h = +\infty\} \uparrow \{h = +\infty\}$ and theorem (7), we obtain:
$$\mu(\{h = +\infty\}) = \lim_{n\to+\infty} \mu(E_n \cap \{h = +\infty\}) = 0$$
We conclude that $h < +\infty$ μ-a.s.

9. Replacing h by $h1_{\{h<+\infty\}}$, we obtain a measurable map with values in $\mathbf{R}^+$, which is μ-almost surely equal to our original h, and therefore such that equation (3) still holds.

10. h has values in $\mathbf{R}^+$ and is measurable, while from 7.:
$$\int h d\mu^{E_n} = \int_{E_n} h d\mu < +\infty$$
So $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu^{E_n})$.

Exercise 8

Exercise 9.

1. Let $n \geq 1$ and $E \in \mathcal{F}$. We have:
$$\int_E h d\mu^{F_n} = \int_{F_n \cap E} h d\mu = \int_{F_n \cap E} h' d\mu = \int_E h' d\mu^{F_n}$$
Furthermore:
$$\int_E h d\mu^{F_n} = \int 1_E h 1_{E_n} 1_{\{h\leq n\}} d\mu \leq n\mu(E_n) < +\infty$$

2. Let $n \geq 1$ and $p \geq 1$. Applying 1. to $E = \{h > h' + 1/p\}$:
$$\int_E h' d\mu^{F_n} = \int_E h d\mu^{F_n} \geq \int_E h' d\mu^{F_n} + \frac{1}{p}\mu(F_n \cap E)$$
and since $\int_E h' d\mu^{F_n} < +\infty$, it follows that $\mu(F_n \cap E) = 0$.

3. Let $n \geq 1$. From the equality:
$$\{h > h'\} = \bigcup_{p=1}^{+\infty} \{h > h' + \frac{1}{p}\}$$
and the fact that $\mu(F_n \cap \{h > h' + 1/p\}) = 0$ for all $p \geq 1$, we have $\mu(F_n \cap \{h > h'\}) = 0$. A similar argument shows that $\mu(F_n \cap \{h' > h\}) = 0$. It follows that $\mu(F_n \cap \{h \neq h'\}) = 0$.

4. By assumption, $F_n = E_n \cap \{h \leq n\}$. Hence, $F_n \subseteq F_{n+1}$ for all $n \geq 1$ and $\cup_{n \geq 1} F_n = \{h < +\infty\}$. In short, $F_n \uparrow \{h < +\infty\}$, and consequently we have $F_n \cap \{h \neq h'\} \uparrow \{h \neq h'\} \cap \{h < +\infty\}$. Applying theorem (7), we conclude that:
$$\mu(\{h \neq h'\} \cap \{h < +\infty\}) = \lim_{n \to +\infty} \mu(F_n \cap \{h \neq h'\}) = 0$$

5. The assumption made on h and h', namely:
$$\forall E \in \mathcal{F} \ , \ \int_E h d\mu = \int_E h' d\mu$$
is symmetric in terms h and h'. Using 4. where h and h' have been interchanged, we obtain $\mu(\{h \neq h'\} \cap \{h' < +\infty\}) = 0$. Since the set $\{h \neq h'\}$ can be decomposed as:
$$\{h \neq h'\} = (\{h \neq h'\} \cap \{h < +\infty\}) \cup (\{h \neq h'\} \cap \{h' < +\infty\})$$
we conclude that $\mu(\{h \neq h'\}) = 0$, i.e. $h = h'$ μ-a.s.

6. Let h and h' be two maps satisfying the conclusion of theorem (61). Then in particular, h and h' are non-negative and measurable, while satisfying:
$$\forall E \in \mathcal{F} \ , \ \int_E h d\mu = \int_E h' d\mu$$
This exercise shows that $h = h'$ μ-a.s. In other words, the *Radon Nikodym derivative of* ν *with respect to* μ (i.e. the map h which satisfies the conclusion of theorem (61)) is unique, up to μ-almost sure equality.

Exercise 9

Exercise 10. The sigma-algebra $\mathcal{F}$ has only two elements, $\emptyset$ and $\{*\}$. If $E = \emptyset$, then:
$$\int_E h d\mu = 0 = \int_E h' d\mu$$

If $E = \{*\}$, then:

$$\int_E hd\mu = 1 \times \mu(\{*\}) = +\infty = 2 \times \mu(\{*\}) = \int_E h'd\mu$$

In any case, we have $\int_E hd\mu = \int_E h'd\mu$. Although h and h' are not μ-almost surely equal, this does not contradict exercise (9), as the measure μ is not sigma-finite.

Exercise 10

Exercise 11.

1. Let $E \in \mathcal{F}$ be such that $|\mu|(E) = 0$. Since $|\mu(E)| \leq |\mu|(E)$ we have $\mu(E) = 0$, and consequently $\mu << |\mu|$.

2. From theorem (57), the total variation $|\mu|$ of μ is a finite measure on $(\Omega, \mathcal{F})$. In particular, it is sigma-finite. μ being a complex measure such that $\mu << |\mu|$, we can apply theorem (60): there exists $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ such that:
$$\forall E \in \mathcal{F}\ ,\ \mu(E) = \int_E hd|\mu|$$

3. If μ is in fact a signed measure, then from theorem (60), h can indeed be chosen to lie in $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, |\mu|)$.

Exercise 11

Exercise 12.

1. Let $A_r = \{|h| < r\}$ (for some $r > 0$) and $(E_n)_{n\geq 1}$ be a measurable partition of A_r. From exercise (11), for all $n \geq 1$:
$$|\mu(E_n)| = \left|\int_{E_n} hd|\mu|\right| \leq \int_{E_n} |h|d|\mu| \leq r|\mu|(E_n)$$
where the first inequality stems from theorem (24), and the second from the fact that $E_n \subseteq \{|h| < r\}$. It follows that:
$$\sum_{n=1}^{+\infty} |\mu(E_n)| \leq r \sum_{n=1}^{+\infty} |\mu|(En) = r|\mu|(A_r)$$

2. $|\mu|(A_r)$ being the least upper bound of all sums $\sum_{n=1}^{+\infty} |\mu(E_n)|$ as $(E_n)_{n\geq 1}$ ranges across all measurable partitions of A_r, it follows from 1. that $|\mu|(A_r) \leq r|\mu|(A_r)$. When $0 < r < 1$, this can only occur if $|\mu|(A_r) = 0$.

3. From the equality:
$$\{|h| < 1\} = \bigcup_{p=2}^{+\infty} \{|h| < 1 - \frac{1}{p}\}$$
and the fact that $|\mu|(\{|h| < 1 - 1/p\}) = |\mu|(A_{1-1/p}) = 0$ for all $p \geq 2$, it follows that $|\mu|(\{|h| < 1\}) = 0$, i.e. $|h| \geq 1$ $|\mu|$-a.s.

4. Let $E \in \mathcal{F}$ be such that $|\mu|(E) > 0$. We have:

$$\left|\frac{1}{|\mu|(E)}\int_E hd|\mu|\right| = \left|\frac{1}{|\mu|(E)}\mu(E)\right| = \frac{|\mu(E)|}{|\mu|(E)} \leq 1$$

5. Applying theorem (59) to the closed disc $S = \{|z| \leq 1\}$ and the finite measure $|\mu|$, we conclude from 4. that $h \in S$ $|\mu|$-a.s. or equivalently that $|h| \leq 1$ $|\mu|$-a.s.

6. Having proved that $|h| \geq 1$ $|\mu|$-a.s. and $|h| \leq 1$ $|\mu|$-a.s., the set $N = \{|h| > 1\} \cup \{|h| < 1\}$ is such that $|\mu|(N) = 0$. Replacing h by $h1_{N^c} + 1_N$, we obtain an element of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ such that $|h| = 1$ (this time *everywhere*), which is almost surely equal to our original h, and therefore such that:

$$\forall E \in \mathcal{F}\ ,\ \mu(E) = \int_E hd|\mu|$$

From 3. of exercise (11), if μ is a signed measure, then h can be chosen to lie in $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, |\mu|)$. This proves theorem (62).

Exercise 12

Exercise 13.

1. Suppose $A_n \uparrow A$. Then $A_n \subseteq A_{n+1}$ for all $n \geq 1$, and furthermore $A = \cup_{n\geq 1} A_n$. Let $\omega \in \Omega$ and $n \geq 1$. If $1_{A_n}(\omega) = 0$, then $1_{A_n}(\omega) \leq 1_{A_{n+1}}(\omega)$ is clear. If $1_{A_n}(\omega) = 1$, then $\omega \in A_n$ and consequently $\omega \in A_{n+1}$, so $1_{A_{n+1}}(\omega) = 1$. In any case, we have $1_{A_n}(\omega) \leq 1_{A_{n+1}}(\omega)$. This shows that $1_{A_n} \leq 1_{A_{n+1}}$ for all $n \geq 1$. Since $A_n \subseteq A$ for all $n \geq 1$, we obtain similarly that $1_{A_n} \leq 1_A$ for all $n \geq 1$, and consequently $\sup_{n\geq 1} 1_{A_n} \leq 1_A$. Let $\omega \in \Omega$. If $1_A(\omega) = 0$, then $1_A(\omega) \leq \sup_{n\geq 1} 1_{A_n}(\omega)$ is clear. If $1_A(\omega) = 1$ then $\omega \in A = \cup_{n\geq 1} A_n$, and there exists $n \geq 1$ such that $\omega \in A_n$. So $1_{A_n}(\omega) = 1 \leq \sup_{n\geq 1} 1_{A_n}(\omega)$. In any case, we have $1_A(\omega) \leq \sup_{n\geq 1} 1_{A_n}(\omega)$. This shows that:

$$1_A = \sup_{n\geq 1} 1_{A_n} = \lim_{n\to+\infty} 1_{A_n}$$

and finally, we have proved that $1_{A_n} \uparrow 1_A$.

2. Suppose that $A_n \downarrow A$. Then $A_{n+1} \subseteq A_n$ for all $n \geq 1$ and $A = \cap_{n\geq 1} A_n$. It follows that $A_n^c \subseteq A_{n+1}^c$ for all $n \geq 1$ and $A^c = \cup_{n\geq 1} A_n^c$, or equivalently that $A_n^c \uparrow A^c$. Applying 1. we obtain that $1_{A_n^c} \uparrow 1_{A^c}$. Since $1_{A_n^c} = 1 - 1_{A_n}$ for all $n \geq 1$ and $1_{A^c} = 1 - 1_A$, we conclude that $1_{A_{n+1}} \leq 1_{A_n}$ for all $n \geq 1$ and $1_A = \lim_n 1_{A_n}$. We have proved that $1_{A_n} \downarrow 1_A$.

3. Suppose that $1_{A_n} \to 1_A$ and let $\mu \in M^1(\Omega, \mathcal{F})$. From theorem (62), there exists $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ such that:

$$\forall E \in \mathcal{F}\ , \mu(E) = \int_E hd|\mu|$$

In particular, $\mu(A_n) = \int 1_{A_n} hd|\mu|$ for all $n \geq 1$. The hypothesis $1_{A_n} \to 1_A$ implies in particular that $1_{A_n} h \to 1_A h$, and since $|1_{A_n} h| \leq |h| \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, |\mu|)$, the dominated convergence theorem (23) allows us to conclude that:

$$\lim_{n\to+\infty} \mu(A_n) = \lim_{n\to+\infty} \int 1_{A_n} hd|\mu| = \int 1_A hd|\mu| = \mu(A)$$

Exercise 13

Exercise 14.

1. Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $\nu : \mathcal{F} \to \mathbf{C}$ be defined by:

$$\forall E \in \mathcal{F}\ ,\ \nu(E) = \int_E f d\mu$$

The fact that $\nu \in M^1(\Omega, \mathcal{F})$ has already been proved in ex. (3) of Tutorial 11. For a slightly leaner proof, here is the following: let $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E. For all $n \geq 1$, we define $A_n = E_1 \uplus \ldots \uplus E_n$. Then, from $1_{A_n} = 1_{E_1} + \ldots + 1_{E_n}$ we obtain:

$$\nu(A_n) = \int 1_{A_n} f d\mu = \sum_{k=1}^{n} \int 1_{E_k} f d\mu = \sum_{k=1}^{n} \nu(E_k) \tag{19}$$

Furthermore, from $A_n \uparrow E$ we have $1_{A_n} \uparrow 1_E$ and consequently $1_{A_n} f \to 1_E f$. Since $|1_{A_n} f| \leq |f| \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ for all $n \geq 1$, it follows from the dominated convergence theorem (23) that:

$$\lim_{n\to+\infty} \nu(A_n) = \lim_{n\to+\infty} \int 1_{A_n} f d\mu = \int 1_E f d\mu = \nu(E) \tag{20}$$

Comparing (19) with (20), it appears that the series $\sum_{k=1}^{+\infty} \nu(E_k)$ converges to $\nu(E)$. So ν is indeed a complex measure on $(\Omega, \mathcal{F})$.

2. From theorem (62), there is $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\nu|)$ with $|h| = 1$ and:

$$\forall E \in \mathcal{F}\ ,\ \nu(E) = \int_E hd|\nu|$$

Let $E, F \in \mathcal{F}$. We have:

$$\int_E f 1_F d\mu = \int_{E\cap F} f d\mu = \nu(E \cap F) = \int_E h 1_F d|\nu|$$

3. Given $g : \Omega \to \mathbf{C}$ bounded and measurable, we claim that:

$$\forall E \in \mathcal{F}\ ,\ \int_E fgd\mu = \int_E hgd|\nu| \tag{21}$$

From 2., equation (21) is true whenever g is of the form $g = 1_F$ with $F \in \mathcal{F}$. By the linearity of the integral, (21) is also true whenever g is a simple function on $(\Omega, \mathcal{F})$. Suppose g is non-negative and measurable, while being bounded. From theorem (18), there exists a sequence $(s_n)_{n\geq 1}$

of simple functions on $(\Omega, \mathcal{F})$, such that $s_n \uparrow g$. Having proved (21) for simple functions, for all $n \geq 1$ we have:

$$\int 1_E f s_n d\mu = \int 1_E h s_n d|\nu| \tag{22}$$

From $s_n \to g$ we obtain $1_E f s_n \to 1_E f g$ and $1_E h s_n \to 1_E h g$. Since $|1_E f s_n| \leq |f| g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ (since g is bounded) and $|1_E h s_n| \leq |h| g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, |\nu|)$, it follows from the dominated convergence theorem (23) that $\int 1_E f s_n d\mu \to \int 1_E f g d\mu$ and $\int 1_E h s_n d|\nu| \to \int 1_E h g d|\nu|$ as $n \to +\infty$. Taking the limit in (22) as $n \to +\infty$, we see that (21) is true whenever g is non-negative and measurable, while being bounded. If g is now an arbitrary $\mathbf{C}$-valued map which is measurable and bounded, then it can be expressed as $g = g_1 - g_2 + i(g_3 - g_4)$ where each g_i is non-negative, measurable and bounded. From the linearity of the integral, we conclude that (21) is also true for g, which completes the proof of our initial claim.

4. Since $|h| = 1$, applying (21) to $g = \bar{h}$, we obtain for all $E \in \mathcal{F}$:

$$\int_E f\bar{h} d\mu = \int_E h\bar{h} d|\nu| = \int_E d|\nu| = |\nu|(E)$$

5. The total variation $|\nu|$ of the complex measure ν being a finite measure on $(\Omega, \mathcal{F})$ (theorem (57)), it has values in $\mathbf{R}^+$. Hence:

$$\int_E Re(f\bar{h}) d\mu = Re\left(\int_E f\bar{h} d\mu\right) = Re(|\nu|(E)) \geq 0$$

and:

$$\int_E Im(f\bar{h}) d\mu = Im\left(\int_E f\bar{h} d\mu\right) = Im(|\nu|(E)) = 0$$

6. Define $g_1 = Re(f\bar{h})$ and $g_2 = Im(f\bar{h})$. Then g_1 and g_2 are elements of $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, and from 5. we have $\int_E g_1 d\mu \geq 0$ while $\int_E g_2 d\mu = 0$ for all $E \in \mathcal{F}$. Since $S = \mathbf{R}^+$ and $S = \{0\}$ are closed subset of $\mathbf{C}$, it is very tempting to apply theorem (59) in an attempt to conclude that $g_1 \in \mathbf{R}^+$ μ-a.s. and $g_2 = 0$ μ-a.s. Unfortunately, μ is not assumed to be a finite measure (it is not even assumed to be sigma-finite) and theorem (59) should therefore be forgotten here. Taking $E = \{g_1 < -1/n\}$ for some $n \geq 1$, we obtain:

$$0 \leq \int_E g_1 d\mu \leq -\frac{1}{n}\mu(\{g_1 < -1/n\}) \leq 0$$

from which we see that $\mu(\{g_1 < -1/n\}) = 0$ for all $n \geq 1$. Since $\{g_1 < 0\} = \cup_{n\geq 1}\{g_1 < -1/n\}$, it follows that $\mu(\{g_1 < 0\}) = 0$ and consequently, $g_1 \in \mathbf{R}^+$ μ-a.s. Similarly, from $\int_E g_2 d\mu = 0$ for all $E \in \mathcal{F}$, we obtain $g_2 \in \mathbf{R}^+$ μ-a.s. and $-g_2 \in \mathbf{R}^+$ μ-a.s. It follows that $g_2 = 0$ μ-a.s. We have proved that $Re(f\bar{h}) \in \mathbf{R}^+$ μ-a.s. while $Im(f\bar{h}) = 0$ μ-a.s., so $f\bar{h} \in \mathbf{R}^+$ μ-a.s.

7. From 6. there exists $N \in \mathcal{F}$ with $\mu(N) = 0$ and $f(\omega)\bar{h}(\omega) \in \mathbf{R}^+$ for all $\omega \in N^c$. In particular, since $|h| = 1$, for all $\omega \in N^c$:

$$f(\omega)\bar{h}(\omega) = |f(\omega)\bar{h}(\omega)| = |f(\omega)|$$

It follows that $f\bar{h} = |f|$ μ-a.s.

8. Let μ be a measure on $(\Omega, \mathcal{F})$ and $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Then, from part 1. of this exercise, $\nu = \int f d\mu$ is a complex measure on $(\Omega, \mathcal{F})$. Furthermore, from 4. we have:

$$\forall E \in \mathcal{F}\ ,\ |\nu|(E) = \int_E f\bar{h} d\mu$$

Finally, from 7. we have $f\bar{h} = |f|$ μ-a.s. We conclude that:

$$\forall E \in \mathcal{F}\ ,\ |\nu|(E) = \int_E |f| d\mu$$

This completes the proof of theorem (63).

Exercise 14

Exercise 15.

1. The positive part μ^+ of the signed measure μ is defined by the formula $\mu^+ = (|\mu| + \mu)/2$ (see exercise (12) of Tutorial 11). It follows that for all $E \in \mathcal{F}$:

$$\mu^+(E) = \frac{1}{2}|\mu|(E) + \frac{1}{2}\int_E h d|\mu| = \int_E \frac{1}{2}(1+h) d|\mu|$$

2. The negative part μ^- of the signed measure μ is defined as $\mu^- = (|\mu| - \mu)/2$. Hence, for all $E \in \mathcal{F}$:

$$\mu^-(E) = \frac{1}{2}|\mu|(E) - \frac{1}{2}\int_E h d|\mu| = \int_E \frac{1}{2}(1-h) d|\mu|$$

3. Since $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, |\mu|)$ is $\mathbf{R}$-valued and $|h| = 1$, h can only assume the values 1 and -1. Having defined $A = \{h = 1\}$, $(1+h)/2 = 0$ on A^c and for all $E \in \mathcal{F}$ we have:

$$\mu^+(E) = \int_E \frac{1}{2}(1+h) d|\mu| = \int_{A\cap E} \frac{1}{2}(1+h) d|\mu| \qquad (23)$$

Furthermore, since $h = (1+h)/2$ on A:

$$\mu(A \cap E) = \int_{A\cap E} h d|\mu| = \int_{A\cap E} \frac{1}{2}(1+h) d|\mu| \qquad (24)$$

Comparing (23) with (24), we obtain $\mu^+ = \mu^A$.

4. Having defined $B = \{h = -1\}$, we have for all $E \in \mathcal{F}$:

$$\mu^-(E) = \int_E \frac{1}{2}(1-h) d|\mu| = \int_{B\cap E} \frac{1}{2}(1-h) d|\mu|$$

since $(1-h)/2 = 0$ on B^c. Furthermore:

$$\mu(B \cap E) = \int_{B\cap E} hd|\mu| = -\int_{B\cap E} \frac{1}{2}(1-h)d|\mu|$$

since $h = -(1-h)/2$ on B. This shows that $\mu^- = -\mu^B$, and completes the proof of theorem (64).

Exercise 15

Exercise 16.

1. Let $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ be measurable. From definition (97), any element of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is an element of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$, and therefore satisfies:

$$\int |f|d|\mu| < +\infty \tag{25}$$

Conversely, if f satisfies the integrability condition (25), then it is an element of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ and therefore an element of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

2. Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. The integral of f w.r. to μ is defined as:

$$\int f d\mu \stackrel{\triangle}{=} \int fhd|\mu| \tag{26}$$

where h is any element of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ with $|h| = 1$ and $\mu = \int hd|\mu|$ (there is at least one such h by virtue of theorem (62)). This definition is potentially ambiguous, as h may not be unique. However, if h' is another element of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ with $|h'| = 1$ and $\mu = \int h'd|\mu|$, then for all $E \in \mathcal{F}$, we have:

$$\mu(E) = \int_E hd|\mu| = \int_E h'd|\mu|$$

which implies that $\int_E (h - h')d|\mu| = 0$. Using exercise (7), it follows that $h = h'$ $|\mu|$-a.s. and consequently the r.h.s integral of equation (26) is unchanged, when replacing h by h'. We conclude that equation (26) is in fact unambiguous, as its r.h.s integral does not depend on the particular choice of element $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ with $|h| = 1$ and $\mu = \int hd|\mu|$.

3. Let $E \in \mathcal{F}$. Then $1_E : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is measurable, and furthermore:

$$\int |1_E|d|\mu| = \int 1_E d|\mu| = |\mu|(E) < +\infty$$

since $|\mu|$ is a finite measure on $(\Omega, \mathcal{F})$ (see theorem (57)). Using 1. it follows that 1_E is an element of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, as defined in definition (97). Moreover, we have:

$$\int 1_E d\mu = \int 1_E hd|\mu| = \int_E hd|\mu| = \mu(E)$$

4. If μ is a finite measure (complex measure with values in $\mathbf{R}^+$), then for all $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ measurable partition of E:

$$\sum_{n=1}^{+\infty} |\mu(E_n)| = \sum_{n=1}^{+\infty} \mu(E_n) = \mu(E)$$

In particular, $\mu(E)$ is an upper bound of all sums $\sum_{n=1}^{+\infty} |\mu(E_n)|$, as $(E_n)_{n\geq 1}$ ranges through all measurable partitions of E. It follows that $|\mu|(E) \leq \mu(E)$. Since $\mu(E) = |\mu(E)| \leq |\mu|(E)$ is clear, we conclude that $|\mu| = \mu$.

5. Suppose that μ is a finite measure. Then μ is not only a measure, but also a complex measure. It follows that definition (97) of the space $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, and of the integral $\int f d\mu$ (valid for complex measures), is potentially in conflict with the definitions already known for measures (definitions (46) and (48)). However, since $\mu = |\mu|$, the space $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ of definition (97) being defined as $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$, coincide with that of definition (46). Furthermore, $h = 1$ being an element of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ with $|h| = 1$ and $\mu = \int h d|\mu|$, the integral $\int f d\mu$ of definition (97) can be expressed as:

$$\int f d\mu = \int f h d|\mu| = \int f d|\mu| = \int f d\mu$$

where the r.h.s integral is that of definition (48). We conclude that definition (97) which extends the notion of integral with respect to complex measures, is consistent with previous definitions laid out for measures.

6. The space $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ being defined as $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$, it is a $\mathbf{C}$-vector space. Let $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ be such that $|h| = 1$ and $\mu = \int h d|\mu|$. Then, for all $f, g \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $\alpha \in \mathbf{C}$, following definition (97) we have:

$$\begin{aligned}\int (f + \alpha g) d\mu &= \int (f + \alpha g) h d|\mu| \\ &= \int f h d|\mu| + \alpha \int g h d|\mu| \\ &= \int f d\mu + \alpha \int g d\mu\end{aligned}$$

where the second equality stems from the linearity of the integral, already established for measures.

7. Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and h be as in definition (97). Then, from theorem (24), we have:

$$\left|\int f d\mu\right| = \left|\int f h d|\mu|\right| \leq \int |f h| d|\mu| = \int |f| d|\mu|$$

Exercise 16

Exercise 17.

1. Let $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E. Then:
$$\sum_{n=1}^{+\infty} |\alpha\nu(E_n)| = |\alpha| \sum_{n=1}^{+\infty} |\nu(E_n)| \leq |\alpha||\nu|(E)$$
It follows that $|\alpha||\nu|(E)$ is an upper bound of all $\sum_{n=1}^{+\infty} |\alpha\nu(E_n)|$ as $(E_n)_{n\geq 1}$ ranges through all measurable partitions of E. Since $|\alpha\nu|(E)$ is the smallest of such upper bounds, we obtain the inequality $|\alpha\nu|(E) \leq |\alpha||\nu|(E)$. This being true for all $E \in \mathcal{F}$, we have proved that $|\alpha\nu| \leq |\alpha||\nu|$ for all $\alpha \in \mathbf{C}$. If $\alpha = 0$, then $|\alpha\nu| = |\alpha||\nu|$ is clear. If $\alpha \neq 0$, then applying what we have just proved to $\nu' = \alpha\nu$ and $\alpha' = 1/\alpha$, we obtain:
$$|\nu| = \left|\frac{1}{\alpha}(\alpha\nu)\right| \leq \frac{1}{|\alpha|}|\alpha\nu|$$
and consequently $|\alpha||\nu| \leq |\alpha\nu|$. This shows that $|\alpha\nu| = |\alpha||\nu|$ for all $\nu \in M^1(\Omega, \mathcal{F})$ and $\alpha \in \mathbf{C}$.

2. Let $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E. Then:
$$\sum_{n=1}^{+\infty} |(\mu+\nu)(E_n)| \leq \sum_{n=1}^{+\infty} |\mu(E_n)| + \sum_{n=1}^{+\infty} |\nu(E_n)| \leq (|\mu| + |\nu|)(E)$$
It follows that $(|\mu| + |\nu|)(E)$ is an upper bound of all sums $\sum_{n=1}^{+\infty} |(\mu + \nu)(E_n)|$ as $(E_n)_{n\geq 1}$ ranges through all measurable partitions of E. $|\mu + \nu|(E)$ being the smallest of such upper bounds, we have $|\mu + \nu|(E) \leq (|\mu|+|\nu|)(E)$. This being true for all $E \in \mathcal{F}$, we have proved that $|\mu+\nu| \leq |\mu| + |\nu|$.

3. Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \cap L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu)$. Then f is $\mathbf{C}$-valued, measurable, and satisfies $\int |f|d|\mu| < +\infty$ with $\int |f|d|\nu| < +\infty$. Using 2. and 1., for all $\alpha \in \mathbf{C}$:
$$|\mu + \alpha\nu| \leq |\mu| + |\alpha\nu| = |\mu| + |\alpha||\nu|$$
Hence, for all $E \in \mathcal{F}$, we have:
$$\int 1_E d|\mu + \alpha\nu| \leq \int 1_E d|\mu| + |\alpha| \int 1_E d|\nu|$$
By linearity, if s is a simple function on $(\Omega, \mathcal{F})$, we obtain:
$$\int s d|\mu + \alpha\nu| \leq \int s d|\mu| + |\alpha| \int s d|\nu|$$
Approximating $|f|$ by a sequence simple functions (see theorem (18)) and using the monotone convergence theorem (19):
$$\int |f| d|\mu + \alpha\nu| \leq \int |f| d|\mu| + |\alpha| \int |f| d|\nu| < +\infty$$
So $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu + \alpha\nu)$, and we have proved the inclusion:
$$L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \cap L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu) \subseteq L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu + \alpha\nu)$$

4. Using 3. of exercise (16), we have:

$$\begin{aligned}\int 1_E d(\mu+\alpha\nu) &= (\mu+\alpha\nu)(E)\\ &= \mu(E)+\alpha\nu(E)\\ &= \int 1_E d\mu + \alpha\int 1_E d\nu\end{aligned}$$

5. Let $f \in L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu)\cap L^1_{\mathbf{C}}(\Omega,\mathcal{F},\nu)$. We claim that:

$$\int f d(\mu+\alpha\nu) = \int f d\mu + \alpha\int f d\nu \tag{27}$$

Note from 3. that $f\in L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu+\alpha\nu)$ and all integrals of equation (27) are therefore well defined. Furthermore from 4., (27) is true whenever f is of the form $f = 1_E$ with $E\in\mathcal{F}$. By linearity (proved in 6. of exercise (16)), equation (27) is in fact true whenever f is a simple function on $(\Omega,\mathcal{F})$. Suppose now that $f:(\Omega,\mathcal{F})\to[0,+\infty]$ is non-negative and measurable, while being an element of $L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu)\cap L^1_{\mathbf{C}}(\Omega,\mathcal{F},\nu)$. From theorem (18), there exists a sequence $(s_n)_{n\geq 1}$ of simple functions on $(\Omega,\mathcal{F})$ such that $s_n\uparrow f$. Let $h\in L^1_{\mathbf{C}}(\Omega,\mathcal{F},|\mu+\alpha\nu|)$ be such that $|h|=1$ and $\mu+\alpha\nu=\int hd|\mu+\alpha\nu|$. Then, $s_nh\to fh$ and furthermore $|s_nh| = |s_n| = s_n \leq f\in L^1_{\mathbf{C}}(\Omega,\mathcal{F},|\mu+\alpha\nu|)$. From the dominated convergence theorem (23), we have:

$$\begin{aligned}\lim_{n\to+\infty}\int s_n d(\mu+\alpha\nu) &= \lim_{n\to+\infty}\int s_n h d|\mu+\alpha\nu|\\ &= \int fhd|\mu+\alpha\nu|\\ &= \int fd(\mu+\alpha\nu)\end{aligned}$$

We show similarly that:

$$\lim_{n\to+\infty}\int s_n d\mu = \int f d\mu$$

and:

$$\lim_{n\to+\infty}\int s_n d\nu = \int f d\nu$$

Having proved (27) for all simple functions on $(\Omega,\mathcal{F})$, we have:

$$\int s_n d(\mu+\alpha\nu) = \int s_n d\mu + \alpha\int s_n d\nu$$

and taking the limit as $n\to+\infty$, we see that (27) is also true for f non-negative, measurable and belonging to $L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu)\cap L^1_{\mathbf{C}}(\Omega,\mathcal{F},\nu)$.

If f is an arbitrary element of $L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu)\cap L^1_{\mathbf{C}}(\Omega,\mathcal{F},\nu)$, then it can be expressed as $f = f_1 - f_2 + i(f_3 - f_4)$ where each f_i is non-negative, measurable and belonging to $L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu)\cap L^1_{\mathbf{C}}(\Omega,\mathcal{F},\nu)$. Equation (27) being true for each

f_i, it follows by linearity that (27) is also true for f. We have proved that (27) is true for all elements f of $\in L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu)\cap L^1_{\mathbf{C}}(\Omega,\mathcal{F},\nu)$.

Exercise 17

Exercise 18.

1. Let μ,ν be two measures on $(\Omega,\mathcal{F})$ and $\alpha\in[0,+\infty]$. Then:

$$(\mu+\alpha\nu)(\emptyset)=\mu(\emptyset)+\alpha\nu(\emptyset)=0 \tag{28}$$

Note that from the convention $(+\infty)\times 0=0$, equation (28) is still true in the case when $\alpha=+\infty$. Furthermore, if $A\in\mathcal{F}$ and $(A_n)_{n\geq 1}$ is a sequence of pairwise disjoint elements of $\mathcal{F}$ with $A=\uplus_{n\geq 1}A_n$, then:

$$\begin{aligned}(\mu+\alpha\nu)(A) &= \mu(A)+\alpha\nu(A)\\ &= \sum_{n=1}^{+\infty}\mu(A_n)+\alpha\sum_{n=1}^{+\infty}\nu(A_n)\\ &= \sum_{n=1}^{+\infty}\mu(A_n)+\sum_{n=1}^{+\infty}\alpha\nu(A_n)\\ &= \sum_{n=1}^{+\infty}\mu(A_n)+\alpha\nu(A_n)\\ &= \sum_{n=1}^{+\infty}(\mu+\alpha\nu)(A_n)\end{aligned}$$

Note that the third equality is still true if $\alpha=+\infty$ or $\nu(A)=\sum_{n\geq 1}\nu(A_n)=+\infty$. It follows that $\mu+\alpha\nu$ is countably additive, and we have proved that it is indeed a measure on $(\Omega,\mathcal{F})$. Now, given $f:(\Omega,\mathcal{F})\to[0,+\infty]$, we claim that:

$$\int f d(\mu+\alpha\nu)=\int f d\mu+\alpha\int f d\nu \tag{29}$$

(29) is obviously true when f is of the form $f=1_E$ with $E\in\mathcal{F}$. By linearity (which is still valid, even if $\alpha=+\infty$), (29) is also true when f is a simple function on $(\Omega,\mathcal{F})$. If f is an arbitrary non-negative and measurable map, from theorem (18) there exists a sequence $(s_n)_{n\geq 1}$ of simple functions on $(\Omega,\mathcal{F})$, such that $s_n\uparrow f$. Having proved (29) for any simple function, for all $n\geq 1$ we have:

$$\int s_n d(\mu+\alpha\nu)=\int s_n d\mu+\alpha\int s_n d\nu \tag{30}$$

From the monotone convergence theorem (19), taking the limit in (30) as $n\to+\infty$, we conclude that (29) is also true for f. Note that if $\alpha=+\infty$ and $(u_n)_{n\geq 1}$ is a sequence in $[0,+\infty]$ converging to some $u\in[0,+\infty]$, then it is not true in general that $\alpha u_n\to\alpha u$. Indeed, consider the case when $u_n=1/n$. Then $\alpha u=(+\infty)\times 0=0$ while $\alpha u_n=(+\infty)\times(1/n)=+\infty$ for all $n\geq 1$, and $(\alpha u_n)_{n\geq 1}$ does not converge to αu. However, if $u_n\leq u_{n+1}$ for all $n\geq 1$, then the convergence $\alpha u_n\to\alpha u$ is true.

Indeed, if $u = \sup_{n\geq 1} u_n = 0$, then $u_n = 0$ for all $n \geq 1$ and consequently $\alpha u_n = 0 = \alpha u$. If $u \neq 0$, then $u_n \neq 0$ for n large enough, and consequently $\alpha u_n = +\infty = \alpha u$ for n large enough. All this to say that even in the case when $\alpha = +\infty$, the convergence $\alpha \int s_n d\nu \to \alpha \int f d\nu$ is true.

2. We claim that:
$$\int f d\mu \leq \int f d\nu \tag{31}$$
Since $\mu \leq \nu$, (31) is true when $f = 1_E$, and $E \in \mathcal{F}$. By linearity, (31) is also true when f is a simple function on $(\Omega, \mathcal{F})$. If f is an arbitrary non-negative and measurable map, from theorem (18) there exists a sequence $(s_n)_{n\geq 1}$ of simple functions on $(\Omega, \mathcal{F})$, such that $s_n \uparrow f$. Having proved (31) for any simple function, for all $n \geq 1$ we have:
$$\int s_n d\mu \leq \int s_n d\nu \tag{32}$$
From the monotone convergence theorem (19), taking the limit in (32) as $n \to +\infty$, we conclude that (31) is also true for f.

Exercise 18

Exercise 19.

1. Since $\mu_1 = Re(\mu)$, for all $F \in \mathcal{F}$ we have $|\mu_1(F)| \leq |\mu(F)|$. Hence, if $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ is a measurable partition of E:
$$\sum_{n=1}^{+\infty} |\mu_1(E_n)| \leq \sum_{n=1}^{+\infty} |\mu(E_n)| \leq |\mu|(E)$$
It follows that $|\mu|(E)$ is an upper bound of all $\sum_{n=1}^{+\infty} |\mu_1(E_n)|$, as $(E_n)_{n\geq 1}$ ranges through all measurable partitions of E. Since $|\mu_1|(E)$ is the smallest of such upper bounds, $|\mu_1|(E) \leq |\mu|(E)$. This being true for all $E \in \mathcal{F}$, we conclude that $|\mu_1| \leq |\mu|$. We show similarly that $|\mu_2| \leq |\mu|$.

2. Let $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E:
$$\sum_{n=1}^{+\infty} |\mu(E_n)| \leq \sum_{n=1}^{+\infty} |\mu_1(E_n)| + |\mu_2(E_n)| \leq |\mu_1|(E) + |\mu_2|(E)$$
$|\mu|(E)$ being the supremum of all sums involved on the l.h.s of this inequality, we conclude that $|\mu|(E) \leq |\mu_1|(E) + |\mu_2|(E)$ for all $E \in \mathcal{F}$, i.e. that $|\mu| \leq |\mu_1| + |\mu_2|$.

3. Let $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ be a measurable map. Proving:
$$L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) = L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_1) \cap L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_2)$$
amounts to showing the equivalence:
$$\int |f| d|\mu| < +\infty \Leftrightarrow \int |f| d|\mu_1| < +\infty, \int |f| d|\mu_2| < +\infty \tag{33}$$

From $|\mu_1| \leq |\mu|$ and $|\mu_2| \leq |\mu|$ using exercise (18) we obtain:

$$\int |f|d|\mu_1| \leq \int |f|d|\mu| \tag{34}$$

and:

$$\int |f|d|\mu_2| \leq \int |f|d|\mu| \tag{35}$$

Furthermore, from $|\mu| \leq |\mu_1| + |\mu_2|$ and exercise (18):

$$\int |f|d|\mu| \leq \int |f|d(|\mu_1| + |\mu_2|) = \int |f|d|\mu_1| + \int |f|d|\mu_2| \tag{36}$$

The equivalence (33) follows easily from (34), (35) and (36).

4. Let $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ be a measurable map. Proving:

$$L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_1) = L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_1^+) \cap L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_1^-)$$

amounts to showing the equivalence:

$$\int |f|d|\mu_1| < +\infty \Leftrightarrow \int |f|d\mu_1^+ < +\infty, \int |f|d\mu_1^- < +\infty \tag{37}$$

The positive and negative parts μ_1^+ and μ_1^- of μ_1 being defined as $\mu_1^+ = (|\mu_1| - \mu_1)/2$ and $\mu_1^- = (|\mu_1| - \mu_1)/2$ (see exercise (12) of Tutorial 11), we have $|\mu_1| = \mu_1^+ + \mu_1^-$. Using exercise (18):

$$\int |f|d|\mu_1| = \int |f|d\mu_1^+ + \int |f|d\mu_1^-$$

Hence, the equivalence (37) is clear. We show similarly that:

$$L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_2) = L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_2^+) \cap L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_2^-)$$

5. Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. We claim that

$$\int f d\mu = \int f d\mu_1^+ - \int f d\mu_1^- + i\left(\int f d\mu_2^+ - \int f d\mu_2^-\right) \tag{38}$$

Note that from 3. and 4. we have:

$$f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_1^+) \cap L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_1^-) \cap L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_2^+) \cap L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_2^-)$$

and consequently all integrals in (38) are well-defined. Applying exercise (17) to the complex measures (in fact signed measures) μ_1, μ_2 and $\alpha = i$, we obtain:

$$\int f d\mu = \int f d\mu_1 + i \int f d\mu_2 \tag{39}$$

Applying exercise (17) to the complex measures (in fact finite measures) μ_1^+, μ_1^- and $\alpha = -1$, we obtain:

$$\int f d\mu_1 = \int f d\mu_1^+ - \int f d\mu_1^- \tag{40}$$

Similarly, we have:

$$\int f d\mu_2 = \int f d\mu_2^+ - \int f d\mu_2^- \tag{41}$$

Equation (38) follows from (39), (40) and (41).

Exercise 19

Exercise 20.

1. By definition, the trace of $\mathcal{F}$ on A is given by:

$$\mathcal{F}_{|A} \stackrel{\triangle}{=} \{A \cap E : E \in \mathcal{F}\}$$

Since A is an element of $\mathcal{F}$, it is clear that $\mathcal{F}_{|A} \subseteq \mathcal{F}'$, where:

$$\mathcal{F}' = \{E : E \in \mathcal{F}, E \subseteq A\}$$

For the reverse inclusion, note that if $E \in \mathcal{F}'$ then E can be written as $E = A \cap E$ and $E \in \mathcal{F}$. So E is an element of $\mathcal{F}_{|A}$.

2. Let $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E. Then $(A\cap E_n)_{n\geq 1}$ is a measurable partition of $A\cap E$. Since μ is a complex measure of $(\Omega, \mathcal{F})$, we have:

$$\mu(A \cap E) = \sum_{n=1}^{+\infty} \mu(A \cap E_n) \tag{42}$$

i.e. the right-hand-side series converges to $\mu(A\cap E)$. By the very definition of μ^A, (42) can be re-expressed as:

$$\mu^A(E) = \sum_{n=1}^{+\infty} \mu^A(E_n) \tag{43}$$

i.e. the right-hand-side series converges to $\mu^A(E)$. This shows that μ^A is a complex measure on $(\Omega, \mathcal{F})$.

Let $E \in \mathcal{F}_{|A}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E, i.e. a sequence of pairwise disjoint elements of $\mathcal{F}_{|A}$ with $E = \uplus_{n=1}^{+\infty} E_n$. From 1., E and every E_n is an element of $\mathcal{F}$, (while being a subset of A). μ being a complex measure on $(\Omega, \mathcal{F})$, we have:

$$\mu(E) = \sum_{n=1}^{+\infty} \mu(E_n) \tag{44}$$

i.e. the right-hand-side series converges to $\mu(E)$. Since $\mu_{|A}$ is defined as the restriction of μ to $\mathcal{F}_{|A}$, and since E and all E_n's are elements of $\mathcal{F}_{|A}$, (45) can be equivalently expressed as:

$$\mu_{|A}(E) = \sum_{n=1}^{+\infty} \mu_{|A}(E_n) \tag{45}$$

i.e. the right-hand-side series converges to $\mu_{|A}(E)$. This shows that $\mu_{|A}$ is a complex measure on $(A, \mathcal{F}_{|A})$.

3. Let $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E. Then $(A \cap E_n)_{n\geq 1}$ is a measurable partition of $A \cap E$. Hence:
$$\sum_{n=1}^{+\infty} |\mu(A \cap E_n)| \leq |\mu|(A \cap E)$$
or equivalently:
$$\sum_{n=1}^{+\infty} |\mu^A(E_n)| \leq |\mu|^A(E)$$

4. From the previous section 3., $|\mu|^A(E)$ is an upper bound of all sums $\sum_{n=1}^{+\infty} |\mu^A(E_n)|$, as $(E_n)_{n\geq 1}$ ranges through all measurable partitions of E. Since $|\mu^A|(E)$ is the smallest of such upper bounds, we have $|\mu^A|(E) \leq |\mu|^A(E)$. This being true for all $E \in \mathcal{F}$, we conclude that $|\mu^A| \leq |\mu|^A$.

5. Let $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of $A \cap E$. For all $n \geq 1$, $E_n \subseteq A$ and consequently $\mu(E_n) = \mu^A(E_n)$. Hence:
$$\sum_{n=1}^{+\infty} |\mu(E_n)| = \sum_{n=1}^{+\infty} |\mu^A(E_n)| \leq |\mu^A|(A \cap E)$$

6. Let $(E_n)_{n\geq 1}$ be a measurable partition of A^c. Then:
$$\sum_{n=1}^{+\infty} |\mu^A(E_n)| = \sum_{n=1}^{+\infty} |\mu(A \cap E_n)| = 0$$
$|\mu^A|(A^c)$ being the supremum of all sums $\sum_{n=1}^{+\infty} |\mu^A(E_n)|$, as $(E_n)_{n\geq 1}$ ranges through all measurable partitions of A^c, we conclude that $|\mu^A|(A^c) = 0$.

7. From 5. it follows that $|\mu^A|(A\cap E)$ is an upper bound of all sums $\sum_{n=1}^{+\infty} |\mu(E_n)|$, as $(E_n)_{n\geq 1}$ ranges through all measurable partitions of $A \cap E$. $|\mu|(A \cap E)$ being the smallest of such upper bounds, we have $|\mu|(A\cap E) \leq |\mu^A|(A\cap E)$. However, from 6. we have $|\mu^A|(A^c) = 0$, and consequently:
$$|\mu^A|(E) = |\mu^A|(A \cap E) + |\mu^A|(A^c \cap E) = |\mu^A|(A \cap E)$$
It follows that $|\mu|(A \cap E) \leq |\mu^A|(E)$. This being true for all $E \in \mathcal{F}$, we see that $|\mu|^A \leq |\mu^A|$. Having proved in 4. that $|\mu^A| \leq |\mu|^A$, we conclude that $|\mu^A| = |\mu|^A$. In other words, the total variation of the *restriction* of μ to A, is equal to the *restriction* of the total variation of μ to A.

8. Let $E \in \mathcal{F}_{|A}$ and $(E_n)_{n\geq 1}$ be an $\mathcal{F}_{|A}$-measurable partition of E. Since $\mathcal{F}_{|A} \subseteq \mathcal{F}$, $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ is also an $\mathcal{F}$-measurable partition of E. Hence:
$$\sum_{n=1}^{+\infty} |\mu(E_n)| \leq |\mu|(E) \tag{46}$$

$\mu_{|A}$ and $|\mu|_{|A}$ being respectively the restrictions of μ and $|\mu|$ to $\mathcal{F}_{|A}$, (46) can be re-expressed as:

$$\sum_{n=1}^{+\infty}|\mu_{|A}(E_n)| \leq |\mu|_{|A}(E)$$

9. Given $E \in \mathcal{F}_{|A}$, it appears from 8. that $|\mu|_{|A}(E)$ is an upper bound of all sums $\sum_{n=1}^{+\infty}|\mu_{|A}(E_n)|$, as $(E_n)_{n\geq 1}$ ranges through all $\mathcal{F}_{|A}$-measurable partitions of E. Since $|\mu_{|A}|(E)$ is the smallest of such upper bounds, we have $|\mu_{|A}|(E) \leq |\mu|_{|A}(E)$. This being true for all $E \in \mathcal{F}_{|A}$, we conclude that $|\mu_{|A}| \leq |\mu|_{|A}$.

10. Let $E \in \mathcal{F}_{|A}$ and $(E_n)_{n\geq 1}$ be an $\mathcal{F}$-measurable partition of E. From 1. we have $E \in \mathcal{F}$ and $E \subseteq A$. It follows that $E_n \subseteq A$ for all $n \geq 1$ and consequently $E_n \in \mathcal{F}_{|A}$. So $(E_n)_{n\geq 1}$ is also an $\mathcal{F}_{|A}$-measurable partition of E. Hence:
$$\sum_{n=1}^{+\infty}|\mu_{|A}(E_n)| \leq |\mu_{|A}|(E)$$
which can be equivalently written as:
$$\sum_{n=1}^{+\infty}|\mu(E_n)| \leq |\mu_{|A}|(E)$$

11. Given $E \in \mathcal{F}_{|A}$, it appears from 10. that $|\mu_{|A}|(E)$ is an upper bound of all sums $\sum_{n=1}^{+\infty}|\mu(E_n)|$, as $(E_n)_{n\geq 1}$ ranges through all $\mathcal{F}$-measurable partitions of E. Since $|\mu|(E)$ is the smallest of such upper bounds, we have $|\mu|(E) \leq |\mu_{|A}|(E)$, or equivalently since $E \in \mathcal{F}_{|A}$, $|\mu|_{|A}(E) \leq |\mu_{|A}|(E)$. This being true for all $E \in \mathcal{F}_{|A}$, $|\mu|_{|A} \leq |\mu_{|A}|$. Having proved in 9. that $|\mu_{|A}| \leq |\mu|_{|A}$, we conclude that $|\mu_{|A}| = |\mu|_{|A}$.

12. By assumption, $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ is such that $|h| = 1$ and $\mu = \int hd|\mu|$. In particular, for all $E \in \mathcal{F}$:
$$\begin{aligned}
\mu^A(E) &= \mu(A\cap E) \\
&= \int_{A\cap E} hd|\mu| \\
&= \int (h1_E)1_A d|\mu| \\
&= \int (h1_E)d|\mu|^A \\
&= \int_E hd|\mu|^A \\
&= \int_E hd|\mu^A|
\end{aligned}$$

where the first equality stems from the definition of μ^A, the second from the fact that $\mu = \int hd|\mu|$, the third, fourth and fifth from a use of definition (49) and finally the sixth from the fact that $|\mu|^A = |\mu^A|$. This being true for all $E \in \mathcal{F}$, we have proved that $\mu^A = \int hd|\mu^A|$.

13. Since $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$, from definition (49), $h_{|A}$ is an element of $L^1_{\mathbf{C}}(A, \mathcal{F}_{|A}, |\mu|_{|A})$. Having proved that $|\mu|_{|A} = |\mu_{|A}|$, it follows that $h_{|A} \in L^1_{\mathbf{C}}(A, \mathcal{F}_{|A}, |\mu_{|A}|)$ [2]. Furthermore, for all $E \in \mathcal{F}_{|A}$:

$$\begin{aligned}
\mu_{|A}(E) &= \mu(E) \\
&= \mu(A \cap E) \\
&= \int_{A\cap E} hd|\mu| \\
&= \int (h1_E)1_A d|\mu| \\
&= \int h_{|A}(1_E)_{|A} d|\mu|_{|A} \\
&= \int_E h_{|A} d|\mu|_{|A} \\
&= \int_E h_{|A} d|\mu_{|A}|
\end{aligned}$$

where the first equality stems from the definition of $\mu_{|A}$, the second from the fact that $E \subseteq A$, the third from the fact that $\mu = \int hd|\mu|$, the fourth, fifth and sixth from definition (49) an finally the seventh from the fact that $|\mu|_{|A} = |\mu_{|A}|$. This being true for all $E \in \mathcal{F}_{|A}$, we conclude that $\mu_{|A} = \int h_{|A} d|\mu_{|A}|$.

14. Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. From definition (97), this is equivalent to $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$. Applying definition (49), we have:

$$f1_A \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|),\, f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|^A),\, f_{|A} \in L^1_{\mathbf{C}}(A, \mathcal{F}_{|A}, |\mu|_{|A})$$

and since $|\mu|^A = |\mu^A|$ and $|\mu|_{|A} = |\mu_{|A}|$, we obtain:

$$f1_A \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|),\, f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu^A|),\, f_{|A} \in L^1_{\mathbf{C}}(A, \mathcal{F}_{|A}, |\mu_{|A}|)$$

Moreover, since $|h| = 1$ and $\mu = \int hd|\mu|$, from definition (97):

$$\int f1_A d\mu = \int fh1_A d|\mu| \tag{47}$$

and similarly, since $\mu^A = \int hd|\mu^A|$ and $|\mu|^A = |\mu^A|$:

$$\int fd\mu^A = \int fhd|\mu^A| = \int fhd|\mu|^A \tag{48}$$

[2] One may argue that $|h_{|A}| = 1$ and $|\mu_{|A}|$ is a finite measure...

Furthermore since $\mu_{|A} = \int h_{|A} d|\mu_{|A}|$ and $|\mu|_{|A} = |\mu_{|A}|$:

$$\int f_{|A} d\mu_{|A} = \int f_{|A} h_{|A} d|\mu_{|A}| = \int (fh)_{|A} d|\mu|_{|A} \tag{49}$$

Finally, from definition (49):

$$\int fh1_A d|\mu| = \int fhd|\mu|^A = \int (fh)_{|A} d|\mu|_{|A} \tag{50}$$

Comparing (47), (48) and (49) with (50), we conclude that:

$$\int f1_A d\mu = \int f d\mu^A = \int f_{|A} d\mu_{|A}$$

Exercise 20

Exercise 21. Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, where μ is a complex measure on $(\Omega, \mathcal{F})$. Let $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ be such that $|h| = 1$ and $\mu = \int hd|\mu|$. Let $\nu = \int f d\mu$, i.e. be the map defined by:

$$\forall E \in \mathcal{F} \ , \ \nu(E) = \int_E f d\mu$$

From definitions (98), (97) and (49), for all $E \in \mathcal{F}$:

$$\nu(E) = \int f1_E d\mu = \int fh1_E d|\mu| = \int_E fhd|\mu|$$

It follows that $\nu = \int fhd|\mu|$, and applying theorem (63), ν is therefore a complex measure on $(\Omega, \mathcal{F})$, with total variation $|\nu|$ given by:

$$\forall E \in \mathcal{F} \ , \ |\nu|(E) = \int_E |fh| d|\mu| = \int_E |f| d|\mu|$$

Let $g : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ be measurable. Applying theorem (21) to $|\nu| = \int |f| d|\mu|$, we obtain:

$$\int |g| d|\nu| = \int |g||f| d|\mu|$$

and therefore we have the equivalence:

$$\int |g| d|\nu| < +\infty \ \Leftrightarrow \ \int |gf| d|\mu| < +\infty$$

i.e.

$$g \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu) \ \Leftrightarrow \ gf \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$$

When such condition is satisfied, we claim that:

$$\int g d\nu = \int gf d\mu \tag{51}$$

This equality is clearly true when g is of the form $g = 1_E$ where $E \in \mathcal{F}$ (such a g would automatically lie in $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu)$ since $|\nu|$ is a finite measure). By the linearity of the integral (with respect to complex measures, such a linearity is proved in exercise (16)), equation (51) is also true when g is a simple function on $(\Omega, \mathcal{F})$. If g is non-negative and measurable, while being an element of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu)$, from theorem (18) there exists a sequence $(s_n)_{n\geq 1}$ of simple functions on $(\Omega, \mathcal{F})$, such that $s_n \uparrow g$. Let k be an arbitrary element of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\nu|)$ with $|k| = 1$ and $\nu = \int kd|\nu|$. Then:

$$\begin{aligned}
\lim_{n\to+\infty} \int s_n d\nu &= \lim_{n\to+\infty} \int s_n k d|\nu| \\
&= \int gkd|\nu| \\
&= \int gd\nu
\end{aligned}$$

where the first and third equalities stem from definition (97), and the second from the dominated convergence theorem (23) (and the fact that $s_n k \to gk$ with $|s_n k| = s_n \leq g \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\nu|)$). Similarly:

$$\begin{aligned}
\lim_{n\to+\infty} \int s_n f d\mu &= \lim_{n\to+\infty} \int s_n f h d|\mu| \\
&= \int gfhd|\mu| \\
&= \int gfd\mu
\end{aligned}$$

where the first and third equalities stem from definition (97), and the second from the dominated convergence theorem (23) (and the fact that $s_n fh \to gfh$ with $|s_n fh| = s_n|f| \leq g|f| \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$). Having proved (51) for simple functions, for all $n \geq 1$:

$$\int s_n d\nu = \int s_n f d\mu$$

and taking the limit as $n \to +\infty$, we see that (51) is also true whenever g is non-negative and measurable, while being an element of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu)$. If g is an arbitrary element $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu)$, then it can be decomposed as $g = g_1 - g_2 + i(g_3 - g_4)$ where each g_i is non-negative and measurable, while being an element of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu)$. By linearity, equation (51) is also true for g.

Exercise 21

Exercise 22.

1. Let $\Omega = \Omega_1 \times \ldots \times \Omega_n$ and $\mathcal{F} = \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$. Then:

$$\forall E \in \mathcal{F} \ , \ \mu(E) = \int_E h d\nu$$

where $h = h_1 \ldots h_n$ and $\nu = |\mu_1| \otimes \ldots \otimes |\mu_n|$ is the product measure, as defined in definition (62). Each total variation $|\mu_i|$ being a finite measure, ν is also a finite measure, and furthermore $|h| = |h_1| \ldots |h_n| = 1$. Moreover, the map h is clearly measurable with respect to $\mathcal{F}$, as the equality:

$$\forall B \in \mathcal{B}(\mathbf{C}) \ , \ h_i^{-1}(B) = \Omega_1 \times \ldots \times h_i^{-1}(B) \times \ldots \times \Omega_n$$

shows that each h_i (viewed as a map defined on the product space $(\Omega, \mathcal{F})$) is measurable. It follows that μ is of the form $\mu = \int h d\nu$, where $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu)$. From theorem (63), we conclude that μ is a complex measure on $(\Omega, \mathcal{F})$. In fact, theorem (63) goes further, asserting that the total variation of μ is:

$$\forall E \in \mathcal{F} \ , \ |\mu|(E) = \int_E |h| d\nu = \int 1_E d\nu = \nu(E)$$

i.e. $|\mu| = \nu = |\mu_1| \otimes \ldots \otimes |\mu_n|$.

2. Let $A = A_1 \times \ldots \times A_n$ be a measurable rectangle. We have:

$$\begin{aligned}
\int_{A_1 \times A_2} h_1 h_2 d|\mu_1| \otimes |\mu_2| &= \int h_1 h_2 1_{A_1 \times A_2} d|\mu_1| \otimes |\mu_2| \\
&= \int (h_1 1_{A_1})(h_2 1_{A_2}) d|\mu_1| \otimes |\mu_2| \\
&= \int \left(\int (h_1 1_{A_1})(h_2 1_{A_2}) d|\mu_2| \right) d|\mu_1| \\
&= \int h_1 1_{A_1} \left(\int h_2 1_{A_2} d|\mu_2| \right) d|\mu_1| \\
&= \int h_1 1_{A_1} \mu_2(A_2) d|\mu_1| \\
&= \mu_2(A_2) \int h_1 1_{A_1} d|\mu_1| \\
&= \mu_1(A_1) \cdot \mu_2(A_2)
\end{aligned}$$

Where crucially, the third equality stems from Fubini theorem (33). If $n = 2$, then we have nothing further to prove. If $n > 2$, we consider the induction hypothesis, for $2 \leq k \leq n$:

$$\int_{B_k} g_k d\nu_k = \mu_1(A_1) \ldots \mu_k(A_k) \tag{52}$$

where $B_k = A_1 \times \ldots \times A_k$, $\nu_k = |\mu_1| \otimes \ldots \otimes |\mu_k|$ and g_k is defined as $g_k = h_1 \ldots h_k$. If we assume that such induction hypothesis is true for some k with $2 \leq k \leq n-1$, then:

$$\begin{aligned}
\int_{B_{k+1}} g_{k+1} d\nu_{k+1} &= \int (g_k 1_{B_k})(h_{k+1} 1_{A_{k+1}}) d\nu_k \otimes |\mu_{k+1}| \\
&= \int g_k 1_{B_k} \left(\int h_{k+1} 1_{A_{k+1}} d|\mu_{k+1}| \right) d\nu_k \\
&= \mu_{k+1}(A_{k+1}) \int_{B_k} g_k d\nu_k \\
&= \mu_1(A_1) \ldots \mu_{k+1}(A_{k+1})
\end{aligned}$$

where the second equality stems from Fubini theorem (33) and the fourth from our induction hypothesis (52). This shows that (52) is in fact true for all $k = 2, \ldots, n$, and finally:

$$\mu(A) = \int_{B_n} g_n d\nu_n = \mu_1(A_1)\ldots\mu_n(A_n)$$

3. We have proved that μ is a complex measure on $(\Omega, \mathcal{F})$ such that for all measurable rectangle $A = A_1 \times \ldots \times A_n$:

$$\mu(A) = \mu_1(A_1)\ldots\mu_n(A_n)$$

In order to prove theorem (66), it remains to show that such a measure is unique. Suppose μ and ν are two complex measures on $(\Omega, \mathcal{F})$ which coincide on the set of measurable rectangles $\mathcal{F}_1 \amalg \ldots \amalg \mathcal{F}_n$. We define:

$$\mathcal{D} = \{E \in \mathcal{F} \ , \ \mu(E) = \nu(E)\}$$

Then $\mathcal{F}_1 \amalg \ldots \amalg \mathcal{F}_n \subseteq \mathcal{D}$, and $\mathcal{D}$ is easily seen to be a Dynkin system on $(\Omega, \mathcal{F})$. Indeed, Ω being a measurable rectangle, we have $\Omega \in \mathcal{D}$. Furthermore, If $A, B \in \mathcal{D}$ and $A \subseteq B$, Then:

$$\begin{aligned}
\mu(B \setminus A) &= \mu(A) + \mu(B \setminus A) - \mu(A) \\
&= \mu(B) - \mu(A) \\
&= \nu(B) - \nu(A) \\
&= \nu(B \setminus A)
\end{aligned}$$

and therefore $B \setminus A \in \mathcal{D}$. Moreover, if $(A_n)_{n\geq 1}$ is a sequence of elements of $\mathcal{D}$ such that $A_n \uparrow A$, then using exercise (13):

$$\mu(A) = \lim_{n\to+\infty} \mu(A_n) = \lim_{n\to+\infty} \nu(A_n) = \nu(A)$$

and therefore $A \in \mathcal{D}$. So $\mathcal{D}$ is indeed a Dynkin system on $(\Omega, \mathcal{F})$. The set of measurable rectangles being closed under finite intersection (and being a subset of $\mathcal{D}$), from the Dynkin system theorem (1), we have:

$$\sigma(\mathcal{F}_1 \amalg \ldots \amalg \mathcal{F}_n) \subseteq \mathcal{D}$$

and consequently $\mathcal{F} = \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n \subseteq \mathcal{D}$. It follows that $\mathcal{D} = \mathcal{F}$ and finally $\mu = \nu$. This proves theorem (66).

Exercise 22

Exercise 23.

1. We saw in exercise (22) that the complex measure μ defined by:

$$\forall E \in \mathcal{F} \ , \ \mu(E) = \int_E h_1 \ldots h_n d|\mu_1| \otimes \ldots \otimes |\mu_n| \tag{53}$$

satisfies the requirement of theorem (66), and is therefore equal to the product measure $\mu_1 \otimes \ldots \otimes \mu_n$. Furthermore, we proved using theorem (63) that $|\mu| = |\mu_1| \otimes \ldots \otimes |\mu_n|$.

2.

$$\begin{aligned} \|\mu_1 \otimes \ldots \otimes \mu_n\| &= |\mu_1 \otimes \ldots \otimes \mu_n|(\Omega) \\ &= |\mu_1| \otimes \ldots \otimes |\mu_n|(\Omega) \\ &= |\mu_1|(\Omega_1) \ldots |\mu_n|(\Omega_n) \\ &= \|\mu_1\| \ldots \|\mu_n\| \end{aligned}$$

3. From (53) and $|\mu| = |\mu_1| \otimes \ldots \otimes |\mu_n|$, we obtain:

$$\forall E \in \mathcal{F} \ , \ \mu(E) = \int_E h_1 \ldots h_n d|\mu|$$

4. Having shown that $\mu = \int hd|\mu|$ with $h = h_1 \ldots h_n$ ($|h| = 1$), it follows from definition (97) that for all $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$:

$$\int f d\mu = \int fhd|\mu|$$

or equivalently:

$$\int f d\mu_1 \otimes \ldots \otimes \mu_n = \int fh_1 \ldots h_n d|\mu_1| \otimes \ldots \otimes |\mu_n|$$

5. Let σ be a permutation of $\mathbf{N}_n$ and $h = h_1 \ldots h_n$. Then:

$$\begin{aligned} \int f d\mu_1 \otimes \ldots \otimes \mu_n &= \int fhd|\mu_1| \otimes \ldots \otimes |\mu_n| \\ &= \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} fhd|\mu_{\sigma(1)}| \ldots d|\mu_{\sigma(n)}| \\ &= \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)} \end{aligned}$$

where the second equality stems from exercise (17) of Tutorial 7, and the third equality from:

$$\begin{aligned} \int_{\Omega_{\sigma(1)}} fhd|\mu_{\sigma(1)}| &= h_{\sigma(2)} \ldots h_{\sigma(n)} \int_{\Omega_{\sigma(1)}} fh_{\sigma(1)} d|\mu_{\sigma(1)}| \\ &= h_{\sigma(2)} \ldots h_{\sigma(n)} \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \end{aligned}$$

followed by an induction argument.

Exercise 23