6. Product Spaces

In the following, I is a non-empty set.

Definition 50 *Let $(\Omega_i)_{i\in I}$ be a family of sets, indexed by a non-empty set I. We call* **Cartesian product** *of the family $(\Omega_i)_{i\in I}$ the set, denoted $\Pi_{i\in I}\Omega_i$, and defined by:*

$$\prod_{i\in I}\Omega_i \stackrel{\triangle}{=} \{\omega : I \to \cup_{i\in I}\Omega_i \ , \ \omega(i) \in \Omega_i \ , \ \forall i \in I\}$$

In other words, $\Pi_{i\in I}\Omega_i$ is the set of all maps ω defined on I, with values in $\cup_{i\in I}\Omega_i$, such that $\omega(i) \in \Omega_i$ for all $i \in I$.

Theorem 25 (Axiom of choice) *Let $(\Omega_i)_{i\in I}$ be a family of sets, indexed by a non-empty set I. Then, $\Pi_{i\in I}\Omega_i$ is non-empty, if and only if Ω_i is non-empty for all $i \in I$[1].*

Exercise 1.

1. Let Ω be a set and suppose that $\Omega_i = \Omega, \forall i \in I$. We use the notation Ω^I instead of $\Pi_{i\in I}\Omega_i$. Show that Ω^I is the set of all maps $\omega : I \to \Omega$.
2. What are the sets $\mathbf{R}^{\mathbf{R}^+}$, $\mathbf{R}^{\mathbf{N}}$, $[0,1]^{\mathbf{N}}$, $\bar{\mathbf{R}}^{\mathbf{R}}$?
3. Suppose $I = \mathbf{N}^*$. We sometimes use the notation $\Pi_{n=1}^{+\infty}\Omega_n$ instead of $\Pi_{n\in\mathbf{N}^*}\Omega_n$. Let $\mathcal{S}$ be the set of all sequences $(x_n)_{n\geq 1}$ such that $x_n \in \Omega_n$ for all $n \geq 1$. Is $\mathcal{S}$ the same thing as the product $\Pi_{n=1}^{+\infty}\Omega_n$?
4. Suppose $I = \mathbf{N}_n = \{1, \ldots, n\}$, $n \geq 1$. We use the notation $\Omega_1 \times \ldots \times \Omega_n$ instead of $\Pi_{i\in\{1,\ldots,n\}}\Omega_i$. For $\omega \in \Omega_1 \times \ldots \times \ \Omega_n$, it is customary to write $(\omega_1, \ldots, \omega_n)$ instead of ω, where we have $\omega_i = \omega(i)$. What is your guess for the definition of sets such as $\mathbf{R}^n, \bar{\mathbf{R}}^n, \mathbf{Q}^n, \mathbf{C}^n$.
5. Let E, F, G be three sets. Define $E \times F \times G$.

Definition 51 *Let I be a non-empty set. We say that a family of sets $(I_\lambda)_{\lambda\in\Lambda}$, where $\Lambda \neq \emptyset$, is a* **partition** *of I, if and only if:*

$$\begin{array}{ll} (i) & \forall\lambda \in \Lambda \ , \ I_\lambda \neq \emptyset \\ (ii) & \forall\lambda, \lambda' \in \Lambda \ , \ \lambda \neq \lambda' \Rightarrow I_\lambda \cap I_{\lambda'} = \emptyset \\ (iii) & I = \cup_{\lambda\in\Lambda} I_\lambda \end{array}$$

Exercise 2. Let $(\Omega_i)_{i\in I}$ be a family of sets indexed by I, and $(I_\lambda)_{\lambda\in\Lambda}$ be a partition of the set I.

1. For each $\lambda \in \Lambda$, recall the definition of $\Pi_{i\in I_\lambda}\Omega_i$.

[1] When I is finite, this theorem is traditionally derived from other axioms.

2. Recall the definition of $\Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}\Omega_i)$.

3. Define a *natural* bijection $\Phi : \Pi_{i\in I}\Omega_i \to \Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}\Omega_i)$.

4. Define a *natural* bijection $\psi : \mathbf{R}^p \times \mathbf{R}^n \to \mathbf{R}^{p+n}$, for all $n, p \geq 1$.

Definition 52 *Let $(\Omega_i)_{i\in I}$ be a family of sets, indexed by a non-empty set I. For all $i \in I$, let $\mathcal{E}_i$ be a set of subsets of Ω_i. We define a* **rectangle** *of the family $(\mathcal{E}_i)_{i\in I}$, as any subset A of $\Pi_{i\in I}\Omega_i$, of the form $A = \Pi_{i\in I}A_i$ where $A_i \in \mathcal{E}_i \cup \{\Omega_i\}$ for all $i \in I$, and such that $A_i = \Omega_i$ except for a finite number of indices $i \in I$. Consequently, the set of all rectangles, denoted $\amalg_{i\in I}\mathcal{E}_i$, is defined as:*

$$\coprod_{i\in I}\mathcal{E}_i \stackrel{\triangle}{=} \left\{\prod_{i\in I}A_i : A_i \in \mathcal{E}_i \cup \{\Omega_i\}\ ,\ A_i \neq \Omega_i \text{ for finitely many } i \in I\right\}$$

Exercise 3. $(\Omega_i)_{i\in I}$ and $(\mathcal{E}_i)_{i\in I}$ being as above:

1. Show that if $I = \mathbf{N}_n$ and $\Omega_i \in \mathcal{E}_i$ for all $i = 1,\ldots,n$, then $\mathcal{E}_1 \amalg \ldots \amalg \mathcal{E}_n = \{A_1 \times \ldots \times A_n\ :\ A_i \in \mathcal{E}_i\ ,\ \forall i \in I\}$.

2. Let A be a rectangle. Show that there exists a finite subset J of I such that: $A = \{\omega \in \Pi_{i\in I}\Omega_i\ :\ \omega(j) \in A_j\ ,\ \forall j \in J\}$ for some A_j's such that $A_j \in \mathcal{E}_j$, for all $j \in J$.

Definition 53 *Let $(\Omega_i, \mathcal{F}_i)_{i\in I}$ be a family of measurable spaces, indexed by a non-empty set I. We call* **measurable rectangle** *, any rectangle of the family $(\mathcal{F}_i)_{i\in I}$. The set of all measurable rectangles is given by* [2]*:*

$$\coprod_{i\in I}\mathcal{F}_i \stackrel{\triangle}{=} \left\{\prod_{i\in I}A_i\ :\ A_i \in \mathcal{F}_i\ ,\ A_i \neq \Omega_i \text{ for finitely many } i \in I\right\}$$

Definition 54 *Let $(\Omega_i, \mathcal{F}_i)_{i\in I}$ be a family of measurable spaces, indexed by a non-empty set I. We define the* **product σ-algebra** *of $(\mathcal{F}_i)_{i\in I}$, as the σ-algebra on $\Pi_{i\in I}\Omega_i$, denoted $\otimes_{i\in I}\mathcal{F}_i$, and generated by all measurable rectangles, i.e.*

$$\bigotimes_{i\in I}\mathcal{F}_i \stackrel{\triangle}{=} \sigma\left(\coprod_{i\in I}\mathcal{F}_i\right)$$

Exercise 4.

1. Suppose $I = \mathbf{N}_n$. Show that $\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$ is generated by all sets of the form $A_1 \times \ldots \times A_n$, where $A_i \in \mathcal{F}_i$ for all $i = 1,\ldots,n$.

2. Show that $\mathcal{B}(\mathbf{R})\otimes\mathcal{B}(\mathbf{R})\otimes\mathcal{B}(\mathbf{R})$ is generated by sets of the form $A\times B\times C$ where $A, B, C \in \mathcal{B}(\mathbf{R})$.

[2] Note that $\Omega_i \in \mathcal{F}_i$ for all $i \in I$.

3. Show that if $(\Omega, \mathcal{F})$ is a measurable space, $\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}$ is the σ-algebra on $\mathbf{R}^+ \times \Omega$ generated by sets of the form $B \times F$ where $B \in \mathcal{B}(\mathbf{R}^+)$ and $F \in \mathcal{F}$.

EXERCISE 5. Let $(\Omega_i)_{i\in I}$ be a family of non-empty sets and $\mathcal{E}_i$ be a subset of the power set $\mathcal{P}(\Omega_i)$ for all $i \in I$.

1. Give a generator of the σ-algebra $\otimes_{i\in I}\sigma(\mathcal{E}_i)$ on $\Pi_{i\in I}\Omega_i$.

2. Show that:
$$\sigma\left(\coprod_{i\in I}\mathcal{E}_i\right) \subseteq \bigotimes_{i\in I}\sigma(\mathcal{E}_i)$$

3. Let A be a rectangle of the family $(\sigma(\mathcal{E}_i))_{i\in I}$. Show that if A is not empty, then the representation $A = \Pi_{i\in I}A_i$ with $A_i \in \sigma(\mathcal{E}_i)$ is unique. Define $J_A = \{i \in I : A_i \neq \Omega_i\}$. Explain why J_A is a well-defined finite subset of I.

4. If $A \in \amalg_{i\in I}\sigma(\mathcal{E}_i)$, Show that if $A = \emptyset$, or $A \neq \emptyset$ and $J_A = \emptyset$, then $A \in \sigma(\amalg_{i\in I}\mathcal{E}_i)$.

EXERCISE 6. Everything being as before, Let $n \geq 0$. We assume that the following induction hypothesis has been proved:
$$A \in \coprod_{i\in I}\sigma(\mathcal{E}_i), A \neq \emptyset, \mathrm{card}J_A = n \;\Rightarrow\; A \in \sigma\left(\coprod_{i\in I}\mathcal{E}_i\right)$$
We assume that A is a non empty measurable rectangle of $(\sigma(\mathcal{E}_i))_{i\in I}$ with $\mathrm{card}J_A = n+1$. Let $J_A = \{i_1, \ldots, i_{n+1}\}$ be an extension of J_A. For all $B \subseteq \Omega_{i_1}$, we define:
$$A^B \stackrel{\triangle}{=} \prod_{i\in I}\bar{A}_i$$
where each $\bar{A}_i$ is equal to A_i except $\bar{A_{i_1}} = B$. We define the set:
$$\Gamma \stackrel{\triangle}{=} \left\{B \subseteq \Omega_{i_1} \;:\; A^B \in \sigma\left(\coprod_{i\in I}\mathcal{E}_i\right)\right\}$$

1. Show that $A^{\Omega_{i_1}} \neq \emptyset$, $\mathrm{card}J_{A^{\Omega_{i_1}}} = n$ and that $A^{\Omega_{i_1}} \in \amalg_{i\in I}\sigma(\mathcal{E}_i)$.

2. Show that $\Omega_{i_1} \in \Gamma$.

3. Show that for all $B \subseteq \Omega_{i_1}$, we have $A^{\Omega_{i_1}\setminus B} = A^{\Omega_{i_1}} \setminus A^B$.

4. Show that $B \in \Gamma \Rightarrow \Omega_{i_1} \setminus B \in \Gamma$.

5. Let $B_n \subseteq \Omega_{i_1}$, $n \geq 1$. Show that $A^{\cup B_n} = \cup_{n\geq 1}A^{B_n}$.

6. Show that Γ is a σ-algebra on Ω_{i_1}.

7. Let $B \in \mathcal{E}_{i_1}$, and for $i \in I$ define $\bar{B}_i = \Omega_i$ for all i's except $\bar{B}_{i_1} = B$. Show that $A^B = A^{\Omega_{i_1}} \cap (\Pi_{i\in I}\bar{B}_i)$.

8. Show that $\sigma(\mathcal{E}_{i_1}) \subseteq \Gamma$.

9. Show that $A = A^{A_{i_1}}$ and $A \in \sigma(\amalg_{i\in I}\mathcal{E}_i)$.

10. Show that $\amalg_{i\in I}\sigma(\mathcal{E}_i) \subseteq \sigma(\amalg_{i\in I}\mathcal{E}_i)$.

11. Show that $\sigma(\amalg_{i\in I}\mathcal{E}_i) = \otimes_{i\in I}\sigma(\mathcal{E}_i)$.

Theorem 26 *Let $(\Omega_i)_{i\in I}$ be a family of non-empty sets indexed by a non-empty set I. For all $i \in I$, let $\mathcal{E}_i$ be a set of subsets of Ω_i. Then, the product σ-algebra $\otimes_{i\in I}\sigma(\mathcal{E}_i)$ on the Cartesian product $\Pi_{i\in I}\Omega_i$ is generated by the rectangles of $(\mathcal{E}_i)_{i\in I}$, i.e. :*

$$\bigotimes_{i\in I}\sigma(\mathcal{E}_i) = \sigma\left(\coprod_{i\in I}\mathcal{E}_i\right)$$

Exercise 7. Let $\mathcal{T}_{\mathbf{R}}$ denote the usual topology in $\mathbf{R}$. Let $n \geq 1$.

1. Show that $\mathcal{T}_{\mathbf{R}} \amalg \ldots \amalg \mathcal{T}_{\mathbf{R}} = \{A_1 \times \ldots \times A_n \ : \ A_i \in \mathcal{T}_{\mathbf{R}}\}$.

2. Show that $\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R}) = \sigma(\mathcal{T}_{\mathbf{R}} \amalg \ldots \amalg \mathcal{T}_{\mathbf{R}})$.

3. Define $\mathcal{C}_2 = \{]a_1, b_1] \times \ldots \times]a_n, b_n] \ : \ a_i, b_i \in \mathbf{R}\}$. Show that $\mathcal{C}_2 \subseteq \mathcal{S} \amalg \ldots \amalg \mathcal{S}$, where $\mathcal{S} = \{]a, b] \ : \ a, b \in \mathbf{R}\}$, but that the inclusion is strict.

4. Show that $\mathcal{S} \amalg \ldots \amalg \mathcal{S} \subseteq \sigma(\mathcal{C}_2)$.

5. Show that $\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R}) = \sigma(\mathcal{C}_2)$.

Exercise 8. Let Ω and Ω' be two non-empty sets. Let A be a subset of Ω such that $\emptyset \neq A \neq \Omega$. Let $\mathcal{E} = \{A\} \subseteq \mathcal{P}(\Omega)$ and $\mathcal{E}' = \emptyset \subseteq \mathcal{P}(\Omega')$.

1. Show that $\sigma(\mathcal{E}) = \{\emptyset, A, A^c, \Omega\}$.

2. Show that $\sigma(\mathcal{E}') = \{\emptyset, \Omega'\}$.

3. Define $\mathcal{C} = \{E \times F \ , \ E \in \mathcal{E}, F \in \mathcal{E}'\}$ and show that $\mathcal{C} = \emptyset$.

4. Show that $\mathcal{E} \amalg \mathcal{E}' = \{A \times \Omega', \Omega \times \Omega'\}$.

5. Show that $\sigma(\mathcal{E}) \otimes \sigma(\mathcal{E}') = \{\emptyset, A \times \Omega', A^c \times \Omega', \Omega \times \Omega'\}$.

6. Conclude that $\sigma(\mathcal{E}) \otimes \sigma(\mathcal{E}') \neq \sigma(\mathcal{C}) = \{\emptyset, \Omega \times \Omega'\}$.

Exercise 9. Let $n \geq 1$ and $p \geq 1$ be two positive integers.

1. Define $\mathcal{F} = \underbrace{\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R})}_{n}$, and $\mathcal{G} = \underbrace{\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R})}_{p}$. Explain why $\mathcal{F} \otimes \mathcal{G}$ can be viewed as a σ-algebra on $\mathbf{R}^{n+p}$.

2. Show that $\mathcal{F} \otimes \mathcal{G}$ is generated by sets of the form $A_1 \times \ldots \times A_{n+p}$ where $A_i \in \mathcal{B}(\mathbf{R}), i = 1, \ldots, n+p$.

3. Show that:

$$\underbrace{\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R})}_{n+p} = \underbrace{(\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R}))}_{n} \otimes \underbrace{(\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R}))}_{p}$$

Exercise 10. Let $(\Omega_i, \mathcal{F}_i)_{i\in I}$ be a family of measurable spaces. Let $(I_\lambda)_{\lambda\in\Lambda}$, where $\Lambda \neq \emptyset$, be a partition of I. Let $\Omega = \Pi_{i\in I}\Omega_i$ and $\Omega' = \Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}\Omega_i)$.

1. Define a *natural* bijection between $\mathcal{P}(\Omega)$ and $\mathcal{P}(\Omega')$.

2. Show that through such bijection, $A = \Pi_{i\in I} A_i \subseteq \Omega$, where $A_i \subseteq \Omega_i$, is identified with $A' = \Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda} A_i) \subseteq \Omega'$.

3. Show that $\amalg_{i\in I}\mathcal{F}_i = \amalg_{\lambda\in\Lambda}(\amalg_{i\in I_\lambda}\mathcal{F}_i)$.

4. Show that $\otimes_{i\in I}\mathcal{F}_i = \otimes_{\lambda\in\Lambda}(\otimes_{i\in I_\lambda}\mathcal{F}_i)$.

Definition 55 *Let Ω be set and $\mathcal{A}$ be a set of subsets of Ω. We call* **topology generated** *by $\mathcal{A}$, the topology on Ω, denoted $\mathcal{T}(\mathcal{A})$, equal to the intersection of all topologies on Ω, which contain $\mathcal{A}$.*

Exercise 11. Let Ω be a set and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$.

1. Explain why $\mathcal{T}(\mathcal{A})$ is indeed a topology on Ω.

2. Show that $\mathcal{T}(\mathcal{A})$ is the smallest topology $\mathcal{T}$ such that $\mathcal{A} \subseteq \mathcal{T}$.

3. Show that the metric topology on a metric space (E, d) is generated by the open balls $\mathcal{A} = \{B(x, \epsilon) : x \in E, \epsilon > 0\}$.

Definition 56 *Let $(\Omega_i, \mathcal{T}_i)_{i\in I}$ be a family of topological spaces, indexed by a non-empty set I. We define the* **product topology** *of $(\mathcal{T}_i)_{i\in I}$, as the topology on $\Pi_{i\in I}\Omega_i$, denoted $\odot_{i\in I}\mathcal{T}_i$, and generated by all rectangles of $(\mathcal{T}_i)_{i\in I}$, i.e.*

$$\bigodot_{i\in I}\mathcal{T}_i \stackrel{\triangle}{=} \mathcal{T}\left(\coprod_{i\in I}\mathcal{T}_i\right)$$

Exercise 12. Let $(\Omega_i, \mathcal{T}_i)_{i\in I}$ be a family of topological spaces.

1. Show that $U \in \odot_{i\in I}\mathcal{T}_i$, if and only if:

$$\forall x \in U\ ,\ \exists V \in \amalg_{i\in I}\mathcal{T}_i\ ,\ x \in V \subseteq U$$

2. Show that $\amalg_{i\in I}\mathcal{T}_i \subseteq \odot_{i\in I}\mathcal{T}_i$.

3. Show that $\otimes_{i\in I}\mathcal{B}(\Omega_i) = \sigma(\amalg_{i\in I}\mathcal{T}_i)$.

4. Show that $\otimes_{i\in I}\mathcal{B}(\Omega_i) \subseteq \mathcal{B}(\Pi_{i\in I}\Omega_i)$.

Exercise 13. Let $n \geq 1$ be a positive integer. For all $x, y \in \mathbf{R}^n$, let:

$$(x,y) \stackrel{\triangle}{=} \sum_{i=1}^{n} x_i y_i$$

and we put $\|x\| = \sqrt{(x,x)}$.

1. Show that for all $t \in \mathbf{R}$, $\|x+ty\|^2 = \|x\|^2 + t^2\|y\|^2 + 2t(x,y)$.
2. From $\|x+ty\|^2 \geq 0$ for all t, deduce that $|(x,y)| \leq \|x\|.\|y\|$.
3. Conclude that $\|x+y\| \leq \|x\| + \|y\|$.

Exercise 14. Let $(\Omega_1, \mathcal{T}_1), \ldots, (\Omega_n, \mathcal{T}_n)$, $n \geq 1$, be metrizable topological spaces. Let $d_1, \ldots, d_n$ be metrics on $\Omega_1, \ldots, \Omega_n$, inducing the topologies $\mathcal{T}_1, \ldots, \mathcal{T}_n$ respectively. Let $\Omega = \Omega_1 \times \ldots \times \Omega_n$ and $\mathcal{T}$ be the product topology on Ω. For all $x, y \in \Omega$, we define:

$$d(x,y) \stackrel{\triangle}{=} \sqrt{\sum_{i=1}^{n} (d_i(x_i, y_i))^2}$$

1. Show that $d : \Omega \times \Omega \to \mathbf{R}^+$ is a metric on Ω.
2. Show that $U \subseteq \Omega$ is open in Ω, if and only if, for all $x \in U$ there are open sets $U_1, \ldots, U_n$ in $\Omega_1, \ldots, \Omega_n$ respectively, such that:
$$x \in U_1 \times \ldots \times U_n \subseteq U$$
3. Let $U \in \mathcal{T}$ and $x \in U$. Show the existence of $\epsilon > 0$ such that:
$$(\forall i = 1, \ldots, n\ d_i(x_i, y_i) < \epsilon) \ \Rightarrow\ y \in U$$
4. Show that $\mathcal{T} \subseteq \mathcal{T}^d_\Omega$.
5. Let $U \in \mathcal{T}^d_\Omega$ and $x \in U$. Show the existence of $\epsilon > 0$ such that:
$$x \in B(x_1, \epsilon) \times \ldots \times B(x_n, \epsilon) \subseteq U$$
6. Show that $\mathcal{T}^d_\Omega \subseteq \mathcal{T}$.
7. Show that the product topological space $(\Omega, \mathcal{T})$ is metrizable.
8. For all $x, y \in \Omega$, define:
$$\begin{aligned} d'(x,y) &\stackrel{\triangle}{=} \sum_{i=1}^{n} d_i(x_i, y_i) \\ d''(x,y) &\stackrel{\triangle}{=} \max_{i=1,\ldots,n} d_i(x_i, y_i) \end{aligned}$$
Show that d', d'' are metrics on Ω.

9. Show the existence of α', β', α'' and $\beta'' > 0$, such that we have $\alpha' d' \leq d \leq \beta' d'$ and $\alpha'' d'' \leq d \leq \beta'' d''$.
10. Show that d' and d'' also induce the product topology on Ω.

EXERCISE 15. Let $(\Omega_n, \mathcal{T}_n)_{n\geq 1}$ be a sequence of metrizable topological spaces. For all $n \geq 1$, let d_n be a metric on Ω_n inducing the topology $\mathcal{T}_n$. Let $\Omega = \Pi_{n=1}^{+\infty}\Omega_n$ be the Cartesian product and $\mathcal{T}$ be the product topology on Ω. For all $x, y \in \Omega$, we define:

$$d(x,y) \triangleq \sum_{n=1}^{+\infty} \frac{1}{2^n}(1 \wedge d_n(x_n, y_n))$$

1. Show that for all $a, b \in \mathbf{R}^+$, we have $1 \wedge (a+b) \leq 1 \wedge a + 1 \wedge b$.
2. Show that d is a metric on Ω.
3. Show that $U \subseteq \Omega$ is open in Ω, if and only if, for all $x \in U$, there is an integer $N \geq 1$ and open sets $U_1, \ldots, U_N$ in $\Omega_1, \ldots, \Omega_N$ respectively, such that:
$$x \in U_1 \times \ldots \times U_N \times \prod_{n=N+1}^{+\infty} \Omega_n \subseteq U$$
4. Show that $d(x,y) < 1/2^n \;\Rightarrow\; d_n(x_n, y_n) \leq 2^n d(x,y)$.
5. Show that for all $U \in \mathcal{T}$ and $x \in U$, there exists $\epsilon > 0$ such that $d(x,y) < \epsilon \;\Rightarrow\; y \in U$.
6. Show that $\mathcal{T} \subseteq \mathcal{T}^d_\Omega$.
7. Let $U \in \mathcal{T}^d_\Omega$ and $x \in U$. Show the existence of $\epsilon > 0$ and $N \geq 1$, such that:
$$\sum_{n=1}^{N} \frac{1}{2^n}(1 \wedge d_n(x_n, y_n)) < \epsilon \;\Rightarrow\; y \in U$$
8. Show that for all $U \in \mathcal{T}^d_\Omega$ and $x \in U$, there is $\epsilon > 0$ and $N \geq 1$ such that:
$$x \in B(x_1, \epsilon) \times \ldots \times B(x_N, \epsilon) \times \prod_{n=N+1}^{+\infty} \Omega_n \subseteq U$$
9. Show that $\mathcal{T}^d_\Omega \subseteq \mathcal{T}$.
10. Show that the product topological space $(\Omega, \mathcal{T})$ is metrizable.

Definition 57 *Let $(\Omega, \mathcal{T})$ be a topological space. A subset $\mathcal{H}$ of $\mathcal{T}$ is called a* **countable base** *of $(\Omega, \mathcal{T})$, if and only if $\mathcal{H}$ is at most countable, and has the property:*

$$\forall U \in \mathcal{T} \;,\; \exists \mathcal{H}' \subseteq \mathcal{H} \;,\; U = \bigcup_{V \in \mathcal{H}'} V$$

Exercise 16.

1. Show that $\mathcal{H} = \{]r, q[\ :\ r, q \in \mathbf{Q}\}$ is a countable base of $(\mathbf{R}, \mathcal{T}_{\mathbf{R}})$.
2. Show that if $(\Omega, \mathcal{T})$ is a topological space with countable base, and $\Omega' \subseteq \Omega$, then the induced topological space $(\Omega', \mathcal{T}_{|\Omega'})$ also has a countable base.
3. Show that $[-1, 1]$ has a countable base.
4. Show that if $(\Omega, \mathcal{T})$ and $(S, \mathcal{T}_S)$ are homeomorphic, then $(\Omega, \mathcal{T})$ has a countable base if and only if $(S, \mathcal{T}_S)$ has a countable base.
5. Show that $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ has a countable base.

Exercise 17. Let $(\Omega_n, \mathcal{T}_n)_{n\geq 1}$ be a sequence of topological spaces with countable base. For $n \geq 1$, Let $\{V_n^k : k \in I_n\}$ be a countable base of $(\Omega_n, \mathcal{T}_n)$ where I_n is a finite or countable set. Let $\Omega = \Pi_{n=1}^{\infty}\Omega_n$ be the Cartesian product and $\mathcal{T}$ be the product topology on Ω. For all $p \geq 1$, we define:

$$\mathcal{H}^p \stackrel{\triangle}{=} \left\{ V_1^{k_1} \times \ldots \times V_p^{k_p} \times \prod_{n=p+1}^{+\infty} \Omega_n \ :\ (k_1, \ldots, k_p) \in I_1 \times \ldots \times I_p \right\}$$

and we put $\mathcal{H} = \cup_{p\geq 1}\mathcal{H}^p$.

1. Show that for all $p \geq 1$, $\mathcal{H}^p \subseteq \mathcal{T}$.
2. Show that $\mathcal{H} \subseteq \mathcal{T}$.
3. For all $p \geq 1$, show the existence of an injection $j_p : \mathcal{H}^p \to \mathbf{N}^p$.
4. Show the existence of a bijection $\phi_2 : \mathbf{N}^2 \to \mathbf{N}$.
5. For $p \geq 1$, show the existence of an bijection $\phi_p : \mathbf{N}^p \to \mathbf{N}$.
6. Show that $\mathcal{H}^p$ is at most countable for all $p \geq 1$.
7. Show the existence of an injection $j : \mathcal{H} \to \mathbf{N}^2$.
8. Show that $\mathcal{H}$ is a finite or countable set of open sets in Ω.
9. Let $U \in \mathcal{T}$ and $x \in U$. Show that there is $p \geq 1$ and $U_1, \ldots, U_p$ open sets in $\Omega_1, \ldots, \Omega_p$ such that:
$$x \in U_1 \times \ldots \times U_p \times \prod_{n=p+1}^{+\infty} \Omega_n \subseteq U$$
10. Show the existence of some $V_x \in \mathcal{H}$ such that $x \in V_x \subseteq U$.
11. Show that $\mathcal{H}$ is a countable base of the topological space $(\Omega, \mathcal{T})$.
12. Show that $\otimes_{n=1}^{+\infty}\mathcal{B}(\Omega_n) \subseteq \mathcal{B}(\Omega)$.

13. Show that $\mathcal{H} \subseteq \otimes_{n=1}^{+\infty}\mathcal{B}(\Omega_n)$.

14. Show that $\mathcal{B}(\Omega) = \otimes_{n=1}^{+\infty}\mathcal{B}(\Omega_n)$

Theorem 27 *Let $(\Omega_n, \mathcal{T}_n)_{n\geq 1}$ be a sequence of topological spaces with countable base. Then, the product space $(\Pi_{n=1}^{+\infty}\Omega_n, \odot_{n=1}^{+\infty}\mathcal{T}_n)$ has a countable base and:*

$$\mathcal{B}\left(\prod_{n=1}^{+\infty}\Omega_n\right) = \bigotimes_{n=1}^{+\infty}\mathcal{B}(\Omega_n)$$

Exercise 18.

1. Show that if $(\Omega, \mathcal{T})$ has a countable base and $n \geq 1$:

$$\mathcal{B}(\Omega^n) = \underbrace{\mathcal{B}(\Omega)\otimes \ldots \otimes \mathcal{B}(\Omega)}_{n}$$

2. Show that $\mathcal{B}(\bar{\mathbf{R}}^n) = \mathcal{B}(\bar{\mathbf{R}})\otimes \ldots \otimes \mathcal{B}(\bar{\mathbf{R}})$.

3. Show that $\mathcal{B}(\mathbf{C}) = \mathcal{B}(\mathbf{R})\otimes \mathcal{B}(\mathbf{R})$.

Definition 58 *We say that a metric space (E, d) is* **separable**, *if and only if there exists a finite or countable dense subset of E, i.e. a finite or countable subset A of E such that $E = \bar{A}$, where $\bar{A}$ is the closure of A in E.*

Exercise 19. Let (E, d) be a metric space.

1. Suppose that (E, d) is separable. Let $\mathcal{H} = \{B(x_n, \frac{1}{p}) : n, p \geq 1\}$, where $\{x_n : n \geq 1\}$ is a countable dense subset in E. Show that $\mathcal{H}$ is a countable base of the metric topological space $(E, \mathcal{T}_E^d)$.

2. Suppose conversely that $(E, \mathcal{T}_E^d)$ has a countable base $\mathcal{H}$. For all $V \in \mathcal{H}$ such that $V \neq \emptyset$, take $x_V \in V$. Show that the set $\{x_V \ : \ V \in \mathcal{H} \ , \ V \neq \emptyset\}$ is at most countable and dense in E.

3. For all $x, y, x', y' \in E$, show that:

$$|d(x, y) - d(x', y')| \leq d(x, x') + d(y, y')$$

4. Let $\mathcal{T}_{E\times E}$ be the product topology on $E \times E$. Show that the map $d : (E \times E, \mathcal{T}_{E\times E}) \to (\mathbf{R}^+, \mathcal{T}_{\mathbf{R}^+})$ is continuous.

5. Show that $d : (E \times E, \mathcal{B}(E \times E)) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

6. Show that $d : (E\times E, \mathcal{B}(E)\otimes\mathcal{B}(E)) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, whenever (E, d) is a separable metric space.

7. Let $(\Omega, \mathcal{F})$ be a measurable space and $f, g : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ be measurable maps. Show that $\Phi : (\Omega, \mathcal{F}) \to E \times E$ defined by $\Phi(\omega) = (f(\omega), g(\omega))$ is measurable with respect to the product σ-algebra $\mathcal{B}(E) \otimes \mathcal{B}(E)$.

8. Show that if (E, d) is separable, then $\Psi : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ defined by $\Psi(\omega) = d(f(\omega), g(\omega))$ is measurable.

9. Show that if (E, d) is separable then $\{f = g\} \in \mathcal{F}$.

10. Let $(E_n, d_n)_{n\geq 1}$ be a sequence of separable metric spaces. Show that the product space $\Pi_{n=1}^{+\infty} E_n$ is metrizable and separable.

Exercise 20. Prove the following theorem.

Theorem 28 *Let $(\Omega_i, \mathcal{F}_i)_{i\in I}$ be a family of measurable spaces and $(\Omega, \mathcal{F})$ be a measurable space. For all $i \in I$, let $f_i : \Omega \to \Omega_i$ be a map, and define $f : \Omega \to \Pi_{i\in I}\Omega_i$ by $f(\omega) = (f_i(\omega))_{i\in I}$. Then, the map:*

$$f : (\Omega, \mathcal{F}) \to \left(\prod_{i\in I} \Omega_i, \bigotimes_{i\in I} \mathcal{F}_i \right)$$

is measurable, if and only if each $f_i : (\Omega, \mathcal{F}) \to (\Omega_i, \mathcal{F}_i)$ is measurable.

Exercise 21.

1. Let $\phi, \psi : \mathbf{R}^2 \to \mathbf{R}$ with $\phi(x, y) = x + y$ and $\psi(x, y) = x.y$. Show that both ϕ and ψ are continuous.

2. Show that $\phi, \psi : (\mathbf{R}^2, \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are measurable.

3. Let $(\Omega, \mathcal{F})$ be a measurable space, and $f, g : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ be measurable maps. Using the previous results, show that $f + g$ and $f.g$ are measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\mathbf{R})$.

Solutions to Exercises

Exercise 1.

1. If $\Omega_i = \Omega$ for all $i \in I$, then $\cup_{i\in I}\Omega_i = \Omega$. For any map $f : I \to \Omega$, the condition $f(i) \in \Omega_i$ for all $i \in I$, is automatically satisfied. Hence, Ω^I is the set of all maps $f : I \to \Omega$.
2. $\mathbf{R}^{\mathbf{R}^+}$ is the set of all maps $f : \mathbf{R}^+ \to \mathbf{R}$. The set $\mathbf{R}^{\mathbf{N}}$ is that of all maps $f : \mathbf{N} \to \mathbf{R}$, or in other words, the set of all sequences $(u_n)_{n\geq 0}$ with values in $\mathbf{R}$. As for $[0, 1]^{\mathbf{N}}$, it is the set of all sequences $(u_n)_{n\geq 0}$ with values in $[0, 1]$. Finally, $\bar{\mathbf{R}}^{\mathbf{R}}$ etc...
3. Yes. Maps defined on $\mathbf{N}^*$ or sequences are the same thing.
4. For any set E, E^n is the set of all maps $f : \mathbf{N}_n \to E$.
5. $E \times F \times G$ is the set of all maps $\omega : \mathbf{N}_3 \to E \cup F \cup G$ such that $\omega_1 \in E$, $\omega_2 \in F$ and $\omega_3 \in G$.

Exercise 1

Exercise 2.

1. $\Pi_{i\in I_\lambda}\Omega_i$ is the set of all maps f defined on I_λ, with $f(i) \in \Omega_i$ for all $i \in I_\lambda$.
2. $\Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}\Omega_i)$ is the set of all maps x defined on Λ, such that $x(\lambda) \in \Pi_{i\in I_\lambda}\Omega_i$, for all $\lambda \in \Lambda$.
3. Given $\omega \in \Pi_{i\in I}\Omega_i$ and $\lambda \in \Lambda$, let $\omega_{|I_\lambda}$ be the restriction of ω to $I_\lambda \subseteq I$. Since $\omega(i) \in \Omega_i$ for all $i \in I$, in particular $\omega(i) \in \Omega_i$ for all $i \in I_\lambda$. Hence, $\omega_{|I_\lambda} \in \Pi_{i\in I_\lambda}\Omega_i$. This being true for all $\lambda \in \Lambda$, the map $\Phi(\omega) = (\omega_{|I_\lambda})_{\lambda\in\Lambda}$ defined on Λ by $\Phi(\omega)(\lambda) = \omega_{|I_\lambda}$, is an element of $\Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}\Omega_i)$. Hence, we have defined a map $\Phi : \Pi_{i\in I}\Omega_i \to \Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}\Omega_i)$. Let $y \in \Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}\Omega_i)$. Since $(I_\lambda)_{\lambda\in\Lambda}$ is a partition of I, for all $i \in I$, there exists a unique $\lambda \in \Lambda$ such that $i \in I_\lambda$. Define $\omega(i) = y(\lambda)(i)$. Then, $\omega(i) \in \Omega_i$ for all $i \in I$, i.e. $\omega \in \Pi_{i\in I}\Omega_i$. Moreover, by construction, $\Phi(\omega)(\lambda) = \omega_{|I_\lambda} = y(\lambda)$, for all $\lambda \in \Lambda$. We have found a map $\omega \in \Pi_{i\in I}\Omega_i$, such that $\Phi(\omega) = y$. So Φ is a surjective map. Suppose that $\Phi(\omega) = \Phi(\omega')$ for some $\omega, \omega' \in \Pi_{i\in I}\Omega_i$. Let $i \in I$, and $\lambda \in \Lambda$ be such that $i \in I_\lambda$. Then, we have:
$$\omega(i) = (\omega_{|I_\lambda})(i) = \Phi(\omega)(\lambda)(i) = \Phi(\omega')(\lambda)(i) = \omega'(i)$$
So $\omega = \omega'$, and Φ is an injective map. We have found a *natural* bijection from $\Pi_{i\in I}\Omega_i$ to $\Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}\Omega_i)$.
Given a map $\omega \in \Pi_{i\in I}\Omega_i$, it is customary to regard ω as the family $(\omega_i)_{i\in I}$ where $\omega_i = \omega(i)$ for all $i \in I$. (A map defined on I is nothing but a family indexed by I). Hence, the restriction $\omega_{|I_\lambda}$ is nothing but the family $(\omega_i)_{i\in I_\lambda}$, and the map $\Phi(\omega)$ can be written as:
$$\Phi((\omega_i)_{i\in I}) = ((\omega_i)_{i\in I_\lambda})_{\lambda\in\Lambda}$$
The mapping Φ looks like a pretty *natural* mapping, given the partition $(I_\lambda)_{\lambda\in\Lambda}$ of the set I.

4. $\mathbf{R}^p \times \mathbf{R}^n$ is the set of all maps $\omega : \mathbf{N}_2 \to \mathbf{R}^p \cup \mathbf{R}^n$ such that $\omega_1 \in \mathbf{R}^p$ and $\omega_2 \in \mathbf{R}^n$[3]. Each $\omega_1 \in \mathbf{R}^p$ is a map $\omega_1 : \mathbf{N}_p \to \mathbf{R}$, and each $\omega_2 \in \mathbf{R}^n$ is a map $\omega_2 : \mathbf{N}_n \to \mathbf{R}$. Given $\omega \in \mathbf{R}^p \times \mathbf{R}^n$, define $\psi(\omega) \in \mathbf{R}^{p+n}$ as:
$$\psi(\omega)(i) = \begin{cases} \omega_1(i) & \text{if } 1 \leq i \leq p \\ \omega_2(i-p) & \text{if } p+1 \leq i \leq p+n \end{cases}$$
i.e. $\psi(\omega) = (\omega_1(1), \ldots, \omega_1(p), \omega_2(1), \ldots, \omega_2(n))$. The mapping $\omega \to \psi(\omega)$ from $\mathbf{R}^p \times \mathbf{R}^n$ to $\mathbf{R}^{p+n}$ is a bijection, which may be regarded as *natural*...

Exercise 2

Exercise 3.

1. Let $A = A_1 \times \ldots \times A_n$ be such that $A_i \in \mathcal{E}_i$ for all $i = 1, \ldots, n$. Then A is of the form $A = \Pi_{i \in \mathbf{N}_n} A_i$ with $A_i \in \mathcal{E}_i \cup \{\Omega_i\}$, and the condition $A_i \neq \Omega_i$ for finitely many $i \in \mathbf{N}_n$, is obviously satisfied. So A is a rectangle of the family $(\mathcal{E}_i)_{i \in \mathbf{N}_n}$, that is $A \in \mathcal{E}_1 \amalg \ldots \amalg \mathcal{E}_n$. Conversely, Let $A = \Pi_{i \in \mathbf{N}_n} A_i$ be a rectangle of the family $(\mathcal{E}_i)_{i \in \mathbf{N}_n}$. Then, each A_i is an element of $\mathcal{E}_i \cup \{\Omega_i\}$. Since $\Omega_i \in \mathcal{E}_i$ for all $i \in \mathbf{N}_n$, each A_i is in fact an element of $\mathcal{E}_i$. So A is of the form $A = A_1 \times \ldots \times A_n$, with $A_i \in \mathcal{E}_i$. We have proved that the set of rectangles of $(\mathcal{E}_i)_{i \in \mathbf{N}_n}$ is given by:
$$\mathcal{E}_1 \amalg \ldots \amalg \mathcal{E}_n = \{A_1 \times \ldots \times A_n : A_i \in \mathcal{E}_i, \forall i \in \mathbf{N}_n\}$$

2. Let A be a rectangle of the family $(\mathcal{E}_i)_{i \in I}$. Then $A = \Pi_{i \in I} A_i$, where $A_i \in \mathcal{E}_i \cup \{\Omega_i\}$, and $A_i \neq \Omega_i$ for finitely many $i \in I$. Let J be the set $J = \{i \in I : A_i \neq \Omega_i\}$. Then J is a finite subset of I. Moreover, for all $j \in J$, $A_j \neq \Omega_j$, yet $A_j \in \mathcal{E}_j \cup \{\Omega_j\}$. So $A_j \in \mathcal{E}_j$. Let $\omega \in A = \Pi_{i \in I} A_i$. Then ω is a map defined on I such that $\omega(i) \in A_i \subseteq \Omega_i$ for all $i \in I$. In particular, $\omega \in \Pi_{i \in I} \Omega_i$, and $\omega(j) \in A_j$ for all $j \in J$. Conversely, suppose $\omega \in \Pi_{i \in I} \Omega_i$ is such that $\omega(j) \in A_j$ for all $j \in J$. Then ω is a map defined on I such that $\omega(i) \in \Omega_i$ for all $i \in I$, and furthermore, $\omega(j) \in A_j$ for all $j \in J$. However, for all $i \in I \setminus J$, we have $A_i = \Omega_i$. It follows that ω is a map defined on I such that $\omega(i) \in A_i$ for all $i \in I$. So $\omega \in \Pi_{i \in I} A_i = A$. We have proved that there exists a finite subset J of I, and a family $(A_j)_{j \in J}$ with $A_j \in \mathcal{E}_j$, such that $A = \{\omega \in \Pi_{i \in I} \Omega_i : \omega(j) \in A_j, \forall j \in J\}$.

Exercise 3

Exercise 4.

1. By definition, $\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$ is generated by the set of measurable rectangles $\mathcal{F}_1 \amalg \ldots \amalg \mathcal{F}_n$. Since $\Omega_i \in \mathcal{F}_i$ for all $i \in \mathbf{N}_n$, and since N_n is finite, these rectangles are of the form $A_1 \times \ldots \times A_n$ where $A_i \in \mathcal{F}_i$, for all $i \in \mathbf{N}_n$.

2. $\mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})$ is generated by the set of measurable rectangles $\mathcal{B}(\mathbf{R}) \amalg \mathcal{B}(\mathbf{R}) \amalg \mathcal{B}(\mathbf{R})$. These rectangles are of the form $A \times B \times C$, where $A, B, C \in \mathcal{B}(\mathbf{R})$.

[3] We view ordered pairs as maps defined on $\mathbf{N}_2$...

3. Since $\mathbf{R}^+ \in \mathcal{B}(\mathbf{R}^+)$ and $\Omega \in \mathcal{F}$, the set of measurable rectangles $\mathcal{B}(\mathbf{R}^+)\amalg\mathcal{F}$ is the set of all $B \times F$, where $B \in \mathcal{B}(\mathbf{R}^+)$ and $F \in \mathcal{F}$. Such sets generate the σ-algebra $\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}$ on $\mathbf{R}^+ \times \Omega$.

Exercise 4

Exercise 5.

1. By definition, a generator of $\otimes_{i\in I}\sigma(\mathcal{E}_i)$ is the set of measurable rectangles of the family $(\sigma(\mathcal{E}_i))_{i\in I}$, i.e. $\amalg_{i\in I}\sigma(\mathcal{E}_i)$.

2. Let $A = \Pi_{i\in I}A_i$ be a rectangle in $\amalg_{i\in I}\mathcal{E}_i$. Then, each A_i is an element of $\mathcal{E}_i \cup \{\Omega_i\}$, and $A_i \neq \Omega_i$ for finitely many $i \in I$. In particular, A is also a rectangle in $\amalg_{i\in I}\sigma(\mathcal{E}_i)$. Hence, we have:

$$\coprod_{i\in I}\mathcal{E}_i \subseteq \coprod_{i\in I}\sigma(\mathcal{E}_i) \subseteq \sigma\left(\coprod_{i\in I}\sigma(\mathcal{E}_i)\right) \stackrel{\triangle}{=} \otimes_{i\in I}\sigma(\mathcal{E}_i)$$

and consequently, $\sigma(\amalg_{i\in I}\mathcal{E}_i) \subseteq \otimes_{i\in I}\sigma(\mathcal{E}_i)$.

3. Let $A \neq \emptyset$ be a rectangle of the family $(\sigma(\mathcal{E}_i))_{i\in I}$. Suppose that $A = \Pi_{i\in I}A_i = \Pi_{i\in I}B_i$ are two representations of A. Since A is non-empty, there exists $f \in A$. The mapping f defined on I, is such that $f(i) \in A_i \cap B_i$ for all $i \in I$. Let $j \in I$ be given. Suppose $x \in A_j$. Define g on I, by $g(i) = f(i)$ if $i \neq j$, and $g(j) = x$. Then, $g(i) \in A_i$ for all $i \in I$. So $g \in \Pi_{i\in I}A_i = A = \Pi_{i\in I}B_i$, and in particular, $x = g(j) \in B_j$. Hence, we see that $A_j \subseteq B_j$, and similarly $B_j \subseteq A_j$. $j \in I$ being arbitrary, we have proved that $A_i = B_i$ for all $i \in I$. The set $J_A = \{i \in I : A_i \neq \Omega_i\}$ is therefore well-defined, as the A_i's are uniquely determined. Furthermore, A being a rectangle, the set J_A is finite.

4. Let $A \in \amalg_{i\in I}\sigma(\mathcal{E}_i)$. If $A = \emptyset$, then A is an element of the σ-algebra $\sigma(\amalg_{i\in I}\mathcal{E}_i)$. If $A \neq \emptyset$ but $J_A = \emptyset$, then $A_i = \Omega_i$ for all $i \in I$, and $A = \Pi_{i\in I}A_i = \Pi_{i\in I}\Omega_i$ is also an element of the σ-algebra $\sigma(\amalg_{i\in I}\mathcal{E}_i)$.

Exercise 5

Exercise 6.

1. By assumption, $A \neq \emptyset$. There exists a map f defined on I, such that $f(i) \in A_i$, for all $i \in I$. Since $A_{i_1} \subseteq \Omega_{i_1}$, f is also an element of $A^{\Omega_{i_1}}$. So $A^{\Omega_{i_1}} \neq \emptyset$. By definition, $J_{A^{\Omega_{i_1}}} = \{i \in I : \bar{A}_i \neq \Omega_i\}$, where each $\bar{A}_i$ is equal to A_i, except $\bar{A}_{i_1} = \Omega_{i_1}$. It follows that $J_{A^{\Omega_{i_1}}} = \{i \in I \setminus \{i_1\} : A_i \neq \Omega_i\} = J_A \setminus \{i_1\}$. Since by assumption, $i_1 \in J_A$, and $\text{card}J_A = n+1$, $\text{card}J_{A^{\Omega_{i_1}}} = n$. Finally, A being a rectangle of the family $(\sigma(\mathcal{E}_i))_{i\in I}$, each A_i is an element of $\sigma(\mathcal{E}_i) \cup \{\Omega_i\} = \sigma(\mathcal{E}_i)$. It follows that $\bar{A}_i \in \sigma(\mathcal{E}_i)$ for all $i \in I$. Since $\bar{A}_i \neq \Omega_i$ for finitely many $i \in I$, we conclude that $A^{\Omega_{i_1}} = \Pi_{i\in I}\bar{A}_i \in \amalg_{i\in I}\sigma(\mathcal{E}_i)$.

2. Our induction hypothesis is that if A is a non-empty rectangle of the family $(\sigma(\mathcal{E}_i))_{i\in I}$ with card$J_A = n$, then $A \in \sigma(\amalg_{i\in I}\mathcal{E}_i)$. Since from 1., $A^{\Omega_{i_1}}$ satisfies such properties, $A^{\Omega_{i_1}} \in \sigma(\amalg_{i\in I}\mathcal{E}_i)$. It follows that $\Omega_{i_1} \in \Gamma$.

3. Let $B \subseteq \Omega_{i_1}$. Let $f \in A^{\Omega_{i_1}\setminus B}$. Then, f is a map defined on I, such that $f(i) \in A_i$ for all $i \in I \setminus \{i_1\}$, and $f(i_1) \in \Omega_{i_1} \setminus B$. In particular, $f \in A^{\Omega_{i_1}}$ and $f \notin A^B$. So $f \in A^{\Omega_{i_1}} \setminus A^B$, and $A^{\Omega_{i_1}\setminus B} \subseteq A^{\Omega_{i_1}} \setminus A^B$. Conversely, suppose $f \in A^{\Omega_{i_1}} \setminus A^B$. f being an element of $A^{\Omega_{i_1}}$, $f(i) \in A_i$ for all $i \in I \setminus \{i_1\}$. Since $f \notin A^B$, $f(i_1)$ cannot be an element of B. It follows that $f(i_1) \in \Omega_{i_1} \setminus B$, and $f \in A^{\Omega_{i_1}\setminus B}$. We have proved that $A^{\Omega_{i_1}\setminus B} = A^{\Omega_{i_1}} \setminus A^B$.

4. Let $B \in \Gamma$. Then, $A^B \in \sigma(\amalg_{i\in I}\mathcal{E}_i)$. All σ-algebras being closed under complementation, we have $(A^B)^c \in \sigma(\amalg_{i\in I}\mathcal{E}_i)$. Moreover, from 2., $A^{\Omega_{i_1}} \in \sigma(\amalg_{i\in I}\mathcal{E}_i)$. It follows that:
$$A^{\Omega_{i_1}\setminus B} = A^{\Omega_{i_1}} \setminus A^B = A^{\Omega_{i_1}} \cap (A^B)^c \in \sigma(\amalg_{i\in I}\mathcal{E}_i)$$
We conclude that $\Omega_{i_1} \setminus B \in \Gamma$.

5. Let $(B_n)_{n\geq 1}$ be a sequence of subsets of Ω_{i_1}. If $f \in A^{\cup B_n}$, then f is a map defined on I, such that $f(i) \in A_i$ for all $i \neq i_1$, and $f(i_1) \in \cup_{n\geq 1} B_n$. There exists $n \geq 1$ such that $f(i_1) \in B_n$, which implies that $f \in A^{B_n}$. So $f \in \cup_{n\geq 1} A^{B_n}$, and we see that $A^{\cup B_n} \subseteq \cup_{n\geq 1} A^{B_n}$. Conversely, suppose that $f \in \cup_{n\geq 1} A^{B_n}$. There exists $n \geq 1$, such that $f \in A^{B_n}$. In particular, $f(i) \in A_i$ for all $i \in I \setminus \{i_1\}$, and $f(i_1) \in B_n \subseteq \cup_{n\geq 1} B_n$. So $f \in A^{\cup B_n}$. We have proved that $A^{\cup B_n} = \cup_{n\geq 1} A^{B_n}$.

6. From 2., $\Omega_{i_1} \in \Gamma$. From 4., Γ is closed under complementation. To show that Γ is a σ-algebra on Ω_{i_1}, it remains to show that Γ is closed under countable union. Let $(B_n)_{n\geq 1}$ be a sequence of elements of Γ. Then, for all $n \geq 1$, $A^{B_n} \in \sigma(\amalg_{i\in I}\mathcal{E}_i)$. It follows that:
$$A^{\cup B_n} = \cup_{n=1}^{+\infty} A^{B_n} \in \sigma(\amalg_{i\in I}\mathcal{E}_i)$$
So $\cup_{n\geq 1} B_n \in \Gamma$, and Γ is indeed closed under countable union. We have proved that Γ is a σ-algebra on Ω_{i_1}.

7. Let $B \in \mathcal{E}_{i_1}$, $\bar{B}_i = \Omega_i$ for all $i \neq i_1$, and $\bar{B}_{i_1} = B$. Let $f \in A^B$. Then, f is a map defined on I, such that $f(i) \in A_i$ for all $i \in I \setminus \{i_1\}$, and $f(i_1) \in B$. In particular, $f \in A^{\Omega_{i_1}}$ and $f(i) \in \bar{B}_i$ for all $i \in I$, i.e. $f \in \Pi_{i\in I}\bar{B}_i$. Hence, $A^B \subseteq A^{\Omega_{i_1}} \cap (\Pi_{i\in I}\bar{B}_i)$. Conversely, suppose that $f \in A^{\Omega_{i_1}} \cap (\Pi_{i\in I}\bar{B}_i)$. Then, $f(i) \in A_i$ for all $i \in I \setminus \{i_1\}$ and $f(i) \in \bar{B}_i$ for all $i \in I$. In particular, $f(i_1) \in \bar{B}_{i_1} = B$. It follows that $f \in A^B$. We have proved that $A^B = A^{\Omega_{i_1}} \cap (\Pi_{i\in I}\bar{B}_i)$.

8. Let $B \in \mathcal{E}_{i_1}$ and $\bar{B}_i = \Omega_i$ for all $i \in I\setminus\{i_1\}$, and $\bar{B}_{i_1} = B$. Then, $\Pi_{i\in I}\bar{B}_i \in \amalg_{i\in I}\mathcal{E}_i$, and in particular, $\Pi_{i\in I}\bar{B}_i \in \sigma(\amalg_{i\in I}\mathcal{E}_i)$. From 2., $\Omega_{i_1} \in \Gamma$, i.e. $A^{\Omega_{i_1}}$ is also an element of $\sigma(\amalg_{i\in I}\mathcal{E}_i)$. It follows from 7. that:
$$A^B = A^{\Omega_{i_1}} \cap (\Pi_{i\in I}\bar{B}_i) \in \sigma(\amalg_{i\in I}\mathcal{E}_i)$$

We conclude that $B \in \Gamma$. This being true for all $B \in \mathcal{E}_{i_1}$, we have $\mathcal{E}_{i_1} \subseteq \Gamma$. However, since Γ is a σ-algebra on Ω_{i_1}, we finally see that $\sigma(\mathcal{E}_{i_1}) \subseteq \Gamma$.

9. Let $f \in A = \amalg_{i\in I} A_i$. Then, $f(i) \in A_i$ for all $i \in I \setminus \{i_1\}$, and $f(i_1) \in A_{i_1}$. So $f \in A^{A_{i_1}}$. Conversely, if $f \in A^{A_{i_1}}$, then $f \in A$. So $A = A^{A_{i_1}}$. Since A is a rectangle of the family $(\sigma(\mathcal{E}_i))_{i\in I}$, $A_{i_1} \in \sigma(\mathcal{E}_{i_1})$. From 8., $\sigma(\mathcal{E}_{i_1}) \subseteq \Gamma$. it follows that $A_{i_1} \in \Gamma$, and consequently $A = A^{A_{i_1}} \in \sigma(\amalg_{i\in I}\mathcal{E}_i)$. This proves our induction hypothesis for $\mathrm{card} J_A = n+1$.

10. Let $A \in \amalg_{i\in I}\sigma(\mathcal{E}_i)$. If $A = \emptyset$, then A is an element of $\sigma(\amalg_{i\in I}\mathcal{E}_i)$. Let $A \neq \emptyset$. If $\mathrm{card} J_A = 0$, then $A = \amalg_{i\in I}\Omega_i \in \sigma(\amalg_{i\in I}\mathcal{E}_i)$. Using an induction argument on $\mathrm{card} J_A$, we have proved that for all $n \geq 0$:

$$\mathrm{card} J_A = n \Rightarrow A \in \sigma(\amalg_{i\in I}\mathcal{E}_i)$$

Since A is a rectangle of the family $(\sigma(\mathcal{E}_i))_{i\in I}$, J_A is a finite set. It follows that $A \in \sigma(\amalg_{i\in I}\mathcal{E}_i)$. Finally, We conclude that $\amalg_{i\in I}\sigma(\mathcal{E}_i) \subseteq \sigma(\amalg_{i\in I}\mathcal{E}_i)$.

11. From 10., we have $\otimes_{i\in I}\sigma(\mathcal{E}_i) = \sigma(\amalg_{i\in I}\sigma(\mathcal{E}_i)) \subseteq \sigma(\amalg_{i\in I}\mathcal{E}_i)$. However, from exercise (5), $\sigma(\amalg_{i\in I}\mathcal{E}_i) \subseteq \otimes_{i\in I}\sigma(\mathcal{E}_i)$. It follows that $\otimes_{i\in I}\sigma(\mathcal{E}_i) = \sigma(\amalg_{i\in I}\mathcal{E}_i)$. The purpose of this difficult exercise is to prove theorem (26). Congratulations !

Exercise 6

Exercise 7.

1. Since $\mathbf{R} \in \mathcal{T}_{\mathbf{R}}$ and $\mathbf{N}_n$ is finite, from definition (52), the set of rectangles $\mathcal{T}_{\mathbf{R}} \amalg \ldots \amalg \mathcal{T}_{\mathbf{R}}$ reduces to all sets of the form $\amalg_{i\in\mathbf{N}_n} A_i$, where $A_i \in \mathcal{T}_{\mathbf{R}}$ for all $i \in \mathbf{N}_n$. In other words:

$$\mathcal{T}_{\mathbf{R}} \amalg \ldots \amalg \mathcal{T}_{\mathbf{R}} = \{A_1 \times \ldots \times A_n : A_i \in \mathcal{T}_{\mathbf{R}}, \forall i \in \mathbf{N}_n\}$$

2. By definition of the Borel σ-algebra, $\mathcal{B}(\mathbf{R})$ is generated by the topology $\mathcal{T}_{\mathbf{R}}$, i.e. $\mathcal{B}(\mathbf{R}) = \sigma(\mathcal{T}_{\mathbf{R}})$. From theorem (26), we have:

$$\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R}) = \sigma(\mathcal{T}_{\mathbf{R}} \amalg \ldots \amalg \mathcal{T}_{\mathbf{R}})$$

3. Let $\mathcal{C}_2 = \{]a_1, b_1] \times \ldots \times]a_n, b_n] : a_i, b_i \in \mathbf{R}\}$, and let $\mathcal{S}$ be the semi-ring on $\mathbf{R}$, $\mathcal{S} = \{]a, b] : a, b \in \mathbf{R}\}$. Since $\mathbf{N}_n$ is finite, from definition (52), the set of rectangles $\mathcal{S} \amalg \ldots \amalg \mathcal{S}$ is made of all sets of the form $\amalg_{i\in\mathbf{N}_n} A_i$, where $A_i \in \mathcal{S} \cup \{\mathbf{R}\}$. Hence, each element of $\mathcal{C}_2$ is an element of $\mathcal{S} \amalg \ldots \amalg \mathcal{S}$, i.e. $\mathcal{C}_2 \subseteq \mathcal{S} \amalg \ldots \amalg \mathcal{S}$. However, $\mathbf{R}^n$ is an element of $\mathcal{S} \amalg \ldots \amalg \mathcal{S}$, but do not belong to $\mathcal{C}_2$. So the inclusion $\mathcal{C}_2 \subseteq \mathcal{S} \amalg \ldots \amalg \mathcal{S}$ is strict.

4. Let $A \in \mathcal{S} \amalg \ldots \amalg \mathcal{S}$. Then A is of the form $A = A_1 \times \ldots \times A_n$, where each A_i is an element of $\mathcal{S}$, or $A_i = \mathbf{R}$. If all A_i's lie in $\mathcal{S}$, then $A \in \mathcal{C}_2 \subseteq \sigma(\mathcal{C}_2)$. Let $J_A^* = \{k \in \mathbf{N}_n : A_k = \mathbf{R}\}$. We have just seen that if $J_A^* = \emptyset$, or equivalently if $\mathrm{card} J_A^* = 0$, then $A \in \sigma(\mathcal{C}_2)$. Suppose we have proved the induction hypothesis, for $k = 0, \ldots, n-1$:

$$A \in \mathcal{S} \amalg \ldots \amalg \mathcal{S}\,,\ \mathrm{card} J_A^* = k \ \Rightarrow\ A \in \sigma(\mathcal{C}_2)$$

and let $A \in \mathcal{S} \amalg \ldots \amalg \mathcal{S}$ be such that $\mathrm{card} J_A^* = k+1$. Let i_1 be an arbitrary element of J_A^*. Then, $A_{i_1} = \mathbf{R} = \cup_{p=1}^{+\infty}]-p,p]$. Hence, A can be written as:

$$A = A_1 \times \ldots \times A_n = \bigcup_{p=1}^{+\infty} A_1 \times \ldots \times]-p,p] \times \ldots \times A_n \tag{1}$$

where $A_1 \times \ldots \times]-p,p] \times \ldots \times A_n = B_p$ is a notation for $\Pi_{i \in \mathbf{N}_n} \bar{A}_i$ where $\bar{A}_i = A_i$ for all $i \neq i_1$, and $\bar{A_{i_1}} =]-p,p]$. Since for all $p \geq 1$, $]-p,p] \in \mathcal{S}$, B_p is an element of $\mathcal{S} \amalg \ldots \amalg \mathcal{S}$, and more importantly $\mathrm{card} J_{B_p}^* = k$. From our induction hypothesis, it follows that $B_p \in \sigma(\mathcal{C}_2)$. Hence, we see from equation (1) that $A \in \sigma(\mathcal{C}_2)$, and we have proved our induction hypothesis for $\mathrm{card} J_A^* = k+1$. We conclude that for all $A \in \mathcal{S} \amalg \ldots \amalg \mathcal{S}$, we have $A \in \sigma(\mathcal{C}_2)$, i.e. $\mathcal{S} \amalg \ldots \amalg \mathcal{S} \subseteq \sigma(\mathcal{C}_2)$.

5. From theorem (6)[4],we know that the semi-ring $\mathcal{S}$ generates the Borel σ-algebra $\mathcal{B}(\mathbf{R})$ on $\mathbf{R}$, i.e. $\mathcal{B}(\mathbf{R}) = \sigma(\mathcal{S})$. Applying theorem (26), we have:

$$\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R}) = \sigma(\mathcal{S} \amalg \ldots \amalg \mathcal{S}) \tag{2}$$

However, from 3., $\mathcal{C}_2 \subseteq \mathcal{S} \amalg \ldots \amalg \mathcal{S}$, hence $\sigma(\mathcal{C}_2) \subseteq \sigma(\mathcal{S} \amalg \ldots \amalg \mathcal{S})$. Moreover, from 4., $\mathcal{S} \amalg \ldots \amalg \mathcal{S} \subseteq \sigma(\mathcal{C}_2)$, and consequently, we have $\sigma(\mathcal{S} \amalg \ldots \amalg \mathcal{S}) \subseteq \sigma(\mathcal{C}_2)$. It follows that $\sigma(\mathcal{S} \amalg \ldots \amalg \mathcal{S}) = \sigma(\mathcal{C}_2)$. Finally, from equation (2), $\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R}) = \sigma(\mathcal{C}_2)$.

Exercise 7

Exercise 8.

1. Let $\Sigma = \sigma(\mathcal{E})$ be the σ-algebra generated by $\mathcal{E} = \{A\}$. Let $\mathcal{F}$ be the set of subsets of Ω defined by $\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$. Note that $\Omega \in \mathcal{F}$, $\mathcal{F}$ is closed under complementation and countable union, so $\mathcal{F}$ is a σ-algebra on Ω. Since $\mathcal{E} \subseteq \mathcal{F}$, we have $\Sigma = \sigma(\mathcal{E}) \subseteq \mathcal{F}$. However, since $\mathcal{E} \subseteq \sigma(\mathcal{E})$, $A \in \Sigma$. So $A^c \in \Sigma$. Furthermore, $\Omega \in \Sigma$ and $\emptyset \in \Sigma$. Finally, $\mathcal{F} \subseteq \Sigma$. We have proved that $\mathcal{F} = \Sigma$.

2. Since $\{\emptyset, \Omega'\}$ is a σ-algebra on Ω' with $\mathcal{E}' \subseteq \{\emptyset, \Omega'\}$, we have $\sigma(\mathcal{E}') \subseteq \{\emptyset, \Omega'\}$. However, $\sigma(\mathcal{E}')$ being a σ-algebra on Ω', we have $\Omega' \in \sigma(\mathcal{E}')$ and $\emptyset \in \sigma(\mathcal{E}')$. Finally, $\sigma(\mathcal{E}') = \{\emptyset, \Omega'\}$.

3. Since $\mathcal{E}' = \emptyset$, $\mathcal{C} = \{E \times F : E \in \mathcal{E}, F \in \mathcal{E}'\} = \emptyset$.

4. The rectangles in $\mathcal{E} \amalg \mathcal{E}'$ are the sets of the form $A_1 \times A_2$, where $A_1 \in \mathcal{E} \cup \{\Omega\}$ and $A_2 \in \mathcal{E}' \cup \{\Omega'\}$. Since $\mathcal{E}' = \emptyset$, the only possible value for A_2 is Ω'. Since $\mathcal{E} = \{A\}$, A_1 can be equal to A or Ω. It follows that $\mathcal{E} \amalg \mathcal{E}' = \{A \times \Omega', \Omega \times \Omega'\}$.

5. From theorem (26), $\sigma(\mathcal{E}) \otimes \sigma(\mathcal{E}') = \sigma(\mathcal{E} \amalg \mathcal{E}')$. Let $\mathcal{F}$ be defined by $\mathcal{F} = \{\emptyset, A \times \Omega', A^c \times \Omega', \Omega \times \Omega'\}$. Note that the complement of $A \times \Omega'$ in

[4]Beware of external links!

$\Omega\times\Omega'$ is $(A\times\Omega')^c = A^c\times\Omega'$. So $\mathcal{F}$ is closed under complementation, and in fact, $\mathcal{F}$ is a σ-algebra on $\Omega\times\Omega'$. However, from 4., $\mathcal{E}\amalg\mathcal{E}' = \{A\times\Omega', \Omega\times\Omega'\}$. So $\mathcal{E}\amalg\mathcal{E}' \subseteq \mathcal{F}$, and consequently $\sigma(\mathcal{E}\amalg\mathcal{E}') \subseteq \mathcal{F}$. Since all elements of $\mathcal{F}$ have to be in $\sigma(\mathcal{E}\amalg\mathcal{E}')$, we also have $\mathcal{F} \subseteq \sigma(\mathcal{E}\amalg\mathcal{E}')$. We have proved that $\mathcal{F} = \sigma(\mathcal{E}\amalg\mathcal{E}')$. We conclude that $\sigma(\mathcal{E})\otimes\sigma(\mathcal{E}') = \mathcal{F}$.

6. Since $\mathcal{C} = \emptyset$, we have $\sigma(\mathcal{C}) = \{\emptyset, \Omega\times\Omega'\}$. It follows from 5. that $\sigma(\mathcal{C}) \neq \sigma(\mathcal{E})\otimes\sigma(\mathcal{E}')$. The purpose of this exercise is to emphasize an easy mistake to make, when applying theorem (26). This theorem states that $\sigma(\mathcal{E})\otimes\sigma(\mathcal{E}') = \sigma(\mathcal{E}\amalg\mathcal{E}')$. It is very tempting to conclude that:

$$\sigma(\mathcal{E})\otimes\sigma(\mathcal{E}') = \sigma(\{E\times F : E\in\mathcal{E}, F\in\mathcal{E}'\})$$

But this is wrong ! The reason being that the set of rectangles $\mathcal{E}\amalg\mathcal{E}'$ is larger than the set of all $E\times F$, where $E\in\mathcal{E}$ and $F\in\mathcal{E}'$. The elements of $\mathcal{E}\amalg\mathcal{E}'$ are indeed of the form $E\times F$, but with $E\in\mathcal{E}\cup\{\Omega\}$ and $F\in\mathcal{E}'\cup\{\Omega'\}$. (Do not forget the '$\cup$'). So $\sigma(\mathcal{E})\otimes\sigma(\mathcal{E}') = \sigma(\{E\times F : E\in\mathcal{E}\cup\{\Omega\}, F\in\mathcal{E}'\cup\{\Omega'\}\})$. You have been warned...

Exercise 8

Exercise 9.

1. Strictly speaking, $\mathcal{F}\otimes\mathcal{G}$ is a σ-algebra on $\mathbf{R}^n\times\mathbf{R}^p$. However, $\mathbf{R}^n\times\mathbf{R}^p$ and $\mathbf{R}^{n+p}$ can be *identified*, through the bijection $\psi : \mathbf{R}^n\times\mathbf{R}^p \to \mathbf{R}^{n+p}$, defined by $\psi(x,y) = (x_1,\ldots,x_n,y_1,\ldots,y_p)$. Hence, $\mathcal{F}\otimes\mathcal{G}$ can be *viewed* as a σ-algebra on $\mathbf{R}^{n+p}$.

2. By definition, $\mathcal{F} = \sigma(\mathcal{C}_1)$, where $\mathcal{C}_1$ is the set of measurable rectangles $\mathcal{C}_1 = \{A_1\times\ldots\times A_n : A_i\in\mathcal{B}(\mathbf{R}), \forall i\in\mathbf{N}_n\}$. Similarly, if $\mathcal{C}_2 = \{A_{n+1}\times\ldots\times A_{n+p} : A_{n+i}\in\mathcal{B}(\mathbf{R}), \forall i\in\mathbf{N}_p\}$, then $\mathcal{G} = \sigma(\mathcal{C}_2)$. From theorem (26), we have $\mathcal{F}\otimes\mathcal{G} = \sigma(\mathcal{C}_1\amalg\mathcal{C}_2)$. Furthermore, since $\mathbf{R}^n\in\mathcal{C}_1$ and $\mathbf{R}^p\in\mathcal{C}_2$, the set of rectangles $\mathcal{C}_1\amalg\mathcal{C}_2$ is given by $\mathcal{C}_1\amalg\mathcal{C}_2 = \{A\times A' : A\in\mathcal{C}_1, A'\in\mathcal{C}_2\}$. If we *identify* sets of the form $(A_1\times\ldots\times A_n)\times(A_{n+1}\times\ldots\times A_{n+p})$ with $A_1\times\ldots\times A_{n+p}$, then $\mathcal{C}_1\amalg\mathcal{C}_2$ can be written as:

$$\mathcal{C}_1\amalg\mathcal{C}_2 = \{A_1\times\ldots\times A_{n+p} : A_i\in\mathcal{B}(\mathbf{R}), \forall i\in\mathbf{N}_{n+p}\}$$

We conclude that $\mathcal{F}\otimes\mathcal{G}$ is generated by the sets of the form $A_1\times\ldots\times A_{n+p}$, where $A_i\in\mathcal{B}(\mathbf{R})$ for all $i\in\mathbf{N}_{n+p}$.

3. Let $\mathcal{C} = \{A_1\times\ldots\times A_{n+p} : A_i\in\mathcal{B}(\mathbf{R}), \forall i\in\mathbf{N}_{n+p}\}$. From 2., $\mathcal{F}\otimes\mathcal{G} = \sigma(\mathcal{C})$. However, $\mathcal{C}$ is the set of measurable rectangles in $\mathbf{R}^{n+p}$. Consequently, $\sigma(\mathcal{C}) = \mathcal{B}(\mathbf{R})\otimes\ldots\otimes\mathcal{B}(\mathbf{R})$ ($n+p$ terms). We conclude that $\mathcal{B}(\mathbf{R})\otimes\ldots\otimes\mathcal{B}(\mathbf{R}) = \mathcal{F}\otimes\mathcal{G}$, i.e.

$$\underbrace{\mathcal{B}(\mathbf{R})\otimes\ldots\otimes\mathcal{B}(\mathbf{R})}_{n+p} = \underbrace{(\mathcal{B}(\mathbf{R})\otimes\ldots\otimes\mathcal{B}(\mathbf{R}))}_{n}\otimes\underbrace{(\mathcal{B}(\mathbf{R})\otimes\ldots\otimes\mathcal{B}(\mathbf{R}))}_{p}$$

Exercise 9

Exercise 10.

1. In exercise (2), we defined a *natural* bijection $\Phi : \Omega \to \Omega'$, by:

$$\Phi((\omega_i)_{i\in I}) \stackrel{\triangle}{=} ((\omega_i)_{i\in I_\lambda})_{\lambda\in\Lambda}$$

This allows us to define $\bar{\Phi} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega')$, by:

$$\bar{\Phi}(A) \stackrel{\triangle}{=} \Phi(A) \stackrel{\triangle}{=} \{\Phi(\omega) : \omega \in A\}$$

for all $A \subseteq \Omega$. In other words, $\bar{\Phi}$ maps every subset A of Ω, with its direct image $\Phi(A)$ by the bijection $\Phi : \Omega \to \Omega'$. Let $A' \subseteq \Omega'$. Since Φ is a bijection, we have $A' = \Phi(\Phi^{-1}(A'))$, i.e. the direct image of the inverse image of A' by Φ is equal to A'. So $A' = \bar{\Phi}(\Phi^{-1}(A'))$, and $\bar{\Phi}$ is a surjective map. If $A, B \subseteq \Omega$ are such that $\bar{\Phi}(A) = \bar{\Phi}(B)$, taking the inverse images of both sides, we have $A = B$. So $\bar{\Phi}$ is an injective map. We have proved that $\bar{\Phi}$ is a bijection from $\mathcal{P}(\Omega)$ to $\mathcal{P}(\Omega')$. Informally, Φ is a bijection allowing us to *identify* an element of $\Pi_{i\in I}\Omega_i$ with an element of $\Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}\Omega_i)$. The bijection $\bar{\Phi}$ allows us to *identify* a subset of $\Pi_{i\in I}\Omega_i$ with a subset of $\Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}\Omega_i)\ldots$

2. Let A be a subset of Ω of the form $A = \Pi_{i\in I}A_i$. Let A' be the *corresponding* set $A' = \Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}A_i)$. Saying that A and A' *are identified through* the bijection $\bar{\Phi}$, is just another way of saying that $A' = \bar{\Phi}(A)$. Suppose $y \in \bar{\Phi}(A)$. There exists $x \in A$ such that $y = \Phi(x)$. For all $\lambda \in \Lambda$, we have $y(\lambda) = \Phi(x)(\lambda) = x_{|I_\lambda}$. Since $x \in A$, each $x_{|I_\lambda}$ is an element of $\Pi_{i\in I_\lambda}A_i$. So $y(\lambda) \in \Pi_{i\in I_\lambda}A_i$ for all $\lambda \in \Lambda$. It follows that $y \in \Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}A_i) = A'$. So $\bar{\Phi}(A) \subseteq A'$. Conversely, suppose $y \in A'$. y is a map defined on Λ, such that $y(\lambda) \in \Pi_{i\in I_\lambda}A_i$ for all $\lambda \in \Lambda$. Each $y(\lambda)$ is a map defined on I_λ, such that $y(\lambda)(i) \in A_i$ for all $i \in I_\lambda$. Let x be the map defined on I by $x(i) = y(\lambda)(i)$, where given $i \in I$, λ is the unique element of Λ such that $i \in I_\lambda$. Then, x is such that $x(i) \in A_i$ for all $i \in I$, so $x \in \Pi_{i\in I}A_i = A$. Moreover, by construction, for all $\lambda \in \Lambda$, $x_{|I_\lambda} = y(\lambda)$. So $y(\lambda) = \Phi(x)(\lambda)$ for all $\lambda \in \Lambda$, i.e. $y = \Phi(x)$. We have found $x \in A$, such that $y = \Phi(x)$. So $y \in \Phi(A) = \bar{\Phi}(A)$. We have proved that $A' \subseteq \bar{\Phi}(A)$. Finally, $A' = \bar{\Phi}(A)$. We have proved that the sets $\Pi_{i\in I}A_i$ and $\Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}A_i)$ are indeed *identified* through the bijection $\bar{\Phi}$.

3. Let $\Pi_{i\in I}A_i \in \amalg_{i\in I}\mathcal{F}_i$. Then, for all $i \in I$, $A_i \in \mathcal{F}_i$, and $A_i \neq \Omega_i$ for finitely many $i \in I$. For each $\lambda \in \Lambda$, $\Pi_{i\in I_\lambda}A_i$ is therefore such that $A_i \in \mathcal{F}_i$ for all $i \in I_\lambda$, and $A_i \neq \Omega_i$ for finitely many $i \in I_\lambda$. So $\Pi_{i\in I_\lambda}A_i \in \amalg_{i\in I_\lambda}\mathcal{F}_i$. It follows that $\Pi_{i\in I}A_i$ can be written as (through identification):

$$\Pi_{i\in I}A_i = \Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}A_i) = \Pi_{\lambda\in\Lambda}B_\lambda$$

where $B_\lambda \in \amalg_{i\in I_\lambda}\mathcal{F}_i$ for all $\lambda \in \Lambda$. Moreover, the set of all $\lambda \in \Lambda$ for which $B_\lambda \neq \Pi_{i\in I_\lambda}\Omega_i$, is necessarily finite. It follows that $\Pi_{i\in I}A_i \in \amalg_{\lambda\in\Lambda}(\amalg_{i\in I_\lambda}\mathcal{F}_i)$. So $\amalg_{i\in I}\mathcal{F}_i \subseteq \amalg_{\lambda\in\lambda}(\amalg_{i\in I_\lambda}\mathcal{F}_i)$. Conversely, let $\Pi_{\lambda\in\Lambda}B_\lambda \in \amalg_{\lambda\in\Lambda}(\amalg_{i\in I_\lambda}\mathcal{F}_i)$. For all $\lambda \in \Lambda$, we have $B_\lambda \in \amalg_{i\in I_\lambda}\mathcal{F}_i$, and $B_\lambda \neq \Pi_{i\in I_\lambda}\Omega_i$ for finitely many $\lambda \in \Lambda$. Hence, each B_λ is of the form $\Pi_{i\in I_\lambda}A_i$, where

$A_i \in \mathcal{F}_i$ for all $i \in I_\lambda$, and $A_i \neq \Omega_i$ for finitely many $i \in I_\lambda$. It follows that $\Pi_{\lambda\in\Lambda}B_\lambda$ can be written (with identification) as:

$$\Pi_{\lambda\in\Lambda}B_\lambda = \Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}A_i) = \Pi_{i\in I}A_i$$

where $A_i \in \mathcal{F}_i$ for all $i \in I$, and $A_i \neq \Omega_i$ for finitely many $i \in I$. So $\Pi_{\lambda\in\Lambda}B_\lambda \in \amalg_{i\in I}\mathcal{F}_i$, and $\amalg_{\lambda\in\Lambda}(\amalg_{i\in I_\lambda}\mathcal{F}_i) \subseteq \amalg_{i\in I}\mathcal{F}_i$. We have proved that $\amalg_{i\in I}\mathcal{F}_i = \amalg_{\lambda\in\Lambda}(\amalg_{i\in I_\lambda}\mathcal{F}_i)$.

4. From definition (54), for all $\lambda \in \Lambda$, $\otimes_{i\in I_\lambda}\mathcal{F}_i = \sigma(\amalg_{i\in I_\lambda}\mathcal{F}_i)$. Using theorem (26), $\otimes_{\lambda\in\Lambda}(\otimes_{i\in I_\lambda}\mathcal{F}_i) = \sigma(\amalg_{\lambda\in\Lambda}(\amalg_{i\in I_\lambda}\mathcal{F}_i))$. Using 3., we conclude that $\otimes_{\lambda\in\Lambda}(\otimes_{i\in I_\lambda}\mathcal{F}_i) = \sigma(\amalg_{i\in I}\mathcal{F}_i) = \otimes_{i\in I}\mathcal{F}_i$.

Exercise 10

Exercise 11.

1. Let $T(\mathcal{A})$ be the set of all topologies $\mathcal{T}$ on Ω, which contain $\mathcal{A}$, i.e. such that $\mathcal{A} \subseteq \mathcal{T}$. Note that $T(\mathcal{A})$ is not the empty set, as the power set $\mathcal{P}(\Omega)$ is clearly a topology on Ω (called the discrete topology) which satisfies $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. By definition (55), the topology $\mathcal{T}(\mathcal{A})$ generated by $\mathcal{A}$, is equal to $\cap_{\mathcal{T}\in T(\mathcal{A})}\mathcal{T}$. In order to show that $\mathcal{T}(\mathcal{A})$ is indeed a topology on Ω, it is sufficient to prove that an arbitrary intersection of topologies on Ω, is also a topology on Ω. Let $(\mathcal{T}_i)_{i\in I}$ be an arbitrary family of topologies on Ω, and let $\mathcal{T} = \cap_{i\in I}\mathcal{T}_i$. Since $\emptyset$ and Ω belong to $\mathcal{T}_i$ for all $i \in I$, $\emptyset$ and Ω are elements of $\mathcal{T}$. If $A, B \in \mathcal{T}$, then $A, B \in \mathcal{T}_i$ for all $i \in I$, and therefore $A \cap B \in \mathcal{T}_i$ for all $i \in I$. It follows that $A \cap B \in \mathcal{T}$, and $\mathcal{T}$ is closed under finite intersection. If $(A_j)_{j\in J}$ is an arbitrary family of elements of $\mathcal{T}$, then for all $i \in I$, $(A_j)_{j\in J}$ is an arbitrary family of elements of $\mathcal{T}_i$, and consequently $\cup_{j\in J}A_j \in \mathcal{T}_i$. This being true for all $i \in I$, $\cup_{j\in J}A_j \in \mathcal{T}$, and $\mathcal{T}$ is closed under arbitrary union. We have proved that $\mathcal{T}$ is a topology on Ω. An arbitrary intersection of topologies on Ω, is a topology on Ω. In particular, the topology $\mathcal{T}(\mathcal{A})$ is a topology on Ω.

2. Given $T(\mathcal{A}) = \{\mathcal{T} : \mathcal{T} \text{ topology on } \Omega\ , \mathcal{A} \subseteq \mathcal{T}\}$, the topology $\mathcal{T}(\mathcal{A})$ generated by $\mathcal{A}$ is given by $\mathcal{T}(\mathcal{A}) = \cap_{\mathcal{T}\in T(\mathcal{A})}\mathcal{T}$. Hence, we have $\mathcal{A} \subseteq \mathcal{T}(\mathcal{A})$. Suppose $\mathcal{T}$ is another topology on Ω, such that $\mathcal{A} \subseteq \mathcal{T}$. Then, $\mathcal{T} \in T(\mathcal{A})$. It follows that $\mathcal{T}(\mathcal{A}) \subseteq \mathcal{T}$. We have proved that $\mathcal{T}(\mathcal{A})$ is the smallest topology on Ω, such that $\mathcal{A} \subseteq \mathcal{T}(\mathcal{A})$.

3. Let (E, d) be a metric space, and $\mathcal{A}$ be the set of all open balls:

$$\mathcal{A} = \{B(x,\epsilon) : x \in E, \epsilon > 0\}$$

Let $\mathcal{T}_E^d$ be the metric topology on E. Since any open ball in E is open with respect to the metric topology, i.e. belongs to $\mathcal{T}_E^d$, we have $\mathcal{A} \subseteq \mathcal{T}_E^d$ and therefore $\mathcal{T}(\mathcal{A}) \subseteq \mathcal{T}_E^d$. Conversely, let $U \in \mathcal{T}_E^d$. Define $\Gamma = \{B(x,\epsilon) : x \in E, \epsilon > 0, B(x,\epsilon) \subseteq U\}$, i.e. let Γ be the set of all open balls in E which are contained in U. Since U is open for the metric topology, from definition (30), for all $x \in U$, there exists $\epsilon > 0$ such that $B(x,\epsilon) \subseteq U$.

In particular, there exists $B \in \Gamma$ such that $x \in B$. Hence, $U \subseteq \cup_{B\in\Gamma} B$. Conversely, for all $x \in \cup_{B\in\Gamma} B$, there exists $B \in \Gamma$ such that $x \in B$. But $B \subseteq U$. So $x \in U$. Hence, we see that $U = \cup_{B\in\Gamma} B$. However, Γ is a subset of $\mathcal{A} \subseteq \mathcal{T}(\mathcal{A})$. It follows that $\cup_{B\in\Gamma} B$ is an element of $\mathcal{T}(\mathcal{A})$. We have proved that $U \in \mathcal{T}(\mathcal{A})$. Hence $\mathcal{T}_E^d \subseteq \mathcal{T}(\mathcal{A})$. Finally, $\mathcal{T}_E^d = \mathcal{T}(\mathcal{A})$, i.e. the metric topology on E is generated by the set of all open balls in E.

Exercise 11

Exercise 12.

1. Let U be a subset of $\Pi_{i\in I}\Omega_i$ with the property:
$$\forall x \in U \ , \ \exists V \in \amalg_{i\in I}\mathcal{T}_i \ : \ x \in V \subseteq U \tag{3}$$
Define $\Gamma = \{V \in \amalg_{i\in I}\mathcal{T}_i : V \subseteq U\}$. Given $x \in U$, since property (3) holds, there exists $V \in \Gamma$ such that $x \in V$. So $U \subseteq \cup_{V\in\Gamma} V$. Conversely, if $x \in \cup_{V\in\Gamma} V$, there exists $V \in \Gamma$ such that $x \in V$. But $V \subseteq U$. So $x \in U$. Hence, we see that $U = \cup_{V\in\Gamma} V$. Since $\Gamma \subseteq \amalg_{i\in I}\mathcal{T}_i \subseteq \odot_{i\in I}\mathcal{T}_i$, each $V \in \Gamma$ is an element of the product topology $\odot_{i\in I}\mathcal{T}_i$. So $\cup_{V\in\Gamma} V$ is also an element of $\odot_{i\in I}\mathcal{T}_i$. We have proved that $U \in \odot_{i\in I}\mathcal{T}_i$, and therefore, any subset of $\Pi_{i\in I}\Omega_i$ with property (3), belongs to the product topology $\odot_{i\in I}\mathcal{T}_i$. Let $\mathcal{T}$ be the set of all U subset of $\Pi_{i\in I}\Omega_i$ which satisfy property (3). We claim that in fact, $\mathcal{T}$ is a topology on $\Pi_{i\in I}\Omega_i$. Indeed, $\emptyset$ satisfies property (3) vacuously. So $\emptyset \in \mathcal{T}$. The set of all rectangles $\amalg_{i\in I}\mathcal{T}_i$ is a subset of $\mathcal{T}$. In particular, $\Pi_{i\in I}\Omega_i \in \mathcal{T}$. Suppose $A, B \in \mathcal{T}$. Let $x \in A \cap B$. Since A satisfies property (3), there exists $V \in \amalg_{i\in I}\mathcal{T}_i$ such that $x \in V \subseteq A$. Similarly, there exists $W \in \amalg_{i\in I}\mathcal{T}_i$ such that $x \in W \subseteq B$. It follows that $x \in V \cap W \subseteq A \cap B$. However, V and W being rectangles of $(\mathcal{T}_i)_{i\in I}$, they can be written as $V = \Pi_{i\in I} A_i$ and $W = \Pi_{i\in I} B_i$, where $A_i, B_i \in \mathcal{T}_i \cup \{\Omega_i\} = \mathcal{T}_i$ and $A_i \neq \Omega_i$ or $B_i \neq \Omega_i$ for finitely many $i \in I$. It follows that $V \cap W = \Pi_{i\in I}(A_i \cap B_i)$, where each $A_i \cap B_i$ lie in $\mathcal{T}_i$ (it is a topology), and $A_i \cap B_i \neq \Omega_i$ for finitely many $i \in I$. So $V \cap W$ is a rectangle of $(\mathcal{T}_i)_{i\in I}$, i.e. $V \cap W \in \amalg_{i\in I}\mathcal{T}_i$, and $x \in V \cap W \subseteq A \cap B$. We have proved that $A \cap B$ satisfies property (3), i.e. $A \cap B \in \mathcal{T}$. So $\mathcal{T}$ is closed under finite intersection. Finally, let $(A_j)_{j\in J}$ be a family of elements of $\mathcal{T}$. Let $x \in \cup_{j\in J} A_j$. There exists $j \in J$ such that $x \in A_j$. Since $A_j \in \mathcal{T}$, there exists $V \in \amalg_{i\in I}\mathcal{T}_i$ such that $x \in V \subseteq A_j$. In particular, $x \in V \subseteq \cup_{j\in J} A_j$. Hence, we see that $\cup_{j\in J} A_j$ satisfies property (3), i.e. $\cup_{j\in J} A_j \in \mathcal{T}$. So $\mathcal{T}$ is closed under arbitrary union. We have proved that $\mathcal{T}$ is a topology on $\Pi_{i\in I}\Omega_i$. Since $\amalg_{i\in I}\mathcal{T}_i \subseteq \mathcal{T}$, we conclude that $\odot_{i\in I}\mathcal{T}_i = \mathcal{T}(\amalg_{i\in I}\mathcal{T}_i) \subseteq \mathcal{T}$. It follows that any element of the product topology satisfies property (3). We have proved that a subset U of $\Pi_{i\in I}\Omega_i$ is an element of $\odot_{i\in I}\mathcal{T}_i$, if and only if it satisfies property (3).

2. $\amalg_{i\in I}\mathcal{T}_i \subseteq \mathcal{T}(\amalg_{i\in I}\mathcal{T}_i) = \odot_{i\in I}\mathcal{T}_i$.

3. From theorem (26), $\otimes_{i\in I}\mathcal{B}(\Omega_i) = \otimes_{i\in I}\sigma(\mathcal{T}_i) = \sigma(\amalg_{i\in I}\mathcal{T}_i)$.

4. From 2., we have $\sigma(\amalg_{i\in I}\mathcal{T}_i) \subseteq \sigma(\odot_{i\in I}\mathcal{T}_i) = \mathcal{B}(\Pi_{i\in I}\Omega_i)$. Using 3., we obtain $\otimes_{i\in I}\mathcal{B}(\Omega_i) \subseteq \mathcal{B}(\Pi_{i\in I}\Omega_i)$.

Exercise 12

Exercise 13.

1. The scalar product (x, y) being semi-linear and commutative:

$$\begin{aligned} \|x+ty\|^2 &= (x+ty, x+ty) \\ &= (x,x) + t(y,x) + t(x,y) + t^2(y,y) \\ &= \|x\|^2 + t^2\|y\|^2 + 2t(x,y) \end{aligned}$$

2. When $y \neq 0$, the polynomial $t \to p(t) = t^2\|y\|^2 + 2t(x,y) + \|x\|^2$ has a minimum attained at $t = -(x,y)/\|y\|^2$. The value of this minimum is $-(x,y)^2/\|y\|^2 + \|x\|^2$. Since $p(t) = \|x+ty\|^2 \geq 0$ for all $t \in \mathbf{R}$, in particular, we have $-(x,y)^2/\|y\|^2 + \|x\|^2 \geq 0$, i.e. $|(x,y)| \leq \|x\|.\|y\|$. This inequality still holds if $y = 0$.

3. We have:

$$\begin{aligned} \|x+y\|^2 &= \|x\|^2 + 2(x,y) + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|.\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2 \end{aligned}$$

Exercise 13

Exercise 14.

1. Each metric d_i has values in $\mathbf{R}^+$. So $d(x,y) < +\infty$ for all x, y, i.e. d also has values in $\mathbf{R}^+$. It is clear that $d(x,y) = d(y,x)$ for all $x, y \in \Omega$. Suppose that $d(x,y) = 0$. Then, for all $i \in \mathbf{N}_n$, we have $d_i(x_i, y_i) = 0$ and consequently $x_i = y_i$. So $x = y$. Conversely, it is clear that $d(x,x) = 0$. Let $x, y, z \in \Omega$. For all $i \in \mathbf{N}_n$, we have:

$$d_i(x_i, y_i) \leq d_i(x_i, z_i) + d_i(z_i, y_i)$$

and therefore:

$$d(x,y) \leq \sqrt{\sum_{i=1}^{n}(d_i(x_i,z_i) + d_i(z_i,y_i))^2}$$

Using exercise (13), we conclude that:

$$d(x,y) \leq \sqrt{\sum_{i=1}^{n}(d_i(x_i,z_i))^2} + \sqrt{\sum_{i=1}^{n}(d_i(z_i,y_i))^2}$$

i.e. $d(x,y) \leq d(x,z) + d(z,y)$. It follows from definition (28)[5] that d is indeed a metric on Ω.

[5]Beware of external links!

2. The set of rectangles $\amalg_{i\in\mathbf{N}_n}\mathcal{T}_i$ is given by:

$$\amalg_{i\in\mathbf{N}_n}\mathcal{T}_i = \{U_1\times\ldots\times U_n : U_i\in\mathcal{T}_i, \forall i\in\mathbf{N}_n\}$$

   It follows from exercise (12) that $U\subseteq\Omega$ is open in Ω, i.e. belongs to the product topology $\mathcal{T}$, if and only if for all $x\in U$, there exist $U_1,\ldots,U_n$ open in $\Omega_1,\ldots,\Omega_n$ respectively, such that:

$$x\in U_1\times\ldots\times U_n\subseteq U$$

3. Let $U\in\mathcal{T}$. From 2., for all $x\in U$, there exist $U_1,\ldots,U_n$ open in $\Omega_1,\ldots,\Omega_n$ respectively, such that $x\in U_1\times\ldots\times U_n\subseteq U$. By assumption, each topology $\mathcal{T}_i$ is induced by the metric d_i, i.e. $\mathcal{T}_i=\mathcal{T}^{d_i}_{\Omega_i}$. For all $i\in\mathbf{N}_n$, $x_i\in U_i$. Hence, there exists $\epsilon_i>0$, such that $B(x_i,\epsilon_i)\subseteq U_i$, where $B(x_i,\epsilon_i)$ denotes the open ball in Ω_i. Let $\epsilon=\min(\epsilon_1,\ldots,\epsilon_n)$. Suppose $y\in\Omega$ is such that $d_i(x_i,y_i)<\epsilon$, for all $i\in\mathbf{N}_n$. Then, $y_i\in B(x_i,\epsilon_i)\subseteq U_i$ for all $i\in\mathbf{N}_n$, and consequently $y\in U_1\times\ldots\times U_n\subseteq U$. We have found $\epsilon>0$ such that:

$$(\forall i\in\mathbf{N}_n, d_i(x_i,y_i)<\epsilon)\ \Rightarrow\ y\in U$$

4. Let $U\in\mathcal{T}$, and $x\in U$. Let $\epsilon>0$ be as in 3. Let $y\in B(x,\epsilon)$, where $B(x,\epsilon)$ denotes the open ball in $\Omega=\Omega_1\times\ldots\times\Omega_n$, with respect to the metric d. Then, $d(x,y)<\epsilon$. Since for all $i\in\mathbf{N}_n$, $d_i(x_i,y_i)\leq d(x,y)$, we have $d_i(x_i,y_i)<\epsilon$ for all $i\in\mathbf{N}_n$. From 3., we see that $y\in U$. So $B(x,\epsilon)\subseteq U$. For all $x\in U$, we have found $\epsilon>0$ such that $B(x,\epsilon)\subseteq U$. It follows that U belongs to the metric topology $\mathcal{T}^d_\Omega$. We have proved that $\mathcal{T}\subseteq\mathcal{T}^d_\Omega$.

5. Let $U\in\mathcal{T}^d_\Omega$ and $x\in U$. From definition (30)[6] of the metric topology, there exists $\epsilon'>0$ such that $B(x,\epsilon')\subseteq U$. Define $\epsilon=\epsilon'/\sqrt{n}$, and let $y\in B(x_1,\epsilon)\times\ldots\times B(x_n,\epsilon)$. Then, for all $i\in\mathbf{N}_n$, $d_i(x_i,y_i)<\epsilon$. Hence, $d(x,y)<\sqrt{n\epsilon^2}=\sqrt{n}\epsilon=\epsilon'$. So $y\in U$. We have found $\epsilon>0$ such that:

$$x\in B(x_1,\epsilon)\times\ldots\times B(x_n,\epsilon)\subseteq U$$

6. Let $U\in\mathcal{T}^d_\Omega$ and $x\in U$. Let $\epsilon>0$ be as in 5. Then, we have $x\in B(x_1,\epsilon)\times\ldots\times B(x_n,\epsilon)\subseteq U$. Each $B(x_i,\epsilon)$ being open in Ω_i, we have found $U_1,\ldots,U_n$ open in $\Omega_1,\ldots,\Omega_n$ respectively, such that $x\in U_1\times\ldots\times U_n\subseteq U$. From 2., we conclude that $U\in\mathcal{T}$. So $\mathcal{T}^d_\Omega\subseteq\mathcal{T}$.

7. From 4. and 6., we have $\mathcal{T}=\mathcal{T}^d_\Omega$. In other words, the product topology $\mathcal{T}=\mathcal{T}_1\odot\ldots\odot\mathcal{T}_n$ is equal to the metric topology $\mathcal{T}^d_\Omega$ on Ω, induced by the metric d. In particular, the topological space $(\Omega,\mathcal{T})$ is metrizable.

8. Both d' and d'' have values in $\mathbf{R}^+$. For all $x,y\in\Omega$, we have $d'(x,y)=d'(y,x)$ and $d''(x,y)=d''(y,x)$. Moreover, it is clear that $d'(x,y)=0$ is equivalent to each $d_i(x_i,y_i)$ being equal to 0, hence equivalent to $x_i=y_i$

[6] Beware of external links!

for all i's, i.e. equivalent to $x = y$. Similarly, $d''(x,y) = 0$ is equivalent to $x = y$. Given $x, y, z \in \Omega$, for all $i \in \mathbf{N}_n$, we have:

$$d_i(x_i, y_i) \leq d_i(x_i, z_i) + d_i(z_i, y_i)$$

It follows immediately that $d'(x,y) \leq d'(x,z) + d'(z,y)$, and furthermore, for all $i = 1, \ldots, n$:

$$d_i(x_i, y_i) \leq d''(x,z) + d''(z,y)$$

From which we conclude that $d''(x,y) \leq d''(x,z) + d''(z,y)$. We have proved that d' and d'' are metrics on Ω.

9. Let $x, y \in \Omega$. For all $i \in \mathbf{N}_n$, define $a_i = d_i(x_i, y_i)$. Let $a, b \in \mathbf{R}^n$ be given $a = (a_1, \ldots, a_n)$ and $b = (1, \ldots, 1)$. From exercise (13), we have $|(a,b)| \leq \|a\|.\|b\|$, and consequently:

$$d'(x,y) \leq \sqrt{n}d(x,y)$$

From $(\sum_{i=1}^n a_i)^2 \geq \sum_{i=1}^n a_i^2$, we obtain:

$$d(x,y) \leq d'(x,y)$$

Hence, $\alpha' d' \leq d \leq \beta' d'$, where $\alpha' = 1/\sqrt{n}$ and $\beta' = 1$.
From $\sum_{i=1}^n a_i^2 \leq n(\max_i a_i)^2$, we obtain:

$$d(x,y) \leq \sqrt{n}d''(x,y)$$

From $(\max_i a_i)^2 \leq \sum_{i=1}^n a_i^2$ we obtain:

$$d''(x,y) \leq d(x,y)$$

Hence, $\alpha'' d'' \leq d \leq \beta'' d''$, where $\alpha'' = 1$ and $\beta'' = \sqrt{n}$.

10. From 9., there exist $\beta' > 0$ such that $d \leq \beta' d'$. Let $U \in \mathcal{T}_\Omega^d$, and $x \in U$. There exists $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$, where $B_d(x, \epsilon)$ denotes the open ball in Ω, relative to the metric d. Suppose $y \in \Omega$ is such that $d'(x,y) < \epsilon/\beta'$. Then, we have $d(x,y) \leq \beta' d'(x,y) < \epsilon$, and it follows that $y \in U$. So $B_{d'}(x, \epsilon/\beta') \subseteq U$. For all $x \in U$, we have found $\epsilon' = \epsilon/\beta' > 0$ such that $B_{d'}(x, \epsilon') \subseteq U$. It follows that $U \in \mathcal{T}_\Omega^{d'}$. We have proved that $\mathcal{T}_\Omega^d \subseteq \mathcal{T}_\Omega^{d'}$. Using 9., from $d' \leq (1/\alpha')d$, we conclude similarly that $\mathcal{T}_\Omega^{d'} \subseteq \mathcal{T}_\Omega^d$. Hence, $\mathcal{T}_\Omega^{d'} = \mathcal{T}_\Omega^d$. Similarly, from $\alpha'' d'' \leq d \leq \beta'' d''$, we have $\mathcal{T}_\Omega^{d''} = \mathcal{T}_\Omega^d$. We have proved that $\mathcal{T}_\Omega^{d'} = \mathcal{T}_\Omega^d = \mathcal{T}_\Omega^{d''}$. Since $\mathcal{T}_\Omega^d = \mathcal{T}$ is the product topology on Ω, we conclude that d' and d'' also induce the product topology $\mathcal{T} = \mathcal{T}_1 \odot \ldots \odot \mathcal{T}_n$ on Ω.

Exercise 14

Exercise 15.

1. For all $a \in \mathbf{R}^+$, $1 \wedge a = \min(1, a)$. Let $a, b \in \mathbf{R}^+$. Suppose $a + b \leq 1$. Then, both $a \leq 1$ and $b \leq 1$, and we have:

$$1 \wedge (a+b) = a + b = 1 \wedge a + 1 \wedge b$$

Suppose $a+b \geq 1$. If both $a \leq 1$ and $b \leq 1$, we have:

$$1 \wedge (a+b) = 1 \leq a+b = 1 \wedge a + 1 \wedge b$$

if $a \geq 1$, we have:

$$1 \wedge (a+b) = 1 = 1 \wedge a \leq 1 \wedge a + 1 \wedge b$$

In any case, we see that:

$$1 \wedge (a+b) \leq 1 \wedge a + 1 \wedge b$$

2. For all $x, y \in \Omega$, we have:

$$d(x,y) = \sum_{n=1}^{+\infty} \frac{1}{2^n}(1 \wedge d_n(x_n, y_n)) \leq \sum_{n=1}^{+\infty} \frac{1}{2^n} < +\infty$$

So d has values in $\mathbf{R}^+$. It is clear that $d(x,y) = d(y,x)$. Moreover, $d(x,y) = 0$ is equivalent to $d_n(x_n, y_n) = 0$ for all $n \geq 1$, which is in turn equivalent to $x = y$. For all $x, y, z \in \Omega$, and $n \geq 1$, we have:

$$d_n(x_n, y_n) \leq d_n(x_n, z_n) + d_n(z_n, y_n)$$

and consequently, using 1.:

$$1 \wedge d_n(x_n, y_n) \leq 1 \wedge d_n(x_n, z_n) + 1 \wedge d_n(z_n, y_n)$$

It follows that $d(x,y) \leq d(x,z) + d(z,y)$. We have proved that d is a metric on Ω.

3. Let $V = \amalg_{n=1}^{+\infty} U_n$ be a rectangle of the family $(\mathcal{T}_n)_{n \geq 1}$. The set $\{n \geq 1 : U_n \neq \Omega_n\}$ being finite, it is either empty or has a maximal element $N \geq 1$. it follows that V can be written as:

$$V = U_1 \times \ldots \times U_N \times \prod_{n=N+1}^{+\infty} \Omega_n \tag{4}$$

where $U_1, \ldots, U_N$ are open in $\Omega_1, \ldots, \Omega_N$ respectively. If the set $\{n \geq 1 : U_n \neq \Omega_n\}$ is empty, then V is also of the form (4), for any $N \geq 1$. Conversely, any set V of the form (4) is a rectangle in $\amalg_{n=1}^{+\infty} \mathcal{T}_n$. From exercise (12), $U \in \mathcal{T} = \odot_{n=1}^{+\infty} \mathcal{T}_n$, if and only if, for all $x \in U$, there exists $V \in \amalg_{n=1}^{+\infty} \mathcal{T}_n$ such that $x \in V \subseteq U$. It follows that $U \subseteq \Omega$ is open in Ω, i.e. belongs to the product topology $\mathcal{T}$, if and only if for all $x \in U$, there exists $N \geq 1$ and open sets $U_1, \ldots, U_N$ in $\Omega_1, \ldots, \Omega_N$ respectively, such that:

$$x \in U_1 \times \ldots \times U_N \times \prod_{n=N+1}^{+\infty} \Omega_n \subseteq U$$

4. Suppose that $d(x,y) < 1/2^n$, for some $n \geq 1$. Then, $d_n(x_n, y_n)$ has to be less than 1. Specifically:

$$d(x,y) \geq \frac{1}{2^n}(1 \wedge d_n(x_n, y_n)) = \frac{1}{2^n} d_n(x_n, y_n)$$

So $d(x,y) < 1/2^n \Rightarrow d_n(x_n,y_n) \leq 2^n d(x,y)$

5. Let $U \in \mathcal{T}$ and $x \in U$. From 3., there exist $N \geq 1$ and $U_1,\ldots,U_N$ open in $\Omega_1,\ldots,\Omega_N$ respectively, such that:

$$x \in U_1 \times \ldots \times U_N \times \prod_{n=N+1}^{+\infty} \Omega_n \subseteq U$$

Let $i \in \{1,\ldots,N\}$. Then $x_i \in U_i \in \mathcal{T}_i$. The topology $\mathcal{T}_i$ being the metric topology $\mathcal{T}_{\Omega_i}^{d_i}$, there exists $\epsilon_i > 0$ such that we have $B(x_i,\epsilon_i) \subseteq U_i$. Let $\epsilon = \min(1/2^N, \epsilon_1/2, \ldots, \epsilon_N/2^N)$ and $y \in \Omega$ be such that $d(x,y) < \epsilon$. In particular, we have $d(x,y) < 1/2^i$, for all $i = 1,\ldots,N$. Hence, from 4., we see that $d_i(x_i,y_i) \leq 2^i d(x,y) < 2^i\epsilon \leq \epsilon_i$. It follows that $y_i \in U_i$ for all $i = 1,\ldots,N$ and consequently $y \in U_1 \times \ldots \times U_N \times \Pi_{n=N+1}^{+\infty}\Omega_n \subseteq U$. We have found $\epsilon > 0$ such that $d(x,y) < \epsilon \Rightarrow y \in U$.

6. From 5. for all $U \in \mathcal{T}$ and $x \in U$, there exists $\epsilon > 0$ such that $B(x,\epsilon) \subseteq U$. It follows that $U \in \mathcal{T}_\Omega^d$. So $\mathcal{T} \subseteq \mathcal{T}_\Omega^d$.

7. Let $U \in \mathcal{T}_\Omega^d$ and $x \in U$. By definition (30) of the metric topology, there exists $\epsilon' > 0$ such that $B(x,\epsilon') \subseteq U$. In other words, there exists $\epsilon' > 0$ such that for all $y \in \Omega$:

$$d(x,y) < \epsilon' \ \Rightarrow\ y \in U$$

Let $\epsilon = \epsilon'/2$ and $N \geq 1$ be such that:

$$\sum_{n=N+1}^{+\infty} \frac{1}{2^n} \leq \epsilon$$

Suppose $y \in \Omega$ is such that:

$$\sum_{n=1}^{N} \frac{1}{2^n}(1 \wedge d_n(x_n,y_n)) < \epsilon$$

Then, we have:

$$d(x,y) < \epsilon + \sum_{n=N+1}^{+\infty} \frac{1}{2^n}(1 \wedge d_n(x_n,y_n)) \leq 2\epsilon = \epsilon'$$

It follows that $y \in U$. We have found $\epsilon > 0$ and $N \geq 1$ such that:

$$\sum_{n=1}^{N} \frac{1}{2^n}(1 \wedge d_n(x_n,y_n)) < \epsilon \ \Rightarrow\ y \in U$$

8. Let $U \in \mathcal{T}_\Omega^d$ and $x \in U$. Let $\epsilon > 0$ an $N \geq 1$ be as in 7. Let $y \in \Omega$ be such that:

$$y \in B(x_1,\epsilon) \times \ldots \times B(x_N,\epsilon) \times \prod_{n=N+1}^{+\infty} \Omega_n$$

For all $n \in \{1, \ldots, N\}$, $d_n(x_n, y_n) < \epsilon$. Hence:

$$\sum_{n=1}^{N} \frac{1}{2^n}(1 \wedge d_n(x_n, y_n)) \leq \epsilon \sum_{n=1}^{N} \frac{1}{2^n} < \epsilon$$

From 7., we conclude that $y \in U$. We have found $\epsilon > 0$ and $N \geq 1$ such that:

$$x \in B(x_1, \epsilon) \times \ldots \times B(x_N, \epsilon) \times \Pi_{n=N+1}^{+\infty}\Omega_n \subseteq U$$

9. Let $U \in \mathcal{T}_\Omega^d$ and $x \in U$. Let $N \geq 1$ and $\epsilon > 0$ be as in 8. Each open ball $B(x_n, \epsilon)$ for $n = 1, \ldots, N$ being open in Ω_n, we have found $U_1, \ldots, U_N$ open in $\Omega_1, \ldots, \Omega_N$ respectively, such that:

$$x \in U_1 \times \ldots \times U_N \times \prod_{n=N+1}^{+\infty} \Omega_n \subseteq U$$

From 3., it follows that $U \in \mathcal{T} = \odot_{n=1}^{+\infty}\mathcal{T}_n$. We have proved that $\mathcal{T}_\Omega^d \subseteq \mathcal{T}$.

10. From 6. and 9., $\mathcal{T}_\Omega^d = \mathcal{T}$. In other words, the product topology $\mathcal{T} = \odot_{n=1}^{+\infty}\mathcal{T}_n$ is induced by the metric d on Ω. In particular, the topological space $(\Omega, \mathcal{T})$ is metrizable. The purpose of this exercise, is to show that a countable product of metrizable topological spaces, is itself a metrizable topological space.

Exercise 15

Exercise 16.

1. $\mathcal{H} = \{]r, q[\colon r, q \in \mathbf{Q}\}$ is a countable subset of $\mathcal{T}_\mathbf{R}$. Let $U \in \mathcal{T}_\mathbf{R}$. Define $\mathcal{H}' = \{V \in \mathcal{H} : V \subseteq U\}$. For all $x \in U$, there exists $\epsilon > 0$ such that $]x - \epsilon, x + \epsilon[\subseteq U$. In fact, the set of rational numbers $\mathbf{Q}$ being dense in $\mathbf{R}$, there exist $r, q \in \mathbf{Q}$ such that $x \in]r, q[\subseteq U$. In other words, there exists $V \in \mathcal{H}'$ such that $x \in V$. Hence, we see that $U \subseteq \cup_{V \in \mathcal{H}'} V$. The reverse inclusion being clearly satisfied, we have $U = \cup_{V \in \mathcal{H}'} V$, i.e. U can be expressed as a union of elements of $\mathcal{H}$. This being true for all open sets $U \in \mathcal{T}_\mathbf{R}$, we have proved that $\mathcal{H}$ is a countable base of $(\mathbf{R}, \mathcal{T}_\mathbf{R})$.

2. Let $\mathcal{H}$ be a countable base of $(\Omega, \mathcal{T})$. Let $\mathcal{H}_{|\Omega'}$ be the trace of $\mathcal{H}$ on Ω', i.e. $\mathcal{H}_{|\Omega'} = \{\Omega' \cap V : V \in \mathcal{H}\}$. Since $\mathcal{H}$ is a countable or finite subset of the topology $\mathcal{T}$, $\mathcal{H}_{|\Omega'}$ is a countable or finite subset of the induced topology $\mathcal{T}_{|\Omega'}$. Let $U' \in \mathcal{T}_{|\Omega'}$ be an open subset in Ω'. Then U' is of the form $U' = \Omega' \cap U$ where $U \in \mathcal{T}$. $\mathcal{H}$ being a countable base of $(\Omega, \mathcal{T})$, there exists a family $(V_i)_{i \in I}$ of elements of $\mathcal{H}$ such that $U = \cup_{i \in I} V_i$. It follows that $(\Omega' \cap V_i)_{i \in I}$ is a family of elements of $\mathcal{H}_{|\Omega'}$ such that $U' = \cup_{i \in I}(\Omega' \cap V_i)$. We conclude that $\mathcal{H}_{|\Omega'}$ is a countable base of the induced topological space $(\Omega', \mathcal{T}_{|\Omega'})$.

3. From 1., $\mathbf{R}$ has a countable base. It follows from 2. that the induced topological space $[-1, 1]$ also has a countable base.

4. Let $h : (\Omega, \mathcal{T}) \to (S, \mathcal{T}_S)$ be a homeomorphism, i.e. a continuous bijection such that h^{-1} is also continuous. Suppose $(\Omega, \mathcal{T})$ has a countable base $\mathcal{H}$. Define $h(\mathcal{H}) = \{h(V) : V \in \mathcal{H}\}$. Since $\mathcal{H}$ is a countable or finite subset of $\mathcal{T}$, h^{-1} being continuous, $h(\mathcal{H})$ is a countable or finite subset of $\mathcal{T}_S$. (Note that each direct image $h(V)$ of V by h can be viewed the inverse image $(h^{-1})^{-1}(V)$ of V by h^{-1}). Let $U' \in \mathcal{T}_S$. h being continuous, $h^{-1}(U') \in \mathcal{T}$. $\mathcal{H}$ being a countable base of $(\Omega, \mathcal{T})$, there exists a family $(V_i)_{i \in I}$ of elements of $\mathcal{H}$, such that $h^{-1}(U') = \cup_{i \in I} V_i$. However, $h(h^{-1}(U')) = U'$, and moreover:
$$h(\cup_{i \in I} V_i) = (h^{-1})^{-1}(\cup_{i \in I} V_i) = \cup_{i \in I}(h^{-1})^{-1}(V_i)$$
So $U' = \cup_{i \in I} h(V_i)$. We conclude that U' can be expressed as a union of elements of $h(\mathcal{H})$. So $h(\mathcal{H})$ is a countable base of $(S, \mathcal{T}_S)$. We have proved that if $(\Omega, \mathcal{T})$ has a countable base, then $(S, \mathcal{T}_S)$ also has a countable base. Using the same argument, switching the roles of h and h^{-1}, we see that conversely, if $(S, \mathcal{T}_S)$ has a countable base, then so does $(\Omega, \mathcal{T})$. We have proved that given two homeomorphic topological spaces, one has a countable base, if and only if the other also has a countable base.

5. The topological spaces $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ and $([-1, 1], \mathcal{T}_{[-1,1]})$ being homeomorphic, we conclude from 3. and 4. that $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ has a countable base.

Exercise 16

Exercise 17.

1. Let $p \geq 1$ and $A \in \mathcal{H}^p$ of the form:
$$A = V_1^{k_1} \times \ldots \times V_p^{k_p} \times \Pi_{n=p+1}^{+\infty} \Omega_n$$
For all $n \geq 1$, the set $\{V_n^k : k \in I_n\}$ being a countable base of $\mathcal{T}_n$, it is a subset of $\mathcal{T}_n$. Hence, for all $i \in \{1, \ldots, p\}$, we have $V_i^{k_i} \in \mathcal{T}_i$. It follows that A is a rectangle of the family $(\mathcal{T}_n)_{n \geq 1}$, i.e. $A \in \amalg_{n=1}^{+\infty} \mathcal{T}_n$. From definition (56), the product topology $\mathcal{T}$ on $\Pi_{n=1}^{+\infty} \Omega_n$ being generated by $\amalg_{n=1}^{+\infty} \mathcal{T}_n$, we have $\amalg_{n=1}^{+\infty} \mathcal{T}_n \subseteq \mathcal{T}$. In particular, $A \in \mathcal{T}$. We have proved that $\mathcal{H}^p \subseteq \mathcal{T}$.

2. Using 1., $\mathcal{H} = \cup_{p \geq 1} \mathcal{H}^p \subseteq \mathcal{T}$.

3. By assumption, for all $n \geq 1$, the index set I_n is finite or countable. There exists an injective map $i_n : I_n \to \mathbf{N}$. Given $p \geq 1$, consider the map $j_p : \mathcal{H}^p \to \mathbf{N}^p$, defined in the following way: for $A = V_1^{k_1} \times \ldots \times V_p^{k_p} \times \Pi_{n=p+1}^{+\infty} \Omega_n \in \mathcal{H}^p$, we put:
$$j_p(A) = (i_1(k_1), \ldots, i_p(k_p))$$
Suppose $B = V_1^{k'_1} \times \ldots \times V_p^{k'_p} \times \Pi_{n=p+1}^{+\infty} \Omega_n$ is another element of $\mathcal{H}^p$ such that $j_p(A) = j_p(B)$. Then:
$$(i_1(k_1), \ldots, i_p(k_p)) = (i_1(k'_1), \ldots, i_p(k'_p))$$

Hence, for all $m \in \mathbf{N}_p$, $i_m(k_m) = i_m(k'_m)$, and i_m being injective, we have $k_m = k'_m$. So $A = B$. We have proved the existence of an injective map $j_p : \mathcal{H}^p \to \mathbf{N}^p$.

4. The existence of a bijection $\phi_2 : \mathbf{N}^2 \to \mathbf{N}$ is a standard result, which we may have used in these tutorials before. Now is a good opportunity to give a formal proof of it. Informally, ϕ_2 is defined as $\phi_2(0,0) = 0$, $\phi_2(1,0) = 1$, $\phi_2(0,1) = 2$, $\phi_2(2,0) = 3$, $\phi_2(1,1) = 4$, $\phi_2(0,2) = 5$, etc... As you can see, going through each diagonal one after the other, we are able to *count* the elements of $\mathbf{N}^2$, thus defining the bijection ϕ_2. Formally, we define the map $\phi_2 : \mathbf{N}^2 \to \mathbf{N}$ as follows:

$$\forall (n,p) \in \mathbf{N}^2 \ , \ \phi_2(n,p) = p + [0 + 1 + \ldots + (n+p)]$$

or equivalently, $\phi_2(n,p) = p + h(n+p)$ where:

$$h(m) = 0 + 1 + \ldots + m$$

Let $N \in \mathbf{N}$. Since $h(m) \uparrow +\infty$, the set $\{m \in \mathbf{N} : h(m) \leq N\}$ is finite and it is also non-empty. Hence, it has a maximal element m, and we have $h(m) \leq N < h(m+1)$. Let $p = N - h(m)$. Then $p \in \mathbf{N}$, and we have $0 \leq p < h(m+1) - h(m) = m + 1$. So $p \leq m$. If we define $n = m - p$, then n is also an element of $\mathbf{N}$. So (n,p) is an element of $\mathbf{N}^2$, such that $m = n + p$, and $N = p + h(m)$. It follows that:

$$\phi_2(n,p) = p + h(n+p) = p + h(m) = N$$

We have proved that ϕ_2 is a surjective map. Suppose (n,p) and (n',p') are elements of $\mathbf{N}^2$, with $\phi_2(n,p) = \phi_2(n',p')$. Since $\phi_2(n,p) = p + h(n+p)$, in particular $h(n+p) \leq \phi_2(n,p)$. However, $h(n+p+1) = p + h(n+p) + n + 1 > \phi_2(n,p)$. It follows that for all $(n,p) \in \mathbf{N}^2$, we have:

$$h(n+p) \leq \phi_2(n,p) < h(n+p+1) \tag{5}$$

Since given $N \in \mathbf{N}$, any $m \in \mathbf{N}$ such that $h(m) \leq N < h(m+1)$ is unique, it follows from $\phi_2(n,p) = \phi_2(n',p')$ and equation (5) that $n + p = n' + p'$. Hence:

$$p = \phi_2(n,p) - h(n+p) = \phi_2(n',p') - h(n'+p') = p'$$

and finally $n = (n+p) - p = (n'+p') - p' = n'$. We have proved that ϕ_2 is an injective map. We conclude that $\phi_2 : \mathbf{N}^2 \to \mathbf{N}$ is a bijection

5. Let $p \geq 1$. The existence of a bijection $\phi_p : \mathbf{N}^p \to \mathbf{N}$ is true for $p = 1$ and $p = 2$. Suppose the existence of ϕ_p has been proved, and let $\phi_2 : \mathbf{N}^2 \to \mathbf{N}$ be as in 4. Let $\phi_{p+1} : \mathbf{N}^{p+1} \to \mathbf{N}$ be defined by:

$$\phi_{p+1}(n_1, \ldots, n_{p+1}) = \phi_2[\phi_p(n_1, \ldots, n_p), n_{p+1}]$$

for all $(n_1, \ldots, n_{p+1}) \in \mathbf{N}^{p+1}$. Let $N \in \mathbf{N}$. ϕ_2 being a surjection, there exists $(n, n_{p+1}) \in \mathbf{N}^2$ with $\phi_2(n, n_{p+1}) = N$. From our induction hypothesis, $\phi_p : \mathbf{N}^p \to \mathbf{N}$ is also a surjective map. There exists $(n_1, \ldots, n_p) \in \mathbf{N}^p$, such that $\phi_p(n_1,\ldots,n_p) = n$. It follows that $(n_1,\ldots,n_{p+1})$ is an element of $\mathbf{N}^{p+1}$ such that $\phi_{p+1}(n_1,\ldots,n_{p+1}) = N$. So ϕ_{p+1} is itself a surjective map. Suppose $(n_1,\ldots,n_{p+1})$ and $(n'_1,\ldots,n'_{p+1})$ are elements of $\mathbf{N}^{p+1}$ such that:

$$\phi_{p+1}(n_1,\ldots,n_{p+1}) = \phi_{p+1}(n'_1,\ldots,n'_{p+1})$$

Then, ϕ_2 being injective, $n_{p+1} = n'_{p+1}$, and:

$$\phi_p(n_1,\ldots,n_p) = \phi_p(n'_1,\ldots,n'_p)$$

ϕ_p being itself injective, $(n_1,\ldots,n_p) = (n'_1,\ldots,n'_p)$, and we conclude that $(n_1,\ldots,n_{p+1}) = (n'_1,\ldots,n'_{p+1})$. So ϕ_{p+1} is an injective map, and finally a bijection. This induction argument proves the existence of a bijection $\phi_p : \mathbf{N}^p \to \mathbf{N}$, for all $p \geq 1$.

6. Let $p \geq 1$. From 3., there exists an injective map $j_p : \mathcal{H}^p \to \mathbf{N}^p$. From 5., there exists a bijection $\phi_p : \mathbf{N}^p \to \mathbf{N}$. It follows that $\phi_p \circ j_p : \mathcal{H}^p \to \mathbf{N}$ is an injective map. This proves that $\mathcal{H}^p$ is finite or countable, i.e. $\mathcal{H}^p$ is at most countable.

7. From 6., for all $p \geq 1$, there exists an injection $\psi_p : \mathcal{H}^p \to \mathbf{N}$. Let $j : \mathcal{H} \to \mathbf{N}^2$ be defined by $j(A) = (p, \psi_p(A))$, where $p \geq 1$ is chosen such that $A \in \mathcal{H}^p$, (there is at least one such p for any $A \in \mathcal{H}$). Suppose $j(A) = j(B)$ for some $A, B \in \mathcal{H}$. Then, there exists $p \geq 1$ such that $A, B \in \mathcal{H}^p$ and $\psi_p(A) = \psi_p(B)$, and consequently $A = B$. So j is an injection. We have proved the existence of an injective map $j : \mathcal{H} \to \mathbf{N}^2$.

8. Let $\phi_2 : \mathbf{N}^2 \to \mathbf{N}$ be a bijection. From 7., there exists an injection $j : \mathcal{H} \to \mathbf{N}^2$. It follows that $\phi_2 \circ j : \mathcal{H} \to \mathbf{N}$ is an injection. This proves that $\mathcal{H}$ is finite or countable, i.e. it is at most countable. From 2., $\mathcal{H} \subseteq \mathcal{T}$. Hence, all elements of $\mathcal{H}$ are open sets in Ω, (with respect to the product topology). We conclude that $\mathcal{H}$ is a finite or countable set of open sets in Ω.

9. From exercise (12), $U \in \mathcal{T} = \odot_{n=1}^{+\infty}\mathcal{T}_n$, if and only if for all $x \in U$, there exists $V \in \amalg_{n=1}^{+\infty}\mathcal{T}_n$ such that $x \in V \subseteq U$. Since all elements of $\amalg_{n=1}^{+\infty}\mathcal{T}_n$ can be written as $U_1 \times \ldots \times U_p \times \Pi_{n=p+1}^{+\infty}\Omega_n$ for some $p \geq 1$ and $U_1,\ldots,U_p$ open in $\Omega_1,\ldots,\Omega_p$ respectively, it follows in particular that if $U \in \mathcal{T}$ and $x \in U$, there exist $p \geq 1$ and $U_1,\ldots,U_p$ open in $\Omega_1,\ldots,\Omega_p$ such that:

$$x \in U_1 \times \ldots \times U_p \times \prod_{n=p+1}^{+\infty} \Omega_n \subseteq U$$

10. Let $U \in \mathcal{T}$ and $x \in U$. Let $p \geq 1$ and $U_1,\ldots,U_p$ open $\Omega_1,\ldots,\Omega_p$ respectively, such that $x \in U_1 \times \ldots \times U_p \times \Pi_{n=p+1}^{+\infty}\Omega_n \subseteq U$. By assumption, for all $n \geq 1$, the set $\{V_n^k : k \in I_n\}$ is a countable base of the topology $\mathcal{T}_n$. Hence, for all $n \in \mathbf{N}_p$, there exists a subset I'_n of I_n, such that

$U_n = \cup_{k\in I'_n} V_n^k$. In particular, since $x_n \in U_n$, there exists $k_n \in I'_n \subseteq I_n$ such that $x_n \in V_n^{k_n} \subseteq U_n$. We have found $k_1,\ldots,k_p$ such that:

$$x \in V_1^{k_1} \times \ldots \times V_p^{k_p} \times \prod_{n=p+1}^{+\infty} \Omega_n \stackrel{\triangle}{=} V_x \subseteq U$$

There exists $V_x \in \mathcal{H}^p \subseteq \mathcal{H}$ such that $x \in V_x \subseteq U$.

11. From 8., $\mathcal{H}$ is a finite or countable subset of the topology $\mathcal{T}$. From 10., for all $U \in \mathcal{T}$, U can be written as $U = \cup_{x\in U} V_x$, where $V_x \in \mathcal{H}$ for all $x \in U$. In other words, any open set U of $\mathcal{T}$ can be written as a union of elements of $\mathcal{H}$. It follows from definition (57) that $\mathcal{H}$ is a countable base of $(\Omega, \mathcal{T})$.

12. From theorem (26), since $\mathcal{B}(\Omega_n) = \sigma(\mathcal{T}_n)$ for all $n \geq 1$:

$$\otimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n) = \sigma(\amalg_{n=1}^{+\infty} \mathcal{T}_n) \subseteq \sigma(\mathcal{T}) = \mathcal{B}(\Omega)$$

13. Let $p \geq 1$ and $A \in \mathcal{H}^p$. Then A is a rectangle of the family $(\mathcal{T}_n)_{n\geq 1}$. Hence $A \in \amalg_{n=1}^{+\infty} \mathcal{T}_n \subseteq \amalg_{n=1}^{+\infty} \mathcal{B}(\Omega_n) \subseteq \otimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n)$. So $\mathcal{H}^p \subseteq \otimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n)$. We conclude that:

$$\mathcal{H} = \bigcup_{p\geq 1} \mathcal{H}^p \subseteq \otimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n)$$

14. Since $\mathcal{H}$ is a countable base of $(\Omega, \mathcal{T})$, any open set U of $\mathcal{T}$ can be expressed as a union of elements of $\mathcal{H}$. Furthermore, $\mathcal{H}$ being at most countable, such union is at most countable. It follows that any open set U in $\mathcal{T}$ is an element of $\sigma(\mathcal{H})$, i.e. $\mathcal{T} \subseteq \sigma(\mathcal{H})$. From 13., we have $\mathcal{H} \subseteq \otimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n)$ and consequently, we have $\sigma(\mathcal{H}) \subseteq \otimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n)$. Hence, we see that $\mathcal{T} \subseteq \otimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n)$, and finally $\mathcal{B}(\Omega) = \sigma(\mathcal{T}) \subseteq \otimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n)$. Using 12., we conclude that:

$$\mathcal{B}(\Omega) = \bigotimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n)$$

The purpose of this exercise is to prove theorem (27).

Exercise 17

Exercise 18.

1. Since $(\Omega, \mathcal{T})$ has a countable base, a *finite version* of theorem (27) would allow us to conclude immediately that:

$$\mathcal{B}(\Omega^n) = \mathcal{B}(\Omega) \otimes \ldots \otimes \mathcal{B}(\Omega)$$

Since $\mathcal{B}(\Omega) = \sigma(\mathcal{T})$, from theorem (26), we have:

$$\mathcal{B}(\Omega) \otimes \ldots \otimes \mathcal{B}(\Omega) = \sigma(\mathcal{T} \amalg \ldots \amalg \mathcal{T}) \subseteq \sigma(\mathcal{T}_{\Omega^n}) = \mathcal{B}(\Omega^n)$$

Let U be open in Ω^n, and $x \in U$. From exercise (12), there exist $V_1,\ldots,V_n$ open in Ω, such that:

$$x \in V_1 \times \ldots \times V_n \subseteq U$$

Since Ω has a countable base, say $\mathcal{H}$, each V_i can be written as a union of elements of $\mathcal{H}$. In particular, there exist $W_1, \ldots, W_n$ in $\mathcal{H}$, such that:

$$x \in W_1 \times \ldots \times W_n \subseteq U$$

Defining $A_x = W_1 \times \ldots \times W_n$, we have $U = \cup_{x \in U} A_x$. Since $\mathcal{H}$ is a subset of $\mathcal{T}$, each A_x is an element of $\mathcal{T} \amalg \ldots \amalg \mathcal{T} \subseteq \mathcal{T}_{\Omega^n}$. Although the set U may not be countable, the set I defined by $I = \{A_x : x \in U\}$ is at most countable, $\mathcal{H}$ being at most countable. So $U = \cup_{x \in U} A_x$ is in fact a countable (or finite) union of elements of $\mathcal{T} \amalg \ldots \amalg \mathcal{T}$. So $U \in \sigma(\mathcal{T} \amalg \ldots \amalg \mathcal{T})$. We have proved that:

$$\mathcal{T}_{\Omega^n} \subseteq \sigma(\mathcal{T} \amalg \ldots \amalg \mathcal{T}) \subseteq \mathcal{B}(\Omega) \otimes \ldots \otimes \mathcal{B}(\Omega)$$

We conclude that:

$$B(\Omega^n) = \sigma(\mathcal{T}_{\Omega^n}) \subseteq \mathcal{B}(\Omega) \otimes \ldots \otimes \mathcal{B}(\Omega)$$

We have proved that $\mathcal{B}(\Omega^n) = \mathcal{B}(\Omega) \otimes \ldots \otimes \mathcal{B}(\Omega)$.

2. This is an immediate consequence of 1. and exercise (16).

3. From 1., $\mathcal{B}(\mathbf{R}^2) = \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})$. $\mathbf{C}$ and $\mathbf{R}^2$ being identified, the usual topology on $\mathbf{C}$ is induced by the metric:

$$d(z, z') = \sqrt{(x - x')^2 + (y - y')^2}$$

with obvious notations. From exercise (14), such metric induces the product topology on $\mathbf{R}^2$. It follows that the usual topology on $\mathbf{C}$ and the product topology on $\mathbf{R}^2$ coincide. So $\mathcal{T}_{\mathbf{C}} = \mathcal{T}_{\mathbf{R}^2}$, and finally $\mathcal{B}(\mathbf{C}) = \mathcal{B}(\mathbf{R}^2) = \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})$.

Exercise 18

Exercise 19.

1. $\mathcal{H} = \{B(x_n, 1/p) : n, p \geq 1\}$ is a finite or countable subset of $\mathcal{T}_E^d$. Let $U \in \mathcal{T}_E^d$ and $x \in U$. There exists $\epsilon > 0$, such that $B(x, \epsilon) \subseteq U$. By assumption, the set $\{x_n : n \geq 1\}$ is dense in E. $p \geq 1$ being such that $1/p \leq \epsilon/2$, there exists $n \geq 1$ such that $x_n \in B(x, 1/p)$. In particular, $x \in B(x_n, 1/p)$. Moreover, for all $y \in B(x_n, 1/p)$, we have:

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < \frac{2}{p} \leq \epsilon$$

So $y \in B(x, \epsilon) \subseteq U$. Hence, we see that $x \in B(x_n, 1/p) \subseteq U$. For all $x \in U$, we have found $V_x \in \mathcal{H}$ such that $x \in V_x \subseteq U$. It follows that $U = \cup_{x \in U} V_x$. So U is a union of elements of $\mathcal{H}$. We have proved that $\mathcal{H}$ is a countable base of $(E, \mathcal{T}_E^d)$.

2. Let $A = \{x_V : V \in \mathcal{H}, V \neq \emptyset\}$. $\mathcal{H}$ being a countable base of $(E, \mathcal{T}_E^d)$, it is at most countable. There exists an injective map $j : \mathcal{H} \to \mathbf{N}$. Let $i : A \to \mathcal{H}$ be defined by $i(x_V) = V$. Then i is clearly an injection, and $j \circ i : A \to \mathbf{N}$

is therefore an injective map. So A is a finite or countable subset of E. Let $x \in E$. Let $U \in \mathcal{T}_E^d$ such that $x \in U$. Since U can be written as a union of elements of $\mathcal{H}$, there exists $V \in \mathcal{H}$, such that $x \in V \subseteq U$. In particular, $V \neq \emptyset$ and x_V is well-defined, with $x_V \in V \subseteq U$. So $x_V \in A \cap U \neq \emptyset$. We have proved that for all $U \in \mathcal{T}_E^d$ such that $x \in U$, $U \cap A \neq \emptyset$. From definition (37)[7], x is an element of $\bar{A}$, the closure of A. We have proved that $E \subseteq \bar{A}$. So $E = \bar{A}$, and A is dense in E. Finally, A is at most countable and dense in E. So (E, d) is a separable metric space. The purpose of 1. and 2. is to show that for metric spaces, being separable, or having a countable base, are equivalent.

3. Let $x, y, x', y' \in E$. We have:

$$d(x, y) \leq d(x, x') + d(x', y') + d(y', y)$$

and therefore:

$$d(x, y) - d(x', y') \leq d(x, x') + d(y, y')$$

Similarly:

$$d(x', y') - d(x, y) \leq d(x, x') + d(y, y')$$

It follows that:

$$|d(x, y) - d(x', y')| \leq d(x, x') + d(y, y')$$

4. Let $\delta : (E \times E)^2 \to \mathbf{R}^+$ be the metric on $E \times E$ defined by:

$$\delta[(x, y), (x', y')] = d(x, x') + d(y, y')$$

From 3., we have:

$$|d(x, y) - d(x', y')| \leq \delta[(x, y), (x', y')] \tag{6}$$

From exercise (14), the product topology $\mathcal{T}_{E \times E}$ on $E \times E$ is induced by the metric δ. Using exercise (4) of Tutorial 4, we conclude from equation (6) that $d : (E \times E, \mathcal{T}_{E \times E}) \to (\mathbf{R}^+, \mathcal{T}_{\mathbf{R}^+})$ is a continuous map.

5. From exercise (13) of Tutorial 4, and the continuity of the map $d : E \times E \to \mathbf{R}^+$ proved in 4., we conclude that:

$$d : (E \times E, \mathcal{B}(E \times E)) \to (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$$

is a measurable map. It follows that:

$$d : (E \times E, \mathcal{B}(E \times E)) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$$

is a also a measurable map.

6. If (E, d) is a separable metric space, from 1. , it has a countable base. From exercise (18), $\mathcal{B}(E \times E) = \mathcal{B}(E) \otimes \mathcal{B}(E)$. We conclude from 5. that $d : (E \times E, \mathcal{B}(E) \otimes \mathcal{B}(E)) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is a measurable map.

[7] Beware of external links!

7. By definition (54), the product σ-algebra $\mathcal{B}(E)\otimes\mathcal{B}(E)$ is generated by the set of measurable rectangles $\mathcal{B}(E)\amalg\mathcal{B}(E)$. From theorem (14), in order to prove the measurability of:
$$\Phi : (\Omega, \mathcal{F}) \to (E \times E, \mathcal{B}(E) \otimes \mathcal{B}(E))$$
it is sufficient to prove that $\Phi^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}(E)\amalg\mathcal{B}(E)$. However, any measurable rectangle B of $\mathcal{B}(E)\amalg\mathcal{B}(E)$ is of the form $B = A_1 \times A_2$, where $A_1, A_2 \in \mathcal{B}(E)$. It follows that:
$$\Phi^{-1}(B) = f^{-1}(A_1) \cap g^{-1}(A_2) \in \mathcal{F}$$
since by assumption, both $f, g : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ are measurable maps. We have proved that $\Phi : \Omega \to E \times E$ is measurable with respect to $\mathcal{F}$ and $\mathcal{B}(E) \otimes \mathcal{B}(E)$.

8. Suppose (E, d) is a separable metric space. From 6., the map:
$$d : (E \times E, \mathcal{B}(E) \otimes \mathcal{B}(E)) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$$
is measurable. However, from 7., the map:
$$\Phi : (\Omega, \mathcal{F}) \to (E \times E, \mathcal{B}(E) \otimes \mathcal{B}(E))$$
is also measurable. It follows that $\Psi = d(f, g) = d \circ \Phi$ is measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\bar{\mathbf{R}})$.

9. From 8., when (E, d) is separable, the map $\Psi = d(f, g)$ is measurable. Hence:
$$\{f = g\} = \Psi^{-1}(\{0\}) \in \mathcal{F}$$

10. Let $(E_n, d_n)_{n\geq 1}$ be a sequence of separable metric spaces. From exercise (15), the product topological space $\Pi_{n=1}^{+\infty} E_n$ is metrizable. From 1., each E_n has a countable base. From theorem (27), $\Pi_{n=1}^{+\infty} E_n$ also has a countable base. Being metrizable, it follows from 2., that it is in fact separable. We have proved that $\Pi_{n=1}^{+\infty} E_n$ is metrizable and separable.

Exercise 19

Exercise 20. Suppose each $f_i : (\Omega, \mathcal{F}) \to (\Omega_i, \mathcal{F}_i)$ is measurable. From theorem (14), in order to prove the measurability of:
$$f : (\Omega, \mathcal{F}) \to (\Pi_{i\in I}\Omega_i, \otimes_{i\in I}\mathcal{F}_i)$$
It is sufficient to show that $f^{-1}(B) \in \mathcal{F}$, for all $B \in \amalg_{i\in I}\mathcal{F}_i$. Let $B = \Pi_{i\in I} A_i$ be a measurable rectangle of the family $(\mathcal{F}_i)_{i\in I}$. For all $i \in I$, $A_i \in \mathcal{F}_i$, and $J = \{i \in I : A_i \neq \Omega_i\}$ is a finite set. Hence:
$$f^{-1}(B) = \bigcap_{i\in I}\{f_i \in A_i\} = \bigcap_{i\in J}\{f_i \in A_i\} \in \mathcal{F}$$
since each f_i is measurable. So f is indeed measurable. Conversely, suppose $f = (f_i)_{i\in I}$ is measurable. Let $j \in I$ and $A_j \in \mathcal{F}_j$. We have:
$$f_j^{-1}(A_j) = f^{-1}(A_j \times \Pi_{i\neq j}\Omega_i) \in \mathcal{F}$$

since $B = A_j \times \Pi_{i \neq j} \Omega_i$ is a measurable rectangle, and lies in $\otimes_{i \in I} \mathcal{F}_i$. So $f_j : (\Omega, \mathcal{F}) \to (\Omega_j, \mathcal{F}_j)$ is a measurable map.

Exercise 20

Exercise 21.

1. Let (x, y) and (x', y') be elements of $\mathbf{R}^2$. We have:
$$|\phi(x, y) - \phi(x', y')| \leq |x - x'| + |y - y'| \tag{7}$$
By definition (17), the usual topology on $\mathbf{R}$ is the metric topology induced by $d(x, y) = |x - y|$. From exercise (14), the product topology on $\mathbf{R}^2$ is induced by:
$$\delta[(x, y), (x', y')] = |x - x'| + |y - y'|$$
It follows from equation (7), and exercise (4) of Tutorial 4 that:
$$\phi : (\mathbf{R}^2, \mathcal{T}_{\mathbf{R}^2}) \to (\mathbf{R}, \mathcal{T}_{\mathbf{R}})$$
is a continuous map.
Let $(x_0, y_0) \in \mathbf{R}^2$ and $\epsilon > 0$. For all $(x, y) \in \mathbf{R}^2$, we have:
$$|\psi(x, y) - \psi(x_0, y_0)| \leq |y|.|x - x_0| + |x_0|.|y - y_0|$$
Suppose $\eta > 0$ is such that:
$$|x - x_0| + |y - y_0| < \eta \leq 1$$
Then in particular, $|y| \leq 1 + |y_0|$, and consequently:
$$|\psi(x, y) - \psi(x_0, y_0)| \leq M.(|x - x_0| + |y - y_0|)$$
where $M = \max(|x_0|, 1 + |y_0|)$. Hence, we see that:
$$\delta[(x, y), (x_0, y_0)] < \eta \;\Rightarrow\; |\psi(x, y) - \psi(x_0, y_0)| < \epsilon$$
where η has been chosen as $\eta = \min(\epsilon/M, 1)$. We conclude from exercise (4) of Tutorial 4 that $\psi : (\mathbf{R}^2, \mathcal{T}_{\mathbf{R}^2}) \to (\mathbf{R}, \mathcal{T}_{\mathbf{R}})$ is a continuous map.

2. ϕ and ψ being continuous, from exercise (13) of Tutorial 4:
$$\phi, \psi : (\mathbf{R}^2, \mathcal{B}(\mathbf{R}^2)) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$$
are measurable maps. Since $(\mathbf{R}, \mathcal{T}_{\mathbf{R}})$ has a countable base, from exercise (18), we have $\mathcal{B}(\mathbf{R}^2) = \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})$. We conclude that:
$$\phi, \psi : (\mathbf{R}^2, \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$$
are measurable maps.

3. Given $f, g : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ measurable, the fact that $f + g$ and $f.g$ are measurable was already proved in Tutorial 4. The purpose of this exercise is to emphasize a more direct proof. From theorem (28), the map:
$$h = (f, g) : (\Omega, \mathcal{F}) \to (\mathbf{R} \times \mathbf{R}, \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R}))$$

is measurable, since both f and g are measurable. From 2:

$$\phi, \psi : (\mathbf{R} \times \mathbf{R}, \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$$

are also measurable. It follows that $f + g = \phi \circ h$ and $f.g = \psi \circ h$ are measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\bar{\mathbf{R}})$. Being real-valued, they are also measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\mathbf{R})$.

Exercise 21