13. Regular Measure

In the following, $\mathbf{K}$ denotes $\mathbf{R}$ or $\mathbf{C}$.

Definition 99 *Let $(\Omega, \mathcal{F})$ be a measurable space. We say that a map $s : \Omega \to \mathbf{C}$ is a* **complex simple function** *on $(\Omega, \mathcal{F})$, if and only if it is of the form:*

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$

where $n \geq 1$, $\alpha_i \in \mathbf{C}$ and $A_i \in \mathcal{F}$ for all $i \in \mathbf{N}_n$. The set of all complex simple functions on $(\Omega, \mathcal{F})$ is denoted $S_{\mathbf{C}}(\Omega, \mathcal{F})$. The set of all $\mathbf{R}$-valued complex simple functions in $(\Omega, \mathcal{F})$ is denoted $S_{\mathbf{R}}(\Omega, \mathcal{F})$.

Recall that a simple function on $(\Omega, \mathcal{F})$, as defined in (40), is just a non-negative element of $S_{\mathbf{R}}(\Omega, \mathcal{F})$.

Exercise 1. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p \in [1, +\infty[$.

1. Suppose $s : \Omega \to \mathbf{C}$ is of the form

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$

   where $n \geq 1$, $\alpha_i \in \mathbf{C}$, $A_i \in \mathcal{F}$ and $\mu(A_i) < +\infty$ for all $i \in \mathbf{N}_n$. Show that $s \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{C}}(\Omega, \mathcal{F})$.

2. Show that any $s \in S_{\mathbf{C}}(\Omega, \mathcal{F})$ can be written as:

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$

   where $n \geq 1$, $\alpha_i \in \mathbf{C} \setminus \{0\}$, $A_i \in \mathcal{F}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$.

3. Show that any $s \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{C}}(\Omega, \mathcal{F})$ is of the form:

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$

   where $n \geq 1$, $\alpha_i \in \mathbf{C}$, $A_i \in \mathcal{F}$ and $\mu(A_i) < +\infty$, for all $i \in \mathbf{N}_n$.

4. Show that $L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{C}}(\Omega, \mathcal{F}) = S_{\mathbf{C}}(\Omega, \mathcal{F})$.

Exercise 2. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p \in [1, +\infty[$. Let f be a non-negative element of $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

1. Show the existence of a sequence $(s_n)_{n \geq 1}$ of non-negative functions in $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{R}}(\Omega, \mathcal{F})$ such that $s_n \uparrow f$.

2. Show that:

$$\lim_{n \to +\infty} \int |s_n - f|^p d\mu = 0$$

3. Show that there exists $s \in L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{R}}(\Omega, \mathcal{F})$ such that $\|f - s\|_p \leq \epsilon$, for all $\epsilon > 0$.

4. Show that $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{K}}(\Omega, \mathcal{F})$ is dense in $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$.

Exercise 3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let f be a non-negative element of $L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. For all $n \geq 1$, we define:

$$s_n \stackrel{\triangle}{=} \sum_{k=0}^{n2^n-1} \frac{k}{2^n} 1_{\{k/2^n \leq f < (k+1)/2^n\}} + n 1_{\{n \leq f\}}$$

1. Show that for all $n \geq 1$, s_n is a simple function.

2. Show there exists $n_0 \geq 1$ and $N \in \mathcal{F}$ with $\mu(N) = 0$, such that:

$$\forall \omega \in N^c \ , \ 0 \leq f(\omega) < n_0$$

3. Show that for all $n \geq n_0$ and $\omega \in N^c$, we have:

$$0 \leq f(\omega) - s_n(\omega) < \frac{1}{2^n}$$

4. Conclude that:

$$\lim_{n \to +\infty} \|f - s_n\|_\infty = 0$$

5. Show the following:

Theorem 67 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p \in [1, +\infty]$. Then, $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{K}}(\Omega, \mathcal{F})$ is dense in $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$.*

Exercise 4. Let $(\Omega, \mathcal{T})$ be a metrizable topological space, and μ be a finite measure on $(\Omega, \mathcal{B}(\Omega))$. We define Σ as the set of all $B \in \mathcal{B}(\Omega)$ such that for all $\epsilon > 0$, there exist F closed and G open in Ω, with:

$$F \subseteq B \subseteq G \ , \ \mu(G \setminus F) \leq \epsilon$$

Given a metric d on $(\Omega, \mathcal{T})$ inducing the topology $\mathcal{T}$, we define:

$$d(x, A) \stackrel{\triangle}{=} \inf\{d(x, y) : \ y \in A\}$$

for all $A \subseteq \Omega$ and $x \in \Omega$.

1. Show that $x \to d(x, A)$ from Ω to $\bar{\mathbf{R}}$ is continuous for all $A \subseteq \Omega$.

2. Show that if F is closed in Ω, $x \in F$ is equivalent to $d(x, F) = 0$.

Exercise 5. Further to exercise (4), we assume that F is a closed subset of Ω. For all $n \geq 1$, we define:

$$G_n \stackrel{\triangle}{=} \{x \in \Omega : \ d(x, F) < \frac{1}{n}\}$$

1. Show that G_n is open for all $n \geq 1$.
2. Show that $G_n \downarrow F$.
3. Show that $F \in \Sigma$.
4. Was it important to assume that μ is finite?
5. Show that $\Omega \in \Sigma$.
6. Show that if $B \in \Sigma$, then $B^c \in \Sigma$.

EXERCISE 6. Further to exercise (5), let $(B_n)_{n\geq 1}$ be a sequence in Σ. Define $B = \cup_{n=1}^{+\infty} B_n$ and let $\epsilon > 0$.

1. Show that for all n, there is F_n closed and G_n open in Ω, with:
$$F_n \subseteq B_n \subseteq G_n \ , \ \mu(G_n \setminus F_n) \leq \frac{\epsilon}{2^n}$$
2. Show the existence of some $N \geq 1$ such that:
$$\mu\left(\left(\bigcup_{n=1}^{+\infty} F_n\right) \setminus \left(\bigcup_{n=1}^{N} F_n\right)\right) \leq \epsilon$$
3. Define $G = \cup_{n=1}^{+\infty} G_n$ and $F = \cup_{n=1}^{N} F_n$. Show that F is closed, G is open and $F \subseteq B \subseteq G$.
4. Show that:
$$G \setminus F \subseteq G \setminus \left(\bigcup_{n=1}^{+\infty} F_n\right) \uplus \left(\bigcup_{n=1}^{+\infty} F_n\right) \setminus F$$
5. Show that:
$$G \setminus \left(\bigcup_{n=1}^{+\infty} F_n\right) \subseteq \bigcup_{n=1}^{+\infty} G_n \setminus F_n$$
6. Show that $\mu(G \setminus F) \leq 2\epsilon$.
7. Show that Σ is a σ-algebra on Ω, and conclude that $\Sigma = \mathcal{B}(\Omega)$.

Theorem 68 *Let $(\Omega, \mathcal{T})$ be a metrizable topological space, and μ be a finite measure on $(\Omega, \mathcal{B}(\Omega))$. Then, for all $B \in \mathcal{B}(\Omega)$ and $\epsilon > 0$, there exist F closed and G open in Ω such that:*
$$F \subseteq B \subseteq G \ , \ \mu(G \setminus F) \leq \epsilon$$

Definition 100 *Let $(\Omega, \mathcal{T})$ be a topological space. We denote $C^b_{\mathbf{K}}(\Omega)$ the* **K**-*vector space of all* **continuous, bounded** *maps $\phi : \Omega \to \mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$.*

Exercise 7. Let $(\Omega, \mathcal{T})$ be a metrizable topological space with some metric d. Let μ be a finite measure on $(\Omega, \mathcal{B}(\Omega))$ and F be a closed subset of Ω. For all $n \geq 1$, we define $\phi_n : \Omega \to \mathbf{R}$ by:

$$\forall x \in \Omega \ , \ \phi_n(x) \stackrel{\triangle}{=} 1 - 1 \wedge (nd(x, F))$$

1. Show that for all $p \in [1, +\infty]$, we have $C^b_{\mathbf{K}}(\Omega) \subseteq L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$.
2. Show that for all $n \geq 1$, $\phi_n \in C^b_{\mathbf{R}}(\Omega)$.
3. Show that $\phi_n \to 1_F$.
4. Show that for all $p \in [1, +\infty[$, we have:
$$\lim_{n\to+\infty} \int |\phi_n - 1_F|^p d\mu = 0$$
5. Show that for all $p \in [1, +\infty[$ and $\epsilon > 0$, there exists $\phi \in C^b_{\mathbf{R}}(\Omega)$ such that $\|\phi - 1_F\|_p \leq \epsilon$.
6. Let $\nu \in M^1(\Omega, \mathcal{B}(\Omega))$. Show that $C^b_{\mathbf{C}}(\Omega) \subseteq L^1_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega), \nu)$ and:
$$\nu(F) = \lim_{n\to+\infty} \int \phi_n d\nu$$
7. Prove the following:

Theorem 69 *Let $(\Omega, \mathcal{T})$ be a metrizable topological space and μ, ν be two complex measures on $(\Omega, \mathcal{B}(\Omega))$ such that:*

$$\forall \phi \in C^b_{\mathbf{R}}(\Omega) \ , \ \int \phi d\mu = \int \phi d\nu$$

Then $\mu = \nu$.

Exercise 8. Let $(\Omega, \mathcal{T})$ be a metrizable topological space and μ be a finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let $s \in S_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega))$ be a complex simple function:

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$

where $n \geq 1$, $\alpha_i \in \mathbf{C}$, $A_i \in \mathcal{B}(\Omega)$ for all $i \in \mathbf{N}_n$. Let $p \in [1, +\infty[$.

1. Show that given $\epsilon > 0$, for all $i \in \mathbf{N}_n$ there is a closed subset F_i of Ω such that $F_i \subseteq A_i$ and $\mu(A_i \setminus F_i) \leq \epsilon$. Let:
$$s' \stackrel{\triangle}{=} \sum_{i=1}^{n} \alpha_i 1_{F_i}$$
2. Show that:
$$\|s - s'\|_p \leq \left(\sum_{i=1}^{n} |\alpha_i| \right) \epsilon^{\frac{1}{p}}$$

3. Conclude that given $\epsilon > 0$, there exists $\phi \in C^b_{\mathbf{C}}(\Omega)$ such that:

$$\|\phi - s\|_p \leq \epsilon$$

4. Prove the following:

Theorem 70 *Let $(\Omega, \mathcal{T})$ be a metrizable topological space and μ be a finite measure on $(\Omega, \mathcal{B}(\Omega))$. Then, for all $p \in [1, +\infty[$, $C^b_{\mathbf{K}}(\Omega)$ is dense in $L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$.*

Definition 101 *A topological space $(\Omega, \mathcal{T})$ is said to be* **σ-compact** *if and only if, there exists a sequence $(K_n)_{n\geq 1}$ of compact subsets of Ω such that $K_n \uparrow \Omega$.*

Exercise 9. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space, with metric d. Let Ω' be open in Ω. For all $n \geq 1$, we define:

$$F_n \stackrel{\triangle}{=} \{x \in \Omega : \ d(x, (\Omega')^c) \geq 1/n\}$$

Let $(K_n)_{n\geq 1}$ be a sequence of compact subsets of Ω such that $K_n \uparrow \Omega$.

1. Show that for all $n \geq 1$, F_n is closed in Ω.
2. Show that $F_n \uparrow \Omega'$.
3. Show that $F_n \cap K_n \uparrow \Omega'$.
4. Show that $F_n \cap K_n$ is closed in K_n for all $n \geq 1$.
5. Show that $F_n \cap K_n$ is a compact subset of Ω' for all $n \geq 1$.
6. Prove the following:

Theorem 71 *Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Then, for all Ω' open subsets of Ω, the induced topological space $(\Omega', \mathcal{T}_{|\Omega'})$ is itself metrizable and σ-compact.*

Definition 102 *Let $(\Omega, \mathcal{T})$ be a topological space and μ be a measure on $(\Omega, \mathcal{B}(\Omega))$. We say that μ is* **locally finite**, *if and only if, every $x \in \Omega$ has an open neighborhood of finite μ-measure, i.e.*

$$\forall x \in \Omega \ , \ \exists U \in \mathcal{T} \ , \ x \in U \ , \ \mu(U) < +\infty$$

Definition 103 *If μ is a measure on a Hausdorff topological space Ω:*
We say that μ is **inner-regular**, *if and only if, for all $B \in \mathcal{B}(\Omega)$:*

$$\mu(B) = \sup\{\mu(K) : \ K \subseteq B \ , \ K \text{ compact}\}$$

We say that μ is **outer-regular**, *if and only if, for all $B \in \mathcal{B}(\Omega)$:*

$$\mu(B) = \inf\{\mu(G) : \ B \subseteq G \ , \ G \text{ open}\}$$

We say that μ is **regular** *if it is both inner and outer-regular.*

Exercise 10. Let $(\Omega, \mathcal{T})$ be a Hausdorff topological space, μ a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$, and K a compact subset of Ω.

1. Show the existence of open sets $V_1, \ldots, V_n$ with $\mu(V_i) < +\infty$ for all $i \in \mathbf{N}_n$ and $K \subseteq V_1 \cup \ldots \cup V_n$, where $n \geq 1$.
2. Conclude that $\mu(K) < +\infty$.

Exercise 11. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Let μ be a finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let $(K_n)_{n\geq 1}$ be a sequence of compact subsets of Ω such that $K_n \uparrow \Omega$. Let $B \in \mathcal{B}(\Omega)$. We define $\alpha = \sup\{\mu(K) : \ K \subseteq B \ , \ K \text{ compact}\}$.

1. Show that given $\epsilon > 0$, there exists F closed in Ω such that $F \subseteq B$ and $\mu(B \setminus F) \leq \epsilon$.
2. Show that $F \setminus (K_n \cap F) \downarrow \emptyset$.
3. Show that $K_n \cap F$ is closed in K_n.
4. Show that $K_n \cap F$ is compact.
5. Conclude that given $\epsilon > 0$, there exists K compact subset of Ω such that $K \subseteq F$ and $\mu(F \setminus K) \leq \epsilon$.
6. Show that $\mu(B) \leq \mu(K) + 2\epsilon$.
7. Show that $\mu(B) \leq \alpha$ and conclude that μ is inner-regular.

Exercise 12. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let $(K_n)_{n\geq 1}$ be a sequence of compact subsets of Ω such that $K_n \uparrow \Omega$. Let $B \in \mathcal{B}(\Omega)$, and $\alpha \in \mathbf{R}$ be such that $\alpha < \mu(B)$.

1. Show that $\mu(K_n \cap B) \uparrow \mu(B)$.
2. Show the existence of $K \subseteq \Omega$ compact, with $\alpha < \mu(K \cap B)$.
3. Let $\mu^K = \mu(K \cap \cdot)$. Show that μ^K is a finite measure, and conclude that $\mu^K(B) = \sup\{\mu^K(K^*) : \ K^* \subseteq B \ , \ K^* \text{ compact}\}$.
4. Show the existence of a compact subset K^* of Ω, such that $K^* \subseteq B$ and $\alpha < \mu(K \cap K^*)$.
5. Show that K^* is closed in Ω.
6. Show that $K \cap K^*$ is closed in K.
7. Show that $K \cap K^*$ is compact.
8. Show that $\alpha < \sup\{\mu(K') : \ K' \subseteq B \ , \ K' \text{ compact}\}$.

9. Show that $\mu(B) \leq \sup\{\mu(K') : K' \subseteq B \ , \ K' \text{ compact}\}$.
10. Conclude that μ is inner-regular.

EXERCISE 13. Let $(\Omega, \mathcal{T})$ be a metrizable topological space.

1. Show that $(\Omega, \mathcal{T})$ is separable if and only if it has a countable base.
2. Show that if $(\Omega, \mathcal{T})$ is compact, for all $n \geq 1$, Ω can be covered by a finite number of open balls with radius $1/n$.
3. Show that if $(\Omega, \mathcal{T})$ is compact, then it is separable.

EXERCISE 14. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space with metric d. Let $(K_n)_{n\geq 1}$ be a sequence of compact subsets of Ω such that $K_n \uparrow \Omega$.

1. For all $n \geq 1$, give a metric on K_n inducing the topology $\mathcal{T}_{|K_n}$.
2. Show that $(K_n, \mathcal{T}_{|K_n})$ is separable.
3. Let $(x_n^p)_{p\geq 1}$ be an at most countable sequence of $(K_n, \mathcal{T}_{|K_n})$, which is dense. Show that $(x_n^p)_{n,p\geq 1}$ is an at most countable dense family of $(\Omega, \mathcal{T})$, and conclude with the following:

Theorem 72 *Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Then, $(\Omega, \mathcal{T})$ is separable and has a countable base.*

EXERCISE 15. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let $\mathcal{H}$ be a countable base of $(\Omega, \mathcal{T})$. We define $\mathcal{H}' = \{V \in \mathcal{H} : \ \mu(V) < +\infty\}$.

1. Show that for all U open in Ω and $x \in U$, there is U_x open in Ω such that $x \in U_x \subseteq U$ and $\mu(U_x) < +\infty$.
2. Show the existence of $V_x \in \mathcal{H}$ such that $x \in V_x \subseteq U_x$.
3. Conclude that $\mathcal{H}'$ is a countable base of $(\Omega, \mathcal{T})$.
4. Show the existence of a sequence $(V_n)_{n\geq 1}$ of open sets in Ω with:

$$\Omega = \bigcup_{n=1}^{+\infty} V_n \ , \ \mu(V_n) < +\infty \ , \ \forall n \geq 1$$

EXERCISE 16. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let $(V_n)_{n\geq 1}$ a sequence of open subsets of Ω such that:

$$\Omega = \bigcup_{n=1}^{+\infty} V_n \ , \ \mu(V_n) < +\infty \ , \ \forall n \geq 1$$

Let $B \in \mathcal{B}(\Omega)$ and $\alpha = \inf\{\mu(G) : \ B \subseteq G \ , \ G \text{ open}\}$.

1. Given $\epsilon > 0$, show that there exists G_n open in Ω such that $B \subseteq G_n$ and $\mu^{V_n}(G_n \setminus B) \leq \epsilon/2^n$, where $\mu^{V_n} = \mu(V_n \cap \cdot)$.
2. Let $G = \cup_{n=1}^{+\infty}(V_n \cap G_n)$. Show that G is open in Ω, and $B \subseteq G$.
3. Show that $G \setminus B = \cup_{n=1}^{+\infty} V_n \cap (G_n \setminus B)$.
4. Show that $\mu(G) \leq \mu(B) + \epsilon$.
5. Show that $\alpha \leq \mu(B)$.
6. Conclude that is μ outer-regular.
7. Show the following:

Theorem 73 *Let μ be a locally finite measure on a metrizable and σ-compact topological space $(\Omega, \mathcal{T})$. Then, μ is regular, i.e.:*

$$\begin{aligned} \mu(B) &= \sup\{\mu(K) : K \subseteq B \ , \ K \text{ compact}\} \\ &= \inf\{\mu(G) : B \subseteq G \ , \ G \text{ open}\} \end{aligned}$$

for all $B \in \mathcal{B}(\Omega)$.

Exercise 17. Show the following:

Theorem 74 *Let Ω be an open subset of $\mathbf{R}^n$, where $n \geq 1$. Any locally finite measure on $(\Omega, \mathcal{B}(\Omega))$ is regular.*

Definition 104 *We call* **strongly σ-compact** *topological space, a topological space $(\Omega, \mathcal{T})$, for which there exists a sequence $(V_n)_{n\geq 1}$ of open sets with compact closure, such that $V_n \uparrow \Omega$.*

Definition 105 *We call* **locally compact** *topological space, a topological space $(\Omega, \mathcal{T})$, for which every $x \in \Omega$ has an open neighborhood with compact closure, i.e. such that:*

$$\forall x \in \Omega \ , \ \exists U \in \mathcal{T} : \ x \in U \ , \ \bar{U} \text{ is compact}$$

Exercise 18. Let $(\Omega, \mathcal{T})$ be a σ-compact and locally compact topological space. Let $(K_n)_{n\geq 1}$ be a sequence of compact subsets of Ω such that $K_n \uparrow \Omega$.

1. Show that for all $n \geq 1$, there are open sets $V_1^n, \ldots, V_{p_n}^n$, $p_n \geq 1$, such that $K_n \subseteq V_1^n \cup \ldots \cup V_{p_n}^n$ and $\bar{V}_1^n, \ldots, \bar{V}_{p_n}^n$ are compact subsets of Ω.
2. Define $W_n = V_1^n \cup \ldots \cup V_{p_n}^n$ and $V_n = \cup_{k=1}^n W_k$, for $n \geq 1$. Show that $(V_n)_{n\geq 1}$ is a sequence of open sets in Ω with $V_n \uparrow \Omega$.
3. Show that $\bar{W}_n = \bar{V}_1^n \cup \ldots \cup \bar{V}_{p_n}^n$ for all $n \geq 1$.

4. Show that $\bar{W}_n$ is compact for all $n \geq 1$.

5. Show that $\bar{V}_n$ is compact for all $n \geq 1$

6. Conclude with the following:

Theorem 75 *A topological space $(\Omega, \mathcal{T})$ is strongly σ-compact, if and only if it is σ-compact and locally compact.*

Exercise 19. Let $(\Omega, \mathcal{T})$ be a topological space and Ω' be a subset of Ω. Let $A \subseteq \Omega'$. We denote $\bar{A}^{\Omega'}$ the closure of A in the induced topological space $(\Omega', \mathcal{T}_{|\Omega'})$, and $\bar{A}$ its closure in Ω.

1. Show that $A \subseteq \Omega' \cap \bar{A}$.

2. Show that $\Omega' \cap \bar{A}$ is closed in Ω'.

3. Show that $\bar{A}^{\Omega'} \subseteq \Omega' \cap \bar{A}$.

4. Let $x \in \Omega' \cap \bar{A}$. Show that if $x \in U' \in \mathcal{T}_{|\Omega'}$, then $A \cap U' \neq \emptyset$.

5. Show that $\bar{A}^{\Omega'} = \Omega' \cap \bar{A}$.

Exercise 20. Let (Ω, d) be a metric space.

1. Show that for all $x \in \Omega$ and $\epsilon > 0$, we have:
$$\overline{B(x,\epsilon)} \subseteq \{y \in \Omega : \ d(x,y) \leq \epsilon\}$$

2. Take $\Omega = [0, 1/2[\cup \{1\}$. Show that $B(0,1) = [0, 1/2[$.

3. Show that $[0, 1/2[$ is closed in Ω.

4. Show that $\overline{B(0,1)} = [0, 1/2[$.

5. Conclude that $\overline{B(0,1)} \neq \{y \in \Omega : \ |y| \leq 1\} = \Omega$.

Exercise 21. Let (Ω, d) be a locally compact metric space. Let Ω' be an open subset of Ω. Let $x \in \Omega'$.

1. Show there exists U open with compact closure, such that $x \in U$.

2. Show the existence of $\epsilon > 0$ such that $B(x,\epsilon) \subseteq U \cap \Omega'$.

3. Show that $\overline{B(x,\epsilon/2)} \subseteq \bar{U}$.

4. Show that $\overline{B(x,\epsilon/2)}$ is closed in $\bar{U}$.

5. Show that $\overline{B(x,\epsilon/2)}$ is a compact subset of Ω.

6. Show that $\overline{B(x,\epsilon/2)} \subseteq \Omega'$.

7. Let $U' = B(x, \epsilon/2) \cap \Omega' = B(x, \epsilon/2)$. Show $x \in U' \in \mathcal{T}_{|\Omega'}$, and:

$$\bar{U}'^{\Omega'} = \overline{B(x, \epsilon/2)}$$

8. Show that the induced topological space Ω' is locally compact.

9. Prove the following:

Theorem 76 *Let $(\Omega, \mathcal{T})$ be a metrizable and strongly σ-compact topological space. Then, for all Ω' open subsets of Ω, the induced topological space $(\Omega', \mathcal{T}_{|\Omega'})$ is itself metrizable and strongly σ-compact.*

Definition 106 *Let $(\Omega, \mathcal{T})$ be a topological space, and $\phi : \Omega \to \mathbf{C}$ be a map. We call* **support** *of ϕ, the closure of the set $\{\phi \neq 0\}$, i.e.:*

$$supp(\phi) \stackrel{\triangle}{=} \overline{\{\omega \in \Omega : \ \phi(\omega) \neq 0\}}$$

Definition 107 *Let $(\Omega, \mathcal{T})$ be a topological space. We denote $C^c_{\mathbf{K}}(\Omega)$ the $\mathbf{K}$-vector space of all* **continuous** *map with* **compact support** *$\phi : \Omega \to \mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$.*

Exercise 22. Let $(\Omega, \mathcal{T})$ be a topological space.

1. Show that $0 \in C^c_{\mathbf{K}}(\Omega)$.

2. Show that $C^c_{\mathbf{K}}(\Omega)$ is indeed a $\mathbf{K}$-vector space.

3. Show that $C^c_{\mathbf{K}}(\Omega) \subseteq C^b_{\mathbf{K}}(\Omega)$.

Exercise 23. let (Ω, d) be a locally compact metric space. let K be a compact subset of Ω, and G be open in Ω, with $K \subseteq G$.

1. Show the existence of open sets $V_1, \ldots, V_n$ such that:

$$K \subseteq V_1 \cup \ldots \cup V_n$$

   and $\bar{V}_1, \ldots, \bar{V}_n$ are compact subsets of Ω.

2. Show that $V = (V_1 \cup \ldots \cup V_n) \cap G$ is open in Ω, and $K \subseteq V \subseteq G$.

3. Show that $\bar{V} \subseteq \bar{V}_1 \cup \ldots \cup \bar{V}_n$.

4. Show that $\bar{V}$ is compact.

5. We assume $K \neq \emptyset$ and $V \neq \Omega$, and we define $\phi : \Omega \to \mathbf{R}$ by:

$$\forall x \in \Omega \ , \ \phi(x) \stackrel{\triangle}{=} \frac{d(x, V^c)}{d(x, V^c) + d(x, K)}$$

   Show that ϕ is well-defined and continuous.

6. Show that $\{\phi \neq 0\} = V$.
7. Show that $\phi \in C^c_{\mathbf{R}}(\Omega)$.
8. Show that $1_K \leq \phi \leq 1_G$.
9. Show that if $K = \emptyset$, there is $\phi \in C^c_{\mathbf{R}}(\Omega)$ with $1_K \leq \phi \leq 1_G$.
10. Show that if $V = \Omega$ then Ω is compact.
11. Show that if $V = \Omega$, there $\phi \in C^c_{\mathbf{R}}(\Omega)$ with $1_K \leq \phi \leq 1_G$.

Theorem 77 *Let $(\Omega, \mathcal{T})$ be a metrizable and locally compact topological space. Let K be a compact subset of Ω, and G be an open subset of Ω such that $K \subseteq G$. Then, there exists $\phi \in C^c_{\mathbf{R}}(\Omega)$, real-valued continuous map with compact support, such that:*

$$1_K \leq \phi \leq 1_G$$

Exercise 24. Let $(\Omega, \mathcal{T})$ be a metrizable and strongly σ-compact topological space. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let $B \in \mathcal{B}(\Omega)$ be such that $\mu(B) < +\infty$. Let $p \in [1, +\infty[$.

1. Show that $C^c_{\mathbf{K}}(\Omega) \subseteq L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$.
2. Let $\epsilon > 0$. Show the existence of K compact and G open, with:
$$K \subseteq B \subseteq G \ , \ \mu(G \setminus K) \leq \epsilon$$
3. Where did you use the fact that $\mu(B) < +\infty$?
4. Show the existence of $\phi \in C^c_{\mathbf{R}}(\Omega)$ with $1_K \leq \phi \leq 1_G$.
5. Show that:
$$\int |\phi - 1_B|^p d\mu \leq \mu(G \setminus K)$$
6. Conclude that for all $\epsilon > 0$, there exists $\phi \in C^c_{\mathbf{R}}(\Omega)$ such that:
$$\|\phi - 1_B\|_p \leq \epsilon$$
7. Let $s \in S_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega)) \cap L^p_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega), \mu)$. Show that for all $\epsilon > 0$, there exists $\phi \in C^c_{\mathbf{C}}(\Omega)$ such that $\|\phi - s\|_p \leq \epsilon$.
8. Prove the following:

Theorem 78 *Let $(\Omega, \mathcal{T})$ be a metrizable and strongly σ-compact topological space*[1]*. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. Then, for all $p \in [1, +\infty[$, the space $C^c_{\mathbf{K}}(\Omega)$ of $\mathbf{K}$-valued, continuous maps with compact support, is dense in $L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$.*

[1] i.e. a metrizable topological space for which there exists a sequence $(V_n)_{n\geq 1}$ of open sets with compact closure, such that $V_n \uparrow \Omega$.

EXERCISE 25. Prove the following:

Theorem 79 *Let* Ω *be an open subset of* $\mathbf{R}^n$, *where* $n \geq 1$. *Then, for any locally finite measure* μ *on* $(\Omega, \mathcal{B}(\Omega))$ *and* $p \in [1, +\infty[$, $C^c_{\mathbf{K}}(\Omega)$ *is dense in* $L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$.

Solutions to Exercises

Exercise 1.

1. From definition (99), s is clearly an element of $S_{\mathbf{C}}(\Omega,\mathcal{F})$. Furthermore, for all $i\in\mathbf{N}_n$, 1_{A_i} is measurable, and:
$$\int |1_{A_i}|^p d\mu = \int 1_{A_i} d\mu = \mu(A_i) < +\infty$$
So $1_{A_i}\in L^p_{\mathbf{C}}(\Omega,\mathcal{F},\mu)$. s being a linear combination of the 1_{A_i}'s is also an element of $L^p_{\mathbf{C}}(\Omega,\mathcal{F},\mu)$. We have proved that s is an element of $L^p_{\mathbf{C}}(\Omega,\mathcal{F},\mu)\cap S_{\mathbf{C}}(\Omega,\mathcal{F})$.

2. Let $s\in S_{\mathbf{C}}(\Omega,\mathcal{F})$. From definition (99), s is of the form:
$$s=\sum_{j=1}^m \beta_j 1_{B_j} \tag{1}$$
where $m\geq 1$, $\beta_j\in\mathbf{C}$, and $B_j\in\mathcal{F}$ for all $j\in\mathbf{N}_m$. If $s=0$, it can be written as $s=1\times 1_{\emptyset}$ and there is nothing further to prove. We assume that $s\neq 0$. The map $\theta:\{0,1\}^m\to\mathbf{C}$ given by $\theta(\epsilon_1,\ldots,\epsilon_m)=\sum_{j=1}^m\beta_j\epsilon_j$ being defined on a finite set, has a finite range. Since $s(\Omega)$ is a subset of $\theta(\{0,1\}^m)$, $s(\Omega)$ is also a finite set. Having assumed that $s\neq 0$, the set $s(\Omega)\setminus\{0\}$ is therefore non-empty and finite. Let $n\geq 1$ be its cardinal, and $\alpha:\mathbf{N}_n\to s(\Omega)\setminus\{0\}$ be an arbitrary bijection. For all $\omega\in\Omega$, we have:
$$s(\omega)=\sum_{i=1}^n \alpha(i)1_{\{s=\alpha(i)\}} \tag{2}$$
Since $B_j\in\mathcal{F}$ for all j's, s is a measurable map. If we define $A_i=\{s=\alpha(i)\}$ for $i\in\mathbf{N}_n$, we have $A_i\in\mathcal{F}$. Furthermore, it is clear that $A_i\cap A_j=\emptyset$ for $i\neq j$. We conclude from (2) that s can be written as:
$$s=\sum_{i=1}^n \alpha(i)1_{A_i}$$
where $n\geq 1$, $\alpha(i)\in\mathbf{C}\setminus\{0\}$, $A_i\in\mathcal{F}$, and $A_i\cap A_j=\emptyset$ for $i\neq j$.

3. Let $s\in L^p_{\mathbf{C}}(\Omega,\mathcal{F},\mu)\cap S_{\mathbf{C}}(\Omega,\mathcal{F})$. From 2. s can be expressed as:
$$s=\sum_{i=1}^n \alpha_i 1_{A_i} \tag{3}$$
where $n\geq 1$, $\alpha_i\neq 0$, $A_i\in\mathcal{F}$ and $A_i\cap A_j=\emptyset$ for $i\neq j$. Let $A=A_1\uplus\ldots\uplus A_n$. Then $s(\omega)=0$ for all $\omega\in A^c$ and furthermore $1_A=1_{A_1}+\ldots+1_{A_n}$. Hence:
$$\int |s|^p d\mu = \sum_{i=1}^n\int |s|^p 1_{A_i}d\mu = \sum_{i=1}^n |\alpha_i|^p\mu(A_i) < +\infty$$

Since $\alpha_i \neq 0$, it follows that $\mu(A_i) < +\infty$ for all $i \in \mathbf{N}_n$. We have been able to express s as (3), where $n \geq 1$, $\alpha_i \in \mathbf{C}$ (in fact $\alpha_i \in \mathbf{C}^*$), $A_i \in \mathcal{F}$ and $\mu(A_i) < +\infty$ for all $i \in \mathbf{N}_n$. This is a converse of 1.

4. Let $s \in S_{\mathbf{C}}(\Omega, \mathcal{F})$. Then s is bounded and measurable.

Exercise 1

Exercise 2.

1. f being non-negative and measurable, from theorem (18) there exists a sequence $(s_n)_{n\geq 1}$ of simple functions on $(\Omega, \mathcal{F})$ such that $s_n \uparrow f$. In particular, each s_n is a non-negative element of $S_{\mathbf{R}}(\Omega, \mathcal{F})$. Furthermore, $s_n \leq f$ for all $n \geq 1$ and having assumed that $f \in L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, we have:

$$\int s_n^p d\mu \leq \int f^p d\mu < +\infty$$

   We conclude that $(s_n)_{n\geq 1}$ is a sequence of non-negative elements of $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{R}}(\Omega, \mathcal{F})$ such that $s_n \uparrow f$.

2. Since $s_n \to f$, we have $|s_n - f|^p \to 0$ as $n \to +\infty$. Furthermore:

$$|s_n - f|^p \leq (s_n + f)^p \leq 2^p f^p \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$$

   From the dominated convergence theorem (23), we obtain:

$$\lim_{n\to+\infty} \int |s_n - f|^p d\mu = 0$$

3. Given $\epsilon > 0$, from 2. there exists $N \geq 1$ such that:

$$n \geq N \;\Rightarrow\; \int |s_n - f|^p d\mu \leq \epsilon^p$$

   In particular, taking $s = s_N$, we have found s belonging to the set $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{R}}(\Omega, \mathcal{F})$ such that $\|f - s\|_p \leq \epsilon$.

4. Let $A_{\mathbf{K}} = L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{K}}(\Omega, \mathcal{F})$. We claim that $A_{\mathbf{K}}$ is dense in $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$, i.e. that $\bar{A}_{\mathbf{K}} = L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$ where $\bar{A}_{\mathbf{K}}$ is the closure of $A_{\mathbf{K}}$ in $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$. Recall from definition (75) that for any open set U in $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$ and $f \in U$, there exists $\epsilon > 0$ such that $B(f, \epsilon) \subseteq U$. Hence, all we need to prove is that given $f \in L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$ and $\epsilon > 0$, there exists $s \in A_{\mathbf{K}}$ such that $\|f - s\|_p \leq \epsilon$. Indeed, if such property is proved, then for any $f \in L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$ and U open containing f, we have $A_{\mathbf{K}} \cap U \neq \emptyset$ and consequently $f \in \bar{A}_{\mathbf{K}}$. So $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \subseteq \bar{A}_{\mathbf{K}}$. Now, if $f \in L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ and $\epsilon > 0$, the existence of $s \in A_{\mathbf{R}}$ such that $\|f - s\|_p \leq \epsilon$ has already been proved when f is non-negative. Suppose $f \in L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. Then $f = f^+ - f^-$ where each f^+, f^- is a non-negative element of $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. There exists $s^+, s^- \in A_{\mathbf{R}}$ such that $\|f^+ - s^+\|_p \leq \epsilon/2$ and $\|f^- - s^-\|_p \leq \epsilon/2$. Taking $s = s^+ - s^-$, we have found $s \in A_{\mathbf{R}}$ such that:

$$\|f - s\|_p \leq \|f^+ - s^+\|_p + \|f^- - s^-\|_p \leq \epsilon$$

and the property is proved for $f \in L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. If f is an element of $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, then $f = f_1 + if_2$ where each f_1, f_2 lies in $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. There exists $s_1, s_2 \in A_{\mathbf{R}}$ such that $\|f_1 - s_1\|_p \leq \epsilon/2$ and $\|f_2 - s_2\|_p \leq \epsilon/2$. Taking $s = s_1 + is_2$, we have found $s \in A_{\mathbf{C}}$ such that:

$$\|f - s\|_p \leq \|f_1 - s_1\|_p + \|f_2 - s_2\|_p \leq \epsilon$$

and the property is proved for $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

Exercise 2

Exercise 3.

1. Given $n \geq 1$, s_n is of the form:

$$s_n = \sum_{i=1}^{p} \alpha_i 1_{A_i}$$

where $p \geq 1$, $\alpha_i \in \mathbf{R}^+$ and $A_i \in \mathcal{F}$ for all $i \in \mathbf{N}_p$. From definition (40), it is therefore a simple function on $(\Omega, \mathcal{F})$ (or indeed a complex simple function on $(\Omega, \mathcal{F})$ with values in $\mathbf{R}^+$).

2. Since f is an element of $L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, we have:

$$\|f\|_\infty \stackrel{\triangle}{=} \inf\{M \in \mathbf{R}^+ : |f| \leq M \ \mu\text{-a.s.}\} < +\infty$$

It is therefore possible to find an integer $n_0 \geq 1$ such that $\|f\|_\infty < n_0$. Since $\|f\|_\infty$ is the greatest lower bound all M's such that $|f| \leq M$ μ-a.s., n_0 cannot be such lower bound. Hence, there exists $M_0 \in \mathbf{R}^+$ such that $|f| \leq M_0$ μ-a.s. and $M_0 < n_0$. Thus, there exists $N \in \mathcal{F}$ with $\mu(N) = 0$, and:

$$\forall \omega \in N^c \ , \ |f(\omega)| \leq M_0 < n_0$$

In particular, since f is a non-negative element of $L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$:

$$\forall \omega \in N^c \ , \ 0 \leq f(\omega) < n_0$$

3. Let $n \geq n_0$ and $\omega \in N^c$. From 2. we have $0 \leq f(\omega) < n_0$ and consequently $s_n(\omega) = k/2^n$, where k is the unique integer of $\{0, \ldots, n2^n - 1\}$ such that $f(\omega) \in [k/2^n, (k+1)/2^n[$. So:

$$0 \leq f(\omega) - s_n(\omega) < \frac{1}{2^n} \tag{4}$$

4. From 3. we have $N \in \mathcal{F}$ with $\mu(N) = 0$ such that for all $\omega \in N^c$, inequality (4) holds for all $n \geq n_0$. So $|f - s_n| < 1/2^n$ μ-a.s. for all $n \geq n_0$. Since $\|f - s_n\|_\infty$ is a lower bound of all M's such that $|f - s_n| \leq M$ μ-a.s., we conclude that $\|f - s_n\|_\infty \leq 1/2^n$ for all $n \geq n_0$, and in particular:

$$\lim_{n \to +\infty} \|f - s_n\|_\infty = 0 \tag{5}$$

5. Let $p \in [1, +\infty]$ be given and $A_{\mathbf{K}} = L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{K}}(\Omega, \mathcal{F})$. If $p \in [1, +\infty[$, we have already proved in exercise (2) that $A_{\mathbf{K}}$ is dense in $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$. We assume that $p = +\infty$ and we claim likewise that $A_{\mathbf{K}}$ is dense in $L^\infty_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$ (note that $A_{\mathbf{K}}$ and $S_{\mathbf{K}}(\Omega, \mathcal{F})$ coincide when $p = +\infty$). Given $f \in L^\infty_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$ and $\epsilon > 0$, we need to show the existence of $s \in A_{\mathbf{K}}$ such that $\|f - s\|_\infty \leq \epsilon$. When $\mathbf{K} = \mathbf{R}$ and f is a non-negative element of $L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, then such existence is guaranteed by (5), (keeping in mind that simple functions on $(\Omega, \mathcal{F})$ are elements of $S_{\mathbf{R}}(\Omega, \mathcal{F}) = A_{\mathbf{R}}$). If $f \in L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, then $f = f^+ - f^-$ where each f^+, f^- is a non-negative element of $L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. There exists s^+, s^- in $A_{\mathbf{R}}$ such that $\|f^+ - s^+\|_\infty \leq \epsilon/2$ and $\|f^- - s^-\|_\infty \leq \epsilon/2$. Taking $s = s^+ - s^-$ we obtain $s \in A_{\mathbf{R}}$ and $\|f - s\|_\infty \leq \epsilon$. This completes the proof of theorem (67) when $\mathbf{K} = \mathbf{R}$. If $f \in L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, then $f = f_1 + if_2$ where each f_1, f_2 is an element of $L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. Approximating f_1 and f_2 by elements s_1, s_2 of $A_{\mathbf{R}}$, we obtain likewise an element $s = s_1 + is_2$ of $A_{\mathbf{C}}$ with $\|f - s\|_\infty \leq \epsilon$. This proves theorem (67).

Exercise 3

Exercise 4.

1. Let $A \subseteq \Omega$. If $A = \emptyset$, then $d(x, A) = +\infty$ for all $x \in \Omega$. In particular, the map $x \to d(x, A)$ is a continuous map. If $A \neq \emptyset$ and $y \in A$, then $d(x, A) \leq d(x, y)$. In particular $d(x, A) < +\infty$ for all $x \in \Omega$. Furthermore, for all $x, x' \in \Omega$ and $y \in A$:

$$d(x, A) \leq d(x, y) \leq d(x, x') + d(x', y)$$

Consequently, $d(x, A) - d(x, x')$ is a lower bound of all $d(x', y)$, as y ranges through A. $d(x', A)$ being the greatest of such lower bounds, we have:

$$d(x, A) \leq d(x, x') + d(x', A)$$

Interchanging the roles of x and x' we obtain:

$$d(x', A) \leq d(x, x') + d(x, A)$$

from which we see that:

$$\forall x, x' \in \Omega \ , \ |d(x, A) - d(x', A)| \leq d(x, x') \tag{6}$$

We conclude from (6) that $x \to d(x, A)$ is continuous.

2. Let F be a closed subset of Ω. If $x \in F$, $d(x, F) \leq d(x, x) = 0$ and consequently $d(x, F) = 0$. Conversely, suppose $d(x, F) = 0$. We shall show that $x \notin F$ is impossible. Indeed, if $x \in F^c$, since F^c is open, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq F^c$. However, $d(x, F) = 0$ implies in particular that $d(x, F) < \epsilon$. Since $d(x, F)$ is the greatest of all lower bounds of $d(x, y)$, as y range through F, ϵ cannot be such a lower bound. Hence, there exists $y \in F$ such that $d(x, y) < \epsilon$. So $y \in B(x, \epsilon) \cap F \neq \emptyset$ which is a contradiction. We have proved that $x \in F$ is equivalent to

$d(x, F) = 0$, whenever F is a closed subset of Ω. This exercise is in fact a repetition of exercise (22) of Tutorial 4.

Exercise 4

Exercise 5.

1. $G_n = \{x \in \Omega : d(x, F) < 1/n\}$ can be written as $\Phi_F^{-1}([-\infty, 1/n[)$ where Φ_F is the map defined on Ω by $\Phi_F(x) = d(x, F)$. Having proved in exercise (4) that Φ_F is continuous, and since $[-\infty, 1/n[$ is open in $\bar{\mathbf{R}}$, we conclude that G_n is an open subset of Ω.

2. It is clear that $G_{n+1} \subseteq G_n$ and $F \subseteq \cap_{n\geq 1} G_n$. Suppose that $x \in \cap_{n\geq 1} G_n$. Then $d(x, F) < 1/n$ for all $n \geq 1$ and consequently $d(x, F) = 0$. From exercise (4), F being a closed subset of Ω, it follows that $x \in F$. This shows that $\cap_{n\geq 1} G_n \subseteq F$ and finally $\cap_{n\geq 1} G_n = F$. So $G_n \downarrow F$.

3. Since μ is a finite measure on $(\Omega, \mathcal{B}(\Omega))$, from theorem (8) and $G_n \downarrow F$ we obtain $\mu(G_n) \to \mu(F)$ as $n \to +\infty$. Furthermore, since $F \subseteq G_n$ for all $n \geq 1$, we have:
$$\mu(G_n \setminus F) = \mu(G_n \setminus F) + \mu(F) - \mu(F) = \mu(G_n) - \mu(F)$$
It follows that $\mu(G_n \setminus F) \to 0$ as $n \to +\infty$. Given $\epsilon > 0$, there exists $N \geq 1$, such that:
$$n \geq N \ \Rightarrow\ \mu(G_n \setminus F) \leq \epsilon$$
In particular, taking $F' = F$ and $G' = G_N$, F' and G' are respectively closed and open subsets of Ω, with $F' \subseteq F \subseteq G'$ and $\mu(G' \setminus F') \leq \epsilon$. This shows that $F \in \Sigma$. We have proved that any closed subset F of Ω is an element of Σ.

4. The application of theorem (8) requires some finiteness property.

5. Ω is a closed subset of Ω. So $\Omega \in \Sigma$.

6. Let $B \in \Sigma$. For all $\epsilon > 0$, there exist F and G respectively closed and open subsets of Ω, such that $F \subseteq B \subseteq G$ and $\mu(G \setminus F) \leq \epsilon$. Since $F^c \setminus G^c = F^c \cap G = G \setminus F$, it follows that $G^c \subseteq B^c \subseteq F^c$ and $\mu(F^c \setminus G^c) \leq \epsilon$. This shows that $B^c \in \Sigma$, since G^c and F^c are respectively closed and open subsets of Ω. We have proved that Σ is closed under complementation.

Exercise 5

Exercise 6.

1. Let $n \geq 1$. By assumption B_n is an element of Σ. For all $\epsilon' > 0$, and in particular for $\epsilon' = \epsilon/2^n$, there exist F_n and G_n respectively closed and open subsets of Ω, with $F_n \subseteq B_n \subseteq G_n$ and $\mu(G_n \setminus F_n) \leq \epsilon'$.

2. Let $H_n = \cup_{k=1}^n F_k$ and $H = \cup_{k\geq 1} F_k$. Then $H_n \uparrow H$, and consequently from theorem (7), $\mu(H_n) \to \mu(H)$ as $n \to +\infty$. μ being a finite measure, we obtain:

$$\lim_{n\to+\infty} \mu(H \setminus H_n) = \lim_{n\to+\infty} \mu(H) - \mu(H_n) = 0$$

In particular, there exists $N \geq 1$ such that $\mu(H \setminus H_N) \leq \epsilon$, or equivalently:

$$\mu\left(\left(\cup_{n=1}^{+\infty} F_n\right) \setminus \left(\cup_{n=1}^{N} F_n\right)\right) \leq \epsilon \tag{7}$$

3. Let $G = \cup_{n\geq 1} G_n$ and $F = \cup_{n=1}^N F_n$. G being a union of open subsets of Ω, is itself an open subset of Ω. F being a finite union of closed subsets of Ω, is itself a closed subset of Ω. Since $F_n \subseteq B_n \subseteq G_n$ for all $n \geq 1$ and $B = \cup_{n\geq 1} B_n$, it is clear that $F \subseteq B \subseteq G$.

4. Let $H = \cup_{n\geq 1} F_n$. The sets $G \setminus H$ and $H \setminus F$ are clearly disjoint. Furthermore if $x \in G \setminus F = G \cap F^c$, then either $x \in H$ or $x \notin H$. If $x \in H$ then $x \in H \setminus F$. If $x \notin H$ then $x \in G \setminus H$. In any case, $x \in G \setminus H \uplus H \setminus F$. This shows that $G \setminus F \subseteq G \setminus H \uplus H \setminus F$.

5. Let $H = \cup_{n\geq 1} F_n$ and $x \in G \setminus H$. Since $x \in G$, there exists $n \geq 1$ such that $x \in G_n$. But $x \in H^c = \cap_{k\geq 1} F_k^c$. So in particular $x \in F_n^c$ and consequently $x \in G_n \setminus F_n$. This shows that $G \setminus H \subseteq \cup_{n\geq 1} G_n \setminus F_n$.

6. Applying 4. and 5. with $H = \cup_{n\geq 1} F_n$, we have:

$$G \setminus F \subseteq (\cup_{n\geq 1} G_n \setminus F_n) \uplus H \setminus F$$

It follows that:

$$\mu(G \setminus F) \leq \sum_{n=1}^{+\infty} \mu(G_n \setminus F_n) + \mu(H \setminus F)$$

Having chosen F_n and G_n such that $\mu(G_n \setminus F_n) \leq \epsilon/2^n$ and having defined F from 2. such that $\mu(H \setminus F) \leq \epsilon$, we conclude that $\mu(G \setminus F) \leq 2\epsilon$.

7. Given a sequence $(B_n)_{n\geq 1}$ in Σ and $B = \cup_{n\geq 1} B_n$, given an arbitrary $\epsilon > 0$, we have shown the existence of F and G respectively closed and open subsets of Ω, such that $F \subseteq B \subseteq G$ (see 3.) and $\mu(G \setminus F) \leq 2\epsilon$ (see 6.). It follows that $B \in \Sigma$. This shows that Σ is closed under countable union. Since $\Omega \in \Sigma$ and Σ is closed under complementation (see exercise (5)), Σ is therefore a σ-algebra on Ω. Furthermore, still from exercise (5), Σ contains every closed subset of Ω. Being closed under complementation, it also contains every open subset of Ω. In other words, the topology $\mathcal{T}$ is a subset of Σ, i.e. $\mathcal{T} \subseteq \Sigma$. The σ-algebra $\sigma(\mathcal{T})$ being the smallest σ-algebra on Ω containing $\mathcal{T}$ (*containing* in the inclusion sense), the fact that Σ is a σ-algebra on Ω implies that $\mathcal{B}(\Omega) = \sigma(\mathcal{T}) \subseteq \Sigma$. Σ being a subset of the Borel σ-algebra $\mathcal{B}(\Omega)$, we conclude that $\Sigma = \mathcal{B}(\Omega)$. Hence, for all $B \in \mathcal{B}(\Omega)$ and $\epsilon > 0$, there exist F and G respectively closed and open subsets of Ω, such that $F \subseteq B \subseteq G$ and $\mu(G \setminus F) \leq \epsilon$. This proves theorem (68).

Exercise 6

Exercise 7.

1. Let $p \in [1, +\infty]$ and $f \in C^b_{\mathbf{K}}(\Omega)$. Since f is continuous, f is Borel measurable. Furthermore, since f is bounded, there exists $M \in \mathbf{R}^+$ such that $|f| \leq M$. This implies that $\|f\|_\infty \leq M$ and in particular $\|f\|_\infty < +\infty$. So $f \in L^\infty_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$. Moreover, if $p \in [1, +\infty[$, μ being a finite measure on $(\Omega, \mathcal{B}(\Omega))$:
$$\int |f|^p d\mu \leq M^p \mu(\Omega) < +\infty$$
so $f \in L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$, and finally $C^b_{\mathbf{K}}(\Omega) \subseteq L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$.

2. Let $n \geq 1$ and ϕ_n be defined by $\phi_n(x) = 1 - 1 \wedge (nd(x, F))$. From exercise (4), the map $x \to d(x, F)$ is continuous. So ϕ_n is also continuous, and furthermore it is clear that $|\phi_n(x)| \leq 1$ for all $x \in \Omega$. So $\phi_n \in C^b_{\mathbf{R}}(\Omega)$.

3. Let $x \in \Omega$. If $x \in F$, then $d(x, F) = 0$ and $\phi_n(x) = 1$ for all $n \geq 1$. In particular, $\phi_n(x) \to 1_F(x)$ as $n \to +\infty$. If $x \notin F$, then from exercise (4), F being a closed subset of Ω, we have $d(x, F) > 0$. It follows that:
$$\lim_{n\to+\infty} \phi_n(x) = 1 - \lim_{n\to+\infty} 1 \wedge (nd(x, F)) = 0$$
In particular, $\phi_n(x) \to 1_F(x)$ as $n \to +\infty$. So $\phi_n \to 1_F$.

4. Let $p \in [1, +\infty[$. From 3. we have $\phi_n \to 1_F$ and consequently $|\phi_n - 1_F|^p \to 0$ as $n \to +\infty$. Furthermore, for all $n \geq 1$:
$$|\phi_n - 1_F|^p \leq (|\phi_n| + |1_F|)^p \leq 2^p$$
μ being a finite measure on $(\Omega, \mathcal{B}(\Omega))$, from the dominated convergence theorem (23) we conclude that:
$$\lim_{n\to+\infty} \int |\phi_n - 1_F|^p d\mu = 0$$

5. Let $p \in [1, +\infty[$ and $\epsilon > 0$. From 4. there is $N \geq 1$ such that:
$$n \geq N \;\Rightarrow\; \int |\phi_n - 1_F|^p d\mu \leq \epsilon^p$$
In particular, taking $\phi = \phi_N$, $\phi \in C^b_{\mathbf{R}}(\Omega)$ and $\|\phi - 1_F\|_p \leq \epsilon$.

6. Let ν be a complex measure on $(\Omega, \mathcal{B}(\Omega))$. From theorem (57), the total variation $|\nu|$ of ν is a finite measure on $(\Omega, \mathcal{B}(\Omega))$. It follows that $C^b_{\mathbf{C}}(\Omega) \subseteq L^1_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega), |\nu|) = L^1_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega), \nu)$. Let $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega), |\nu|)$ be such that $|h| = 1$ and $\nu = \int h d|\nu|$. Then:
$$\begin{aligned} \left| \int \phi_n d\nu - \nu(F) \right| &= \left| \int \phi_n d\nu - \int 1_F d\nu \right| \\ &= \left| \int (\phi_n - 1_F) h d|\nu| \right| \end{aligned}$$

$$\leq \int |\phi_n - 1_F| d|\nu|$$

where the second equality stems from definition (97), and the last inequality from theorem (24). We conclude from 4. applied to $\mu = |\nu|$ and $p = 1$, that:

$$\nu(F) = \lim_{n \to +\infty} \int \phi_n d\nu$$

7. Let $(\Omega, \mathcal{T})$ be a metrizable topological space, and μ, ν be two complex measures on $(\Omega, \mathcal{B}(\Omega))$. We assume that:

$$\forall \phi \in C^b_{\mathbf{R}}(\Omega) \ , \ \int \phi d\mu = \int \phi d\nu \tag{8}$$

and we claim that $\mu = \nu$. We define:

$$\mathcal{D} = \{E \in \mathcal{B}(\Omega) \ : \ \mu(E) = \nu(E)\}$$

Let F be a closed subset of Ω. From 6. and (8) we have:

$$\mu(F) = \lim_{n \to +\infty} \int \phi_n d\mu = \lim_{n \to +\infty} \int \phi_n d\nu = \nu(F)$$

So $F \in \mathcal{D}$. Hence, any closed subset of Ω is an element of $\mathcal{D}$. In particular, $\Omega \in \mathcal{D}$. Furthermore, if $A, B \in \mathcal{D}$ with $A \subseteq B$, then:

$$\mu(B \setminus A) = \mu(B) - \mu(A) = \nu(B) - \nu(A) = \nu(B \setminus A)$$

So $B \setminus A \in \mathcal{D}$. Finally, if $(E_n)_{n \geq 1}$ is a sequence of elements of $\mathcal{D}$ with $E_n \uparrow E$, then using exercise (13) of Tutorial 12 we have:

$$\mu(E) = \lim_{n \to +\infty} \mu(E_n) = \lim_{n \to +\infty} \nu(E_n) = \nu(E)$$

So $E \in \mathcal{D}$, and we have proved that $\mathcal{D}$ is a Dynkin system on Ω. In particular, $\mathcal{D}$ is closed under complementation, and since it contains every closed subset of Ω, it also contains every open subset of Ω. So $\mathcal{T} \subseteq \mathcal{D}$ and finally, since $\mathcal{T}$ is closed under finite intersection, from the Dynkin system theorem (1) we conclude that $\mathcal{B}(\Omega) = \sigma(\mathcal{T}) \subseteq \mathcal{D}$. It follows that $\mathcal{B}(\Omega) = \mathcal{D}$ and consequently $\mu = \nu$, which completes the proof of theorem (69).

Exercise 7

Exercise 8.

1. Let $\epsilon > 0$ and $i \in \mathbf{N}_n$. Since $A_i \in \mathcal{B}(\Omega)$, μ is a finite measure on $(\Omega, \mathcal{B}(\Omega))$ and $(\Omega, \mathcal{T})$ is metrizable, from theorem (68) there exist F_i, G_i respectively closed and open subsets of Ω, such that $F_i \subseteq A_i \subseteq G_i$ and $\mu(G_i \setminus F_i) \leq \epsilon$. In particular, $A_i \setminus F_i \subseteq G_i \setminus F_i$ and we have $\mu(A_i \setminus F_i) \leq \epsilon$.

2. From $s = \sum_{i=1}^n \alpha_i 1_{A_i}$ and $s' = \sum_{i=1}^n \alpha_i 1_{F_i}$ we obtain:

$$\begin{aligned}
\|s - s'\|_p &= \left\|\sum_{i=1}^n \alpha_i (1_{A_i} - 1_{F_i})\right\|_p \\
&\leq \sum_{i=1}^n |\alpha_i| \cdot \|1_{A_i} - 1_{F_i}\|_p \\
&= \sum_{i=1}^n |\alpha_i| \left(\int |1_{A_i} - 1_{F_i}|^p d\mu\right)^{\frac{1}{p}} \\
&= \sum_{i=1}^n |\alpha_i| \left(\int 1_{A_i \setminus F_i} d\mu\right)^{\frac{1}{p}} \\
&= \sum_{i=1}^n |\alpha_i| \mu(A_i \setminus F_i)^{\frac{1}{p}} \\
&\leq \left(\sum_{i=1}^n |\alpha_i|\right) \epsilon^{\frac{1}{p}}
\end{aligned}$$

3. Let $\epsilon > 0$. Choosing $\epsilon' > 0$ sufficiently small such that:

$$\left(\sum_{i=1}^n \|\alpha_i|\right) \epsilon'^{1/p} \leq \epsilon/2$$

and applying 2. to ϵ', there exist closed subsets $F_1, \ldots, F_n$ of Ω, such that $\|s - s'\|_p \leq \epsilon/2$, where s' is defined as:

$$s' = \sum_{i=1}^n \alpha_i 1_{F_i}$$

Furthermore for all $i \in \mathbf{N}_n$, from 5. of exercise (7) there exists $\phi_i \in C^b_{\mathbf{R}}(\Omega)$ such that $|\alpha_i| \cdot \|\phi_i - 1_{F_i}\|_p \leq \epsilon/2n$. We Define:

$$\phi = \sum_{i=1}^n \alpha_i \phi_i$$

Then $\phi \in C^b_{\mathbf{C}}(\Omega)$ (in fact $\phi \in C^b_{\mathbf{R}}(\Omega)$ if $\alpha_i \in \mathbf{R}$ for all i's), and:

$$\begin{aligned}
\|\phi - s'\|_p &= \left\|\sum_{i=1}^n \alpha_i (\phi_i - 1_{F_i})\right\|_p \\
&\leq \sum_{i=1}^n |\alpha_i| \cdot \|\phi_i - 1_{F_i}\|_p \\
&\leq \epsilon/2
\end{aligned}$$

Finally, we obtain $\|\phi - s\|_p \leq \|\phi - s'\|_p + \|s - s'\|_p \leq \epsilon$.

4. Suppose $(\Omega, \mathcal{T})$ is a metrizable topological space, and μ is a finite measure on $(\Omega, \mathcal{B}(\Omega))$. For all $p \in [1, +\infty[$, we clearly have $C^b_{\mathbf{K}}(\Omega) \subseteq L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$ and we claim that $C^b_{\mathbf{K}}(\Omega)$ is in fact dense in $L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$. Given $f \in L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$ and $\epsilon > 0$, we have to prove the existence of $\phi \in C^b_{\mathbf{K}}(\Omega)$ such that $\|f - \phi\|_p \leq \epsilon$. From theorem (67), the set $S_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega))$ (which is a subset of $L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$ since μ is finite) is dense in $L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$. There exists $s \in S_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega))$ such that $\|f - s\|_p \leq \epsilon/2$. Applying 3. to the $\mathbf{K}$-valued simple function s, there exists $\phi \in C^b_{\mathbf{K}}(\Omega)$ (ϕ can indeed be chosen $\mathbf{R}$-valued if $\mathbf{K} = \mathbf{R}$), such that $\|\phi - s\|_p \leq \epsilon/2$. It follows that:

$$\|f - \phi\|_p \leq \|f - s\|_p + \|\phi - s\|_p \leq \epsilon$$

which completes the proof of theorem (70).

Exercise 8

Exercise 9.

1. $F_n = \phi^{-1}([1/n, +\infty])$ where ϕ is the continuous map defined by $\phi(x) = d(x, \Omega'^c)$. Since $[1/n, +\infty]$ is a closed subset of $\bar{\mathbf{R}}$, we conclude that F_n is a closed subset of Ω.

2. For all $n \geq 1$ it is clear that $F_n \subseteq F_{n+1}$. Let $x \in \Omega'$. Since Ω' is an open subset of Ω, Ω'^c is a closed subset of Ω and $x \notin \Omega'^c$. It follows from exercise (4) that $d(x, \Omega'^c) > 0$. Hence, there exists $n \geq 1$ such that $d(x, \Omega'^c) \geq 1/n$. So $x \in F_n$ and we have proved that $\Omega' \subseteq \cup_{n\geq 1} F_n$. To prove the reverse inclusion, suppose $x \in F_n$ for a some $n \geq 1$. Then in particular $d(x, \Omega'^c) > 0$ and x cannot be an element of Ω'^c. So $x \in \Omega'$. This shows that $F_n \subseteq \Omega'$ for all $n \geq 1$, and we have proved that $F_n \uparrow \Omega'$.

3. Since $F_n \subseteq F_{n+1}$ and $K_n \subseteq K_{n+1}$, $F_n \cap K_n \subseteq F_{n+1} \cap K_{n+1}$. Furthermore, it is clear that $\cup_{n\geq 1} F_n \cap K_n \subseteq \Omega'$ since $F_n \subseteq \Omega'$ for all $n \geq 1$. Finally if $x \in \Omega'$, since $F_n \uparrow \Omega'$ there exists $p \geq 1$ such that $x \in F_p$. Since $K_n \uparrow \Omega$ there exists $q \geq 1$ such that $x \in K_q$. Taking $n = \max(p, q)$, we have $x \in F_n \cap K_n$. So $\Omega' \subseteq \cup_{n\geq 1} F_n \cap K_n$ and we have proved that $F_n \cap K_n \uparrow \Omega'$.

4. Let $n \geq 1$. Since F_n is closed in Ω, F_n^c is open in Ω. By the very definition of the induced topology on K_n, $K_n \setminus F_n = K_n \cap F_n^c$ is an open subset of K_n. We conclude that $F_n \cap K_n$ is a closed subset of K_n.

5. By assumption, each K_n is a compact subset of Ω. Equivalently, the induced topological space $(K_n, \mathcal{T}_{|K_n})$ is compact. Having proved that $F_n \cap K_n$ is a closed subset of K_n, from exercise (2) of Tutorial 8, $F_n \cap K_n$ is a compact subset of K_n, or equivalently a compact subset of Ω'.

6. We have found a sequence $(F_n \cap K_n)_{n\geq 1}$ of compact subsets of Ω', such that $F_n \cap K_n \uparrow \Omega'$. This shows that the induced topological space $(\Omega', \mathcal{T}_{|\Omega'})$ is σ-compact. From theorem (12), it is also metrizable, which completes the proof of theorem (71).

Exercise 9

Exercise 10.

1. Let $x \in K$. Since μ is locally finite, there exists U_x open subset of Ω, such that $x \in U_x$ and $\mu(U_x) < +\infty$. It is clear that $K \subseteq \cup_{x\in K} U_x$, and K being a compact subset of Ω, there exists a finite subset $\{x_1, \ldots, x_n\}$ of K such that $K \subseteq U_{x_1} \cup \ldots \cup U_{x_n}$. Taking $V_i = U_{x_i}$, we have found $V_1, \ldots, V_n$ open subsets of Ω, such that $\mu(V_i) < +\infty$ for all $i \in \mathbf{N}_n$ and:
$$K \subseteq V_1 \cup \ldots \cup V_n \tag{9}$$
Note that if $n = 0$, $K = \emptyset$ and it is always possible to assume $n = 1$ by taking $V_1 = \emptyset$ (not a very important comment).

2. From (9) and exercise (13) of Tutorial 5, we obtain:
$$\mu(K) \leq \mu(V_1 \cup \ldots \cup V_n) \leq \sum_{i=1}^{n} \mu(V_i) < +\infty$$

Exercise 10

Exercise 11.

1. Let $\epsilon > 0$. Since $(\Omega, \mathcal{T})$ is metrizable and μ is a finite measure, from theorem (68) there exist F, G respectively closed and open subsets of Ω, such that $F \subseteq B \subseteq G$ and $\mu(G \setminus F) \leq \epsilon$. In particular, there exists F closed with $F \subseteq B$ and $\mu(B \setminus F) \leq \epsilon$.

2. Since $K_n \subseteq K_{n+1}$, $F \setminus (K_{n+1} \cap F) \subseteq F \setminus (K_n \cap F)$ for all $n \geq 1$. Moreover, we have:
$$\bigcap_{n=1}^{+\infty} F \setminus (K_n \cap F) = \bigcap_{n=1}^{+\infty} F \cap (K_n^c \cup F^c) = F \cap \left(\bigcup_{n=1}^{+\infty} K_n\right)^c = \emptyset$$
It follows that $F \setminus (K_n \cap F) \downarrow \emptyset$.

3. F being a closed subset of Ω, $K_n \cap F$ is closed with respect to the induced topology on K_n. In other words, $K_n \cap F$ is a closed subset of K_n.

4. Since K_n is compact, and $K_n \cap F$ is closed in K_n, from exercise (2) of Tutorial 8, $K_n \cap F$ is itself compact.

5. Since $F \setminus (K_n \cap F) \downarrow \emptyset$ and μ is a finite measure, from theorem (8) we have $\mu(F \setminus (K_n \cap F)) \to 0$ as $n \to +\infty$. In particular, there exists $n \geq 1$ such that $\mu(F \setminus (K_n \cap F)) \leq \epsilon$. Taking $K = K_n \cap F$, from 4. K is a compact subset of K_n, or equivalently a compact subset of Ω. Hence, we have found a compact subset K of Ω, such that $K \subseteq F$ and $\mu(F \setminus K) \leq \epsilon$.

6. Since $\mu(B \setminus F) \leq \epsilon$ and $\mu(F \setminus K) \leq \epsilon$, we have:

$$\begin{aligned} \mu(B) &= \mu(B \setminus F) + \mu(F) \\ &= \mu(B \setminus F) + \mu(F \setminus K) + \mu(K) \\ &\leq \mu(K) + 2\epsilon \end{aligned}$$

7. We have proved in 6. that for all $B \in \mathcal{B}(\Omega)$, there exists K compact with $K \subseteq B$ and $\mu(B) \leq \mu(K) + 2\epsilon$. α being an upper bound of all $\mu(K)$, as K ranges through all compacts subsets with $K \subseteq B$, we have $\mu(K) \leq \alpha$. So $\mu(B) \leq \alpha + 2\epsilon$. This being true for all $\epsilon > 0$, it follows that $\mu(B) \leq \alpha$. Moreover, for all K compact with $K \subseteq B$, we have $\mu(K) \leq \mu(B)$. So $\mu(B)$ is an upper bound of all $\mu(K)$, as K ranges through compacts with $K \subseteq B$. α being the smallest of such upper bounds, we have $\alpha \leq \mu(B)$ and finally:

$$\mu(B) = \alpha = \sup\{\mu(K) : K \subseteq B\ ,\ K \text{ compact}\}$$

This being true for all $B \in \mathcal{B}(\Omega)$, from definition (103), μ is inner-regular. We have proved that any finite measure on a metrizable, σ-compact topological space is inner-regular.

Exercise 11

Exercise 12.

1. Since $K_n \uparrow \Omega$, we have $K_n \cap B \uparrow B$. From theorem (7), it follows that $\mu(K_n \cap B) \uparrow \mu(B)$.

2. Since $\alpha < \mu(B)$ and $\mu(K_n \cap B) \to \mu(B)$, there exists $n \geq 1$ such that $\alpha < \mu(K_n \cap B)$. Taking $K = K_n$, we have found K compact subset of Ω such that $\alpha < \mu(K \cap B)$.

3. From exercise (10), μ being a locally finite measure and K being compact, we have $\mu(K) < +\infty$. Hence, for all $A \in \mathcal{B}(\Omega)$:

$$\mu^K(A) = \mu(K \cap A) \leq \mu(K) < +\infty$$

So μ^K is a finite measure on $(\Omega, \mathcal{B}(\Omega))$. Since $(\Omega, \mathcal{T})$ is metrizable and σ-compact, from exercise (11) it follows that μ^K is inner-regular. In particular:

$$\mu^K(B) = \sup\{\mu^K(K^*) : K^* \subseteq B\ ,\ K^* \text{ compact}\}$$

4. It appears from 3. that $\mu^K(B)$ is the smallest upper bound of all $\mu^K(K^*)$, as K^* ranges through compacts with $K^* \subseteq B$. Since $\alpha < \mu^K(B)$, α cannot be such an upper bound. Hence, there exists K^* compact with $K^* \subseteq B$, such that $\alpha < \mu(K \cap K^*)$.

5. $(\Omega, \mathcal{T})$ being metrizable, it is a Hausdorff topological space. K^* being a compact subset of Ω, we conclude from theorem (35) that K^* is a closed subset of Ω.

6. Having proved that K^* is a closed subset of Ω, $K \cap K^*$ is closed relative to the induced topology on K. In other words, $K \cap K^*$ is a closed subset of K.

7. $K \cap K^*$ being a closed subset of K, and K being compact, from exercise (2) of Tutorial 8 we conclude that $K \cap K^*$ is itself compact.

8. We have shown that $\alpha < \mu(K \cap K^*)$ and that $K \cap K^*$ is a compact subset of Ω. Since $K^* \subseteq B$, we have $K \cap K^* \subseteq B$ and we conclude that:
$$\alpha < \mu(K \cap K^*) \leq \sup\{\mu(K') : K' \subseteq B \ , \ K' \text{ compact}\} \tag{10}$$

9. For all $\alpha \in \bar{\mathbf{R}}$ with $\alpha < \mu(B)$, inequality (10) holds. Hence:
$$\mu(B) \leq \sup\{\mu(K') : K' \subseteq B \ , \ K' \text{ compact}\}$$

10. Is is clear that:
$$\sup\{\mu(K') : K' \subseteq B \ , \ K' \text{ compact}\} \leq \mu(B)$$
We conclude that:
$$\mu(B) = \sup\{\mu(K') : K' \subseteq B \ , \ K' \text{ compact}\}$$
This being true for all $B \in \mathcal{B}(\Omega)$, from definition (103), μ is inner-regular. We have proved that any locally finite measure on a metrizable and σ-compact topological space, is inner-regular.

Exercise 12

Exercise 13.

1. Let $(\Omega, \mathcal{T})$ be a metrizable topological space. Suppose $(\Omega, \mathcal{T})$ is separable. From definition (58), there exists a sequence $(x_n)_{n\geq 1}$ of elements of Ω, which are dense in Ω. The set of open balls:
$$\mathcal{H} = \{B(x_n, 1/p) \ : \ n \geq 1, \ p \geq 1\}$$
is easily seen to be a countable base of $(\Omega, \mathcal{T})$. Indeed, it is a subset of the topology $\mathcal{T}$ which is at most countable, and for any open set U and any $x \in U$, on can easily find $n \geq 1$ and $p \geq 1$ such that:
$$x \in B(x_n, 1/p) \subseteq U$$
So U is a union of elements of $\mathcal{H}$. We have proved that if $(\Omega, \mathcal{T})$ is separable, then it has a countable base. Conversely, suppose $(\Omega, \mathcal{T})$ has a countable base, say $\mathcal{H}$. For all $V \in \mathcal{H}$, $V \neq \emptyset$, let x_V be an arbitrary element of V. Then, the set:
$$A = \{x_V \ : \ V \in \mathcal{H}, \ V \neq \emptyset\}$$
is at most countable, and is easily seen to be dense in Ω. Indeed, for all $x \in \Omega$ and $\epsilon > 0$, the open ball $B(x, \epsilon)$ being a union of elements of $\mathcal{H}$ (see definition (57) of a countable base), we have $x \in V \subseteq B(x, \epsilon)$ for

some $V \in \mathcal{H}$, $V \neq \emptyset$. In particular, we have found $x_V \in A$, such that $d(x, x_V) < \epsilon$. This shows that $(\Omega, \mathcal{T})$ is separable, and we have proved the equivalence between the separability of $(\Omega, \mathcal{T})$, and the fact that it has a countable base. This equivalence was already proved in slightly more detail, as part of exercise (19) of Tutorial 6.

2. We assume that $(\Omega, \mathcal{T})$ is not only metrizable, but also compact. Let $n \geq 1$. Then $(B(x, 1/n))_{x\in\Omega}$ is a family of open sets whose union is equal to Ω itself. In other words, it is an open covering of Ω. Since $(\Omega, \mathcal{T})$ is compact, this open covering has a finite sub-covering. In other words, there exists an integer $p \geq 1$ and $x_1, \ldots, x_p$ in Ω, such that:
$$\Omega = B(x_1, 1/n) \cup \ldots \cup B(x_p, 1/n)$$
We have proved that Ω can be covered by a finite number of open balls with radius $1/n$.

3. We assume that $(\Omega, \mathcal{T})$ is not only metrizable but also compact. From 2. given $n \geq 1$, Ω can be covered by a finite number, say $p_n \geq 1$, of open balls with radius $1/n$. Let $x_{1,n}, \ldots, x_{p_n,n}$ be the centers of such open balls. Then, the set $A = \{x_{k,n} \ : \ n \geq 1, \ k = 1, \ldots, p_n\}$ is at most countable, and we claim that it is dense in Ω. Let $x \in \Omega$. We have to show that $x \in \bar{A}$, i.e. that given U open containing x, we have $U \cap A \neq \emptyset$. $(\Omega, \mathcal{T})$ being metrizable, it is sufficient to show that given $\epsilon > 0$, $B(x, \epsilon) \cap A \neq \emptyset$. Let $n \geq 1$ be such that $1/n \leq \epsilon$. Since x belongs to an open ball $B(x_{k,n}, 1/n)$ for some $k = 1, \ldots, p_n$, in particular we have $d(x, x_{k,n}) < \epsilon$. This shows that $B(x, \epsilon) \cap A \neq \emptyset$ and we have proved that A is dense in Ω. This shows that $(\Omega, \mathcal{T})$ is separable. The purpose of this exercise is to show that a metrizable compact topological space is also separable.

Exercise 13

Exercise 14.

1. From theorem (12), the induced metric $d_{|K_n}$ induces the induced topology $\mathcal{T}_{|K_n}$ on K_n.

2. By assumption, each K_n is a compact subset of Ω. In other words, the topological space $(K_n, \mathcal{T}_{|K_n})$ is compact. However from 1. it is also metrizable. It follows from exercise (13) that $(K_n, \mathcal{T}_{|K_n})$ is separable.

3. Let $A = \{x_n^p \ : \ n \geq 1, \ p \geq 1\}$. Then A is an at most countable set, and we claim that A is dense in Ω. Since $(\Omega, \mathcal{T})$ is metrizable, given $x \in \Omega$ and $\epsilon > 0$, it is sufficient to show that $A \cap B(x, \epsilon) \neq \emptyset$. Since $\Omega = \cup_{n\geq 1} K_n$, there is $n \geq 1$ such that $x \in K_n$. By assumption, the sequence $(x_n^p)_{p\geq 1}$ is dense in K_n. Hence, there exists $p \geq 1$ such that $d_{|Kn}(x, x_n^p) < \epsilon$. Equivalently, we have $d(x, x_n^p) < \epsilon$. It follows that $A \cap B(x, \epsilon) \neq \emptyset$ and we have proved that A is dense in Ω. This shows that $(\Omega, \mathcal{T})$ is separable. The purpose of this exercise is to prove that a metrizable and σ-compact topological space, is also separable. This is the objective of theorem (72).

Exercise 14

Exercise 15.

1. Let U be open in Ω and $x \in U$. The measure μ being locally finite, there exists some open set W_x such that $x \in W_x$ and $\mu(W_x) < +\infty$. Defining $U_x = U \cap W_x$, U_x is an open set in Ω such that $x \in U_x \subseteq U$ and $\mu(U_x) < +\infty$.

2. Since U_x is open, and $\mathcal{H}$ is a countable base of $(\Omega, \mathcal{T})$, U_x can be expressed as a union of elements of $\mathcal{H}$. In particular, since $x \in U_x$, there exists some $V_x \in \mathcal{H}$ such that $x \in V_x \subseteq U_x$.

3. $\mathcal{H}'$ being a subset of $\mathcal{H}$, and $\mathcal{H}$ being a countable base of $(\Omega, \mathcal{T})$, $\mathcal{H}'$ is an at most countable set of open sets in Ω. Furthermore, given U open in Ω and $x \in U$, it follows from 1. and 2. that there exists $V_x \in \mathcal{H}$ such that $x \in V_x \subseteq U$ and $\mu(V_x) < +\infty$. In other words, there exists $V_x \in \mathcal{H}'$ such that $x \in V_x \subseteq U$. Consequently, U can be expressed as $U = \cup_{x \in U} V_x$ and we have proved that any open set in Ω can be written as a union of elements of $\mathcal{H}'$. This shows that $\mathcal{H}'$ is a countable base of $(\Omega, \mathcal{T})$.

4. Since Ω is an open set in Ω, and $\mathcal{H}'$ is a countable base of $(\Omega, \mathcal{T})$, Ω can be written as a union of elements of $\mathcal{H}'$. In other words, there exists a subset $\mathcal{G} \subseteq \mathcal{H}'$ such that $\Omega = \cup_{V \in \mathcal{G}} V$. $\mathcal{H}'$ being at most countable, $\mathcal{G}$ is itself at most countable. There exists a map $\phi : \mathbf{N}^* \to \mathcal{G}$ which is surjective. So $\Omega = \cup_{n \geq 1} \phi(n)$, and defining $V_n = \phi(n)$ we obtain $\Omega = \cup_{n \geq 1} V_n$ where each V_n is an element of $\mathcal{G} \subseteq \mathcal{H}'$. In particular, each V_n is an open set in Ω with $\mu(V_n) < +\infty$.

Exercise 15

Exercise 16.

1. Let $\mu^{V_n} = \mu(V_n \cap \cdot)$. Since $\mu(V_n) < +\infty$, μ^{V_n} is a finite measure on $(\Omega, \mathcal{B}(\Omega))$. Furthermore, $(\Omega, \mathcal{T})$ is a metrizable topological space. Applying theorem (68), since $B \in \mathcal{B}(\Omega)$, there exist F_n closed and G_n open such that $F_n \subseteq B \subseteq G_n$ and $\mu^{V_n}(G_n \setminus F_n) \leq \epsilon/2^n$. In particular, since $G_n \setminus B \subseteq G_n \setminus F_n$, there exists G_n open such that $B \subseteq G_n$ and $\mu^{V_n}(G_n \setminus B) \leq \epsilon/2^n$.

2. Let $G = \cup_{n \geq 1} V_n \cap G_n$. Each V_n and G_n is an open set in Ω. So G is a union of open sets in Ω. It follows that G is an open set in Ω. Furthermore, since $\Omega = \cup_{n \geq 1} V_n$ and $B \subseteq G_n$ for all $n \geq 1$, we have:
$$B = \bigcup_{n=1}^{+\infty} V_n \cap B \subseteq \bigcup_{n=1}^{+\infty} V_n \cap G_n = G$$

3. We have:
$$G \setminus B = G \cap B^c = \bigcup_{n=1}^{+\infty} V_n \cap G_n \cap B^c = \bigcup_{n=1}^{+\infty} V_n \cap (G_n \setminus B)$$

4. From 3. and 1. we obtain:
$$\mu(G \setminus B) \leq \sum_{n=1}^{+\infty} \mu(V_n \cap (G_n \setminus B)) = \sum_{n=1}^{+\infty} \mu^{V_n}(G_n \setminus B) \leq \epsilon$$
Since $B \subseteq G$, we have $\mu(G) = \mu(B) + \mu(G \setminus B)$ and consequently $\mu(G) \leq \mu(B) + \epsilon$.

5. Since G is open and $B \subseteq G$, we have $\alpha \leq \mu(G)$. Using 4. it follows that $\alpha \leq \mu(B) + \epsilon$. This being true for all $\epsilon > 0$, we conclude that $\alpha \leq \mu(B)$.

6. For all G open with $B \subseteq G$, we have $\mu(B) \leq \mu(G)$. It follows that $\mu(B)$ is a lower bound of all $\mu(G)$'s where G is open with $B \subseteq G$. α being the greatest of such lower bounds, we have $\mu(B) \leq \alpha$. However, from 5. we have $\alpha \leq \mu(B)$. It follows that $\alpha = \mu(B)$. We have proved that for all $B \in \mathcal{B}(\Omega)$:
$$\mu(B) = \inf\{\mu(G) : \ B \subseteq G \ , \ G \text{ open}\}$$
This shows that μ is outer-regular.

7. In this exercise, we proved that a locally finite measure on a metrizable and σ-compact topological space is outer-regular. However, in exercise (12), we proved that it is also inner-regular. It follows that a locally finite measure on a metrizable and σ-compact topological space is regular. This proves theorem (73).

Exercise 16

Exercise 17. Let Ω be an open subset of $\mathbf{R}^n$, and μ be a locally finite measure in $(\Omega, \mathcal{B}(\Omega))$. $\mathbf{R}^n$ is a metrizable topological space, and furthermore from theorem (48) any closed and bounded subset of $\mathbf{R}^n$ is compact. In particular, $K_p = [-p, p]^n$ is a compact subset of $\mathbf{R}^n$ for all $p \geq 1$. So $\mathbf{R}^n$ is both metrizable and σ-compact. From theorem (71) it follows that the induced topological space $(\Omega, (\mathcal{T}_{\mathbf{R}^n})_{|\Omega})$ is also metrizable and σ-compact. Applying theorem (73), we conclude that μ being locally finite, is a regular measure. We have proved that any locally finite measure on an open subset of $\mathbf{R}^n$ is regular. This is the objective of theorem (74).

Exercise 17

Exercise 18.

1. Since $(\Omega, \mathcal{T})$ is locally compact, for all $x \in \Omega$, there exists W_x open in Ω such that $x \in W_x$ and $\bar{W}_x$ is compact. Let $n \geq 1$. K_n is a compact subset of Ω. Furthermore, $(K_n \cap W_x)_{x \in K_n}$ is an open covering of K_n, from which therefore we can extract a finite sub-covering. There exists an integer $p_n \geq 1$ and $x_1^n, \ldots, x_{p_n}^n$ elements of K_n, such that:
$$K_n = (K_n \cap W_{x_1^n}) \cup \ldots \cup (K_n \cap W_{x_{p_n}^n})$$
Setting $V_k^n = W_{x_k^n}$ for $k = 1, \ldots, p_n$, we have found $V_1^n, \ldots, V_{p_n}^n$ open subsets of Ω such that $K_n \subseteq V_1^n \cup \ldots \cup V_{p_n}^n$ and $\bar{V}_1^n, \ldots, \bar{V}_{p_n}^n$ are compact subsets of Ω.

2. Let $W_n = V_1^n \cup \ldots \cup V_{p_n}^n$ and $V_n = \cup_{k=1}^n W_k$ for $n \geq 1$. Since $V_1^n, \ldots, V_{p_n}^n$ are open, each W_n is open, and consequently each V_n is open. So $(V_n)_{n\geq 1}$ is a sequence of open sets in Ω, and it is clear that $V_n \subseteq V_{n+1}$ for all $n \geq 1$. Let $x \in \Omega$. Since $K_n \uparrow \Omega$, in particular $\Omega = \cup_{n\geq 1} K_n$ and there exists $n \geq 1$ such that $x \in K_n$. From 1. we have $K_n \subseteq W_n$, and since $W_n \subseteq V_n$, it follows that $x \in V_n$. This shows that $\Omega = \cup_{n\geq 1} V_n$ and we have proved that $(V_n)_{n\geq 1}$ is a sequence of open sets such that $V_n \uparrow \Omega$.

3. In order to show that $\bar{W}_n = \bar{V}_1^n \cup \ldots \cup \bar{V}_{p_n}^n$ it is sufficient to prove that for all A, B subsets of Ω, we have $\overline{A \cup B} = \bar{A} \cup \bar{B}$. Recall from exercise (21) of Tutorial 4 that the closure in Ω of any set A, is the smallest closed set containing A (in the sense of inclusion). In particular, we have $A \subseteq \bar{A}$ and $B \subseteq \bar{B}$ and consequently $A \cup B \subseteq \bar{A} \cup \bar{B}$. However, $\bar{A} \cup \bar{B}$ being closed, this implies that $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$. Furthermore since $A \subseteq A \cup B \subseteq \overline{A \cup B}$ and $\overline{A \cup B}$ is closed, we have $\bar{A} \subseteq \overline{A \cup B}$ and likewise $\bar{B} \subseteq \overline{A \cup B}$. It follows that $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$ and we have proved the equality $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

4. Since $\bar{W}_n = \bar{V}_1^n \cup \ldots \cup \bar{V}_{p_n}^n$ and each $\bar{V}_k^n$ is a compact subset of Ω, in order to prove that $\bar{W}_n$ is compact, it is sufficient to show that if A and B are compact subsets of Ω, then $A \cup B$ is also a compact subset of Ω. For that purpose we shall use the characterization of compact subsets proved in exercise (2) of Tutorial 8. Let $(U_i)_{i\in I}$ be a family of open sets in Ω such that $A \cup B \subseteq \cup_{i\in I} U_i$. Then in particular $A \subseteq \cup_{i\in I} U_i$ and A being a compact subset of Ω, there exists I_1 finite subset of I such that $A \subseteq \cup_{i\in I_1} U_i$. Similarly, there exists I_2 finite subset of I such that $B \subseteq \cup_{i\in I_2} U_i$, It follows that $A \cup B \subseteq \cup_{i\in I_1\cup I_2} U_i$ and $I_1 \cup I_2$ being finite, we conclude that $A \cup B$ is a compact subset of Ω.

5. Let $n \geq 1$. From 2. we have $V_n = \cup_{k=1}^n W_k$. Using a similar argument as in 3. we see that $\bar{V}_n = \cup_{k=1}^n \bar{W}_k$. Using a similar argument as in 4., each $\bar{W}_k$ being compact by virtue of 4. itself, we conclude that $\bar{V}_n$ is itself compact.

6. Let $(\Omega, \mathcal{T})$ be a topological space. If $(\Omega, \mathcal{T})$ is σ-compact and locally compact, we have been able to construct a sequence $(V_n)_{n\geq 1}$ of open sets in Ω, such that $V_n \uparrow \Omega$ and $\bar{V}_n$ is compact for all $n \geq 1$. So $(\Omega, \mathcal{T})$ is strongly σ-compact. Conversely, suppose that $(\Omega, \mathcal{T})$ is strongly σ-compact, and let $(V_n)_{n\geq 1}$ be a sequence of open sets in Ω, such that $V_n \uparrow \Omega$ and each $\bar{V}_n$ is compact. Then $\bar{V}_n \uparrow \Omega$ and Ω is therefore σ-compact. Furthermore, for all $x \in \Omega$, there exists $n \geq 1$ such that $x \in V_n$. Since V_n is open and $\bar{V}_n$ is compact, this shows that Ω is locally compact. This completes the proof of theorem (75).

Exercise 18

Exercise 19.

1. Since $A \subseteq \Omega'$ and $A \subseteq \bar{A}$, we have $A \subseteq \Omega' \cap \bar{A}$.

2. The complement of $\Omega' \cap \bar{A}$ in Ω' is:
$$\Omega' \setminus (\Omega' \cap \bar{A}) = \Omega' \cap (\Omega'^c \cup \bar{A}^c) = \Omega' \cap \bar{A}^c$$
Since $\bar{A}$ is closed in Ω, $\bar{A}^c$ is open in Ω and consequently by definition of the induced topology, $\Omega' \cap \bar{A}^c$ is open in Ω'. It follows that $\Omega' \cap \bar{A}$ is closed in Ω'. Note more generally that if F is closed in Ω, then $\Omega' \cap F$ is closed in Ω'.

3. The closure $\bar{A}^{\Omega'}$ of A in Ω' being the smallest closed subset of Ω' containing A, we conclude from $A \subseteq \Omega' \cap \bar{A}$ and $\Omega' \cap \bar{A}$ closed in Ω', that $\bar{A}^{\Omega'} \subseteq \Omega' \cap \bar{A}$.

4. Let $x \in \Omega' \cap \bar{A}$. Suppose $U' \in \mathcal{T}_{|\Omega'}$ and $x \in U'$. There exists $U \in \mathcal{T}$ such that $U' = U \cap \Omega'$. From $x \in U'$, we have $x \in U$ and since $x \in \bar{A}$, we obtain that $A \cap U \neq \emptyset$. However by assumption, A is a subset of Ω'. Hence:
$$A \cap U' = A \cap (U \cap \Omega') = (A \cap \Omega') \cap U = A \cap U \neq \emptyset$$
So we have proved that $A \cap U' \neq \emptyset$.

5. It follows from 4. that $\Omega' \cap \bar{A} \subseteq \bar{A}^{\Omega'}$. However from 3. we have $\bar{A}^{\Omega'} \subseteq \Omega' \cap \bar{A}$. We conclude that $\bar{A}^{\Omega'} = \Omega' \cap \bar{A}$.

Exercise 19

Exercise 20.

1. Let $x \in \Omega$ and $\epsilon > 0$. Let $y \in \overline{B(x,\epsilon)}$. For all U open in Ω such that $y \in U$, we have $U \cap B(x,\epsilon) \neq \emptyset$. In particular, for all $\eta > 0$, we have $B(y,\eta) \cap B(x,\epsilon) \neq \emptyset$. Let $z \in \Omega$ be such that $d(y,z) < \eta$ and $d(x,z) < \epsilon$. From the triangle inequality:
$$d(x,y) \leq d(x,z) + d(y,z) < \epsilon + \eta$$
This being true for all $\eta > 0$, it follows that $d(x,y) \leq \epsilon$. We have proved that:
$$\overline{B(x,\epsilon)} \subseteq \{y \in \Omega \ : \ d(x,y) \leq \epsilon\}$$

2. Let $\Omega = [0,1/2[\cup\{1\}$ together with its usual metric. Then, the open ball $B(0,1)$ is given by:
$$B(0,1) = \{x \in \Omega \ : \ |x| < 1\} = [0,1/2[$$

3. The complement of $[0,1/2[$ in Ω is $\{1\}$, which can be written as $]1/2,2[\cap\Omega$ and is therefore open in Ω, since $]1/2,2[$ is open in $\mathbf{R}$. It follows that $[0,1/2[$ is closed in Ω.

4. From 2. we have $B(0,1) = [0,1/2[$ and from 3. $[0,1/2[$ is a closed subset of Ω, and is therefore equal to its closure. Hence:
$$\overline{B(0,1)} = \overline{[0,1/2[} = [0,1/2[$$

5. Since $\Omega = \{y \in \Omega \ : \ |y| \leq 1\}$ and $[0, 1/2[\neq \Omega$, we conclude that:
$$\overline{B(0,1)} \neq \{y \in \Omega \ : \ |y| \leq 1\}$$
The purpose of this exercise is to provide a counter-example to the belief that the inclusion proved in 1.:
$$\overline{B(x,\epsilon)} \subseteq \{y \in \Omega \ : \ d(x,y) \leq \epsilon\}$$
can be shown to be an equality.

Exercise 20

Exercise 21.

1. Ω being locally compact, there exists U open with compact closure such that $x \in U$.
2. Since $x \in \Omega'$ and $x \in U$, we have $x \in U \cap \Omega'$. Furthermore, both U and Ω' being open in Ω, $U \cap \Omega'$ is open in Ω. The topology on Ω being metric, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U \cap \Omega'$.
3. From $B(x, \epsilon/2) \subseteq B(x, \epsilon) \subseteq U \cap \Omega' \subseteq U$ we conclude that $\overline{B(x, \epsilon/2)} \subseteq \bar{U}$.
4. From 3. we have $\overline{B(x, \epsilon/2)} = \overline{B(x, \epsilon/2)} \cap \bar{U}$ and $\overline{B(x, \epsilon/2)}$ being closed in Ω, we conclude that it is also closed in $\bar{U}$.
5. Since $\bar{U}$ is compact and $\overline{B(x, \epsilon/2)}$ is a closed subset of $\bar{U}$, it follows from exercise (2) of Tutorial 8 that $\overline{B(x, \epsilon/2)}$ is a compact subset of $\bar{U}$, and consequently also a compact subset of Ω.
6. Let $y \in \overline{B(x, \epsilon/2)}$. From 1. of exercise (20), $d(x, y) \leq \epsilon/2$ and in particular $d(x, y) < \epsilon$. From 2. we have $B(x, \epsilon) \subseteq \Omega'$ and consequently $y \in \Omega'$. This shows that $\overline{B(x, \epsilon/2)} \subseteq \Omega'$.
7. Let $U' = B(x, \epsilon/2) \cap \Omega' = B(x, \epsilon/2)$. It is clear that $x \in U'$ and furthermore $B(x, \epsilon/2)$ being open in Ω, U' is open in Ω', i.e. $U' \in \mathcal{T}_{|\Omega'}$. Using 6. and exercise (19), we obtain:
$$\bar{U}'^{\Omega'} = \bar{U}' \cap \Omega' = \overline{B(x, \epsilon/2)} \cap \Omega' = \overline{B(x, \epsilon/2)}$$
In particular $\bar{U}'^{\Omega'}$ is compact, as can be seen from 5.
8. Given $x \in \Omega'$, we have found U' open in Ω' such that $x \in U'$ and $\bar{U}'^{\Omega'}$ is compact. This shows that $(\Omega', \mathcal{T}_{|\Omega'})$ is locally compact.
9. Let $(\Omega, \mathcal{T})$ be a metrizable and strongly σ-compact topological space. Let Ω' be an open subset of Ω. From theorem (75), $(\Omega, \mathcal{T})$ is metrizable, σ-compact and locally compact. Since Ω' is open, it follows from theorem (71) that the induced topological space $(\Omega', \mathcal{T}_{|\Omega'})$ is itself metrizable and σ-compact. Furthermore, we have proved in this exercise that $(\Omega', \mathcal{T}_{|\Omega'})$ is also locally compact. So $(\Omega', \mathcal{T}_{|\Omega'})$ is metrizable, σ-compact and locally compact. Using theorem (75) once more, we conclude that

$(\Omega', \mathcal{T}_{|\Omega'})$ is metrizable and strongly σ-compact. This completes the proof of theorem (76).

Exercise 21

Exercise 22.

1. The constant map $\phi : x \to 0$ is continuous. Indeed for any U open in $\mathbf{K}$, $\phi^{-1}(U)$ is either equal to $\emptyset$ or to Ω itself. In any case $\phi^{-1}(U)$ is an open subset of Ω. Furthermore, $\text{supp}(\phi) = \emptyset$ and is therefore compact (see exercise (2) of Tutorial 8). This shows that $\phi \in C^c_{\mathbf{K}}(\Omega)$.

2. $C^c_{\mathbf{K}}(\Omega)$ being a non-empty subset of the set of all maps $\phi : \Omega \to \mathbf{K}$, to show that $C^c_{\mathbf{K}}(\Omega)$ is a $\mathbf{K}$-vector space, it is sufficient to show that given $\phi, \psi \in C^c_{\mathbf{K}}(\Omega)$ and $\lambda \in \mathbf{K}$, the map $\phi + \lambda\psi$ is also an element of $C^c_{\mathbf{K}}(\Omega)$. To show that $\phi + \lambda\psi$ is continuous, one may proceed as follows: define $\Phi : \mathbf{K}^2 \to \mathbf{K}$ by $\Phi(x, y) = x+\lambda y$, and $\Psi : \Omega \to \mathbf{K}^2$ by $\Psi(\omega) = (\phi(\omega), \psi(\omega))$. Then $\phi+\lambda\psi = \Phi \circ \Psi$ and Φ being continuous, it is sufficient to show that Ψ is itself a continuous map. However, the continuity of Ψ follows from the fact that each coordinate mapping ϕ and ψ is continuous. Indeed if $U \times V$ is an open rectangle in $\mathbf{K}^2$, then $\Psi^{-1}(U \times V) = \phi^{-1}(U) \cap \psi^{-1}(V)$ and is therefore open in Ω. Any open set W in $\mathbf{K}^2$ being a union of open rectangles, it is clear that $\Psi^{-1}(W)$ is open in Ω. So much for the continuity of $\phi + \lambda\psi$. From the inclusion:

$$\{\phi + \lambda\psi \neq 0\} \subseteq \{\phi \neq 0\} \cup \{\psi \neq 0\}$$

and the fact that given A, B subsets of Ω, $\overline{A \cup B} = \bar{A} \cup \bar{B}$ (see the proof of 3. in exercise (18)), we obtain:

$$\text{supp}(\phi + \lambda\psi) \subseteq \text{supp}(\phi) \cup \text{supp}(\psi)$$

Since ϕ and ψ lie in $C^c_{\mathbf{K}}(\Omega)$, both $\text{supp}(\phi)$ and $\text{supp}(\psi)$ are compact and consequently $A = \text{supp}(\phi) \cup \text{supp}(\psi)$ is itself compact (see the proof of 4. in exercise (18)). Furthermore, $\text{supp}(\phi + \lambda\psi)$ being closed in Ω while being a subset of A, it is also closed in A. From exercise (2) of Tutorial 8, $\text{supp}(\phi+\lambda\psi)$ is therefore compact. We have proved that $\phi+\lambda\psi \in C^c_{\mathbf{K}}(\Omega)$.

3. Let $\phi \in C^c_{\mathbf{K}}(\Omega)$. If $\phi = 0$ then $\phi \in C^b_{\mathbf{K}}(\Omega)$. We assume that $\phi \neq 0$. Let $A = \text{supp}(\phi)$. Then $|\phi|_{|A}$ is a continuous map defined on the non-empty compact topological space $(A, \mathcal{T}_{|A})$. From theorem (37), $|\phi|_{|A}$ attains its maximum, i.e. there exists $x_M \in A$ such that:

$$|\phi(x_M)| = \sup_{x\in A} |\phi(x)|$$

Since $\phi(x) = 0$ for all $x \in A^c$, we have:

$$|\phi(x_M)| = \sup_{x\in\Omega} |\phi(x)|$$

which shows in particular that $\sup_{x\in\Omega} |\phi(x)| < +\infty$. So $\phi \in C^b_{\mathbf{K}}(\Omega)$ and we have proved that $C^c_{\mathbf{K}}(\Omega) \subseteq C^b_{\mathbf{K}}(\Omega)$.

Exercise 22

Exercise 23.

1. Since Ω is locally compact, for all $x \in \Omega$ there exists an open set W_x such that $x \in W_x$ and $\bar{W}_x$ is compact. From $K \subseteq \cup_{x\in K} W_x$ and the fact that K is a compact subset of Ω, we deduce the existence of $n \geq 1$ and $x_1, \ldots, x_n \in K$ such that $K \subseteq \cup_{k=1}^n W_{x_k}$. Setting $V_k = W_{x_k}$ for all $k = 1, \ldots, n$, we have found open sets $V_1, \ldots, V_n$ such that:
$$K \subseteq V_1 \cup \ldots \cup V_n \tag{11}$$
and each $\bar{V}_k$ is compact.

2. An arbitrary union of open sets is open. A finite intersection of open sets is open. Since $V_1, \ldots, V_n$ and G are open, the set $V = (V_1 \cup \ldots \cup V_n) \cap G$ is an open set in Ω. By assumption, $K \subseteq G$ and it therefore follows from (11) that $K \subseteq V$. The fact that $V \subseteq G$ is clear. We have proved that V is open and $K \subseteq V \subseteq G$.

3. Given A, B subsets of Ω, $\overline{A \cup B} = \bar{A} \cup \bar{B}$ (see proof of 3. in exercise (18)). From $V \subseteq V_1 \cup \ldots \cup V_n$ we obtain:
$$\bar{V} \subseteq \overline{V_1 \cup \ldots \cup V_n} = \bar{V}_1 \cup \ldots \cup \bar{V}_n$$

4. If A, B are compact subsets of Ω, $A \cup B$ is a compact subset of Ω (see proof of 4. in exercise (18)). It follows that $K' = \bar{V}_1 \cup \ldots \cup \bar{V}_n$ is a compact subset of Ω. Furthermore from 3. $\bar{V}$ is a subset of K'. Being closed in Ω, $\bar{V}$ is also closed in K' (it can be written as $\bar{V} = F \cap K'$ where F is closed in Ω, take $F = \bar{V}$). Using exercise (2) of Tutorial 8, it follows that $\bar{V}$ is compact.

5. Given A subset of Ω, $d(x, A)$ is well defined for all $x \in \Omega$ as:
$$d(x, A) = \inf\{d(x, y) \ : \ y \in A\}$$
where it is understood that $\inf \emptyset = +\infty$. Since $K \neq \emptyset$ and $V \neq \Omega$, $d(x, K)$ and $d(x, V^c)$ are well-defined real numbers for all $x \in \Omega$. Furthermore, for all A closed in Ω, $d(x, A) = 0$ is equivalent to $x \in A$ (see exercise (22) of Tutorial 4). V being open in Ω, V^c is a closed subset of Ω. So $d(x, V^c) = 0$ is equivalent to $x \in V^c$. K being a compact subset of Ω and Ω being a Hausdorff topological space (it is metric), K is a closed subset of Ω (see theorem (35)). So $d(x, K) = 0$ is equivalent to $x \in K$. It follows that $d(x, V^c) + d(x, K) = 0$ is equivalent to $x \in K \cap V^c$, which can never happen since $K \subseteq V$. We have proved that for all $x \in \Omega$, $\phi(x)$ is a well-defined real number. So $\phi : \Omega \to \mathbf{R}$ is well-defined. For all A subsets of Ω, the map $x \to d(x, A)$ is continuous (see exercise (22) of Tutorial 4). We conclude that ϕ is also continuous.

6. $\phi(x) \neq 0$ is equivalent to $d(x, V^c) \neq 0$ which is itself equivalent to $x \notin V^c$ (since V^c is closed), i.e. $x \in V$. We have proved that $\{\phi \neq 0\} = V$.

7. From 7. $\{\phi \neq 0\} = V$ and consequently $\text{supp}(\phi) = \bar{V}$. Having proved in 4. that $\bar{V}$ is compact, it follows that ϕ has compact support. So $\phi : \Omega \to \mathbf{R}$ is continuous with compact support, i.e. $\phi \in C^c_{\mathbf{R}}(\Omega)$.

8. To show that $1_K \leq \phi$ it is sufficient to show that $x \in K$ implies $1 \leq \phi(x)$. However, K being closed in Ω, $x \in K$ is equivalent to $d(x, K) = 0$. In particular, $x \in K$ implies that $\phi(x) = 1$. It is clear that $\phi(x) \leq 1$ for all $x \in \Omega$. To show that $\phi \leq 1_G$, it is sufficient to show that $x \notin G$ implies $\phi(x) = 0$. But $V \subseteq G$ and consequently $x \notin G$ implies $x \notin V$, i.e. $x \in V^c$. And V^c being closed, $x \in V^c$ is equivalent to $d(x, V^c) = 0$. In particular, we see that $x \notin G$ implies $\phi(x) = 0$. So $1_K \leq \phi \leq 1_G$.

9. Suppose $K = \emptyset$. With $\phi = 0$, $\phi \in C^c_{\mathbf{R}}(\Omega)$ and $1_K \leq \phi \leq 1_G$.

10. Suppose $V = \Omega$. Then $\bar{V} = \bar{\Omega} = \Omega$. $\bar{V}$ being compact (see 4.), it follows that Ω is compact.

11. Suppose $V = \Omega$. Since $V \subseteq G$, we have $G = \Omega$, i.e. $1_G = 1$. Take $\phi = 1$. Then ϕ is continuous and $\text{supp}(\phi) = \Omega$ is compact (see 10.). So $\phi \in C^c_{\mathbf{R}}(\Omega)$ and $1_K \leq \phi \leq 1_G$. This proves theorem (77).

Exercise 23

Exercise 24.

1. Let $\phi \in C^c_{\mathbf{K}}(\Omega)$. Then ϕ is continuous and from exercise (13) of Tutorial 4, the map $\phi : (\Omega, \mathcal{B}(\Omega)) \to (\mathbf{K}, \mathcal{B}(\mathbf{K}))$ is therefore measurable. Furthermore from exercise (22), $C^c_{\mathbf{K}}(\Omega) \subseteq C^b_{\mathbf{K}}(\Omega)$. So ϕ is also bounded. There exists $m \in \mathbf{R}^+$ such that $|\phi| \leq m$. Let $A = \text{supp}(\phi)$. Then A is a compact subset of Ω, and from exercise (10), μ being locally finite, $\mu(A) < +\infty$. Since $\{\phi \neq 0\} \subseteq A$, we have $A^c \subseteq \{\phi = 0\}$ and consequently $\phi = \phi 1_A$. Hence:
$$\int |\phi|^p d\mu = \int 1_A |\phi|^p d\mu \leq m^p \mu(A) < +\infty$$
So $\phi \in L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$ and finally $C^c_{\mathbf{K}}(\Omega) \subseteq L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$.

2. Let $\epsilon > 0$. Since $(\Omega, \mathcal{T})$ is metrizable and strongly σ-compact, in particular from theorem (75), it is metrizable and σ-compact. Since μ is a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$, from theorem (73) μ is regular. Having assumed that $\mu(B) < +\infty$, we have $\mu(B) < \mu(B) + \epsilon/2$. From the outer-regularity of μ, $\mu(B)$ is the greatest lower-bound of all $\mu(G)$'s where G is open with $B \subseteq G$. So $\mu(B) + \epsilon/2$ cannot be such lower-bound. There exists G open with $B \subseteq G$ such that:
$$\mu(G) < \mu(B) + \frac{\epsilon}{2} \tag{12}$$
Likewise, $\mu(B) - \epsilon/2 < \mu(B)$ and from the inner-regularity of μ, $\mu(B)$ is the lowest upper-bound of all $\mu(K)$'s where K is compact with $K \subseteq B$.

So $\mu(B)-\epsilon/2$ cannot be such upper-bound, and consequently, there exists K compact with $K\subseteq B$ such that:

$$\mu(B)-\frac{\epsilon}{2}<\mu(K) \tag{13}$$

Hence, we have found K compact and G open with $K\subseteq B\subseteq G$, and furthermore from (12) and (13) we have:

$$\mu(G)<\mu(B)+\frac{\epsilon}{2}<\mu(K)+\epsilon$$

and consequently:

$$\mu(K)+\mu(G\setminus K)=\mu(G)<\mu(K)+\epsilon$$

K being compact and μ locally finite, from exercise (10) we have $\mu(K)<+\infty$, and we conclude that $\mu(G\setminus K)<\epsilon$. In particular $\mu(G\setminus K)\leq\epsilon$.

3. The fact that $\mu(B)<+\infty$ was used when writing the inequalities $\mu(B)<\mu(B)+\epsilon/2$ and $\mu(B)-\epsilon/2<\mu(B)$. Without this assumption, these inequalities would not be strict, and the argument developed in 2. would fail.

4. Since $(\Omega,\mathcal{T})$ is metrizable and strongly σ-compact, in particular from theorem (75), it is metrizable and locally compact. K being compact and G open with $K\subseteq G$, from theorem (77), there exists $\phi\in C^c_{\mathbf{R}}(\Omega)$ such that $1_K\leq\phi\leq 1_G$.

5. Since $1_K\leq\phi\leq 1_G$, in particular $0\leq\phi\leq 1$ and consequently we have $|\phi-1_B|^p\leq 1$. Suppose $x\notin G$. Then $1_G(x)=0$ and therefore $\phi(x)=0$. Since $B\subseteq G$, we also have $1_B(x)=0$ and consequently $|\phi(x)-1_B(x)|^p=0$. Suppose $x\in K$. Then $1_K(x)=1$ and therefore $\phi(x)=1$. Since $K\subseteq B$ we also have $1_B(x)=1$ and consequently $|\phi(x)-1_B(x)|^p=0$. We have proved that $x\notin G\setminus K$ implies that $|\phi(x)-1_B(x)|^p=0$. It follows that $|\phi-1_B|^p\leq 1_{G\setminus K}$ and finally:

$$\int|\phi-1_B|^p d\mu\leq\int 1_{G\setminus K}d\mu=\mu(G\setminus K)$$

6. Let $\epsilon>0$. Applying 2. to ϵ^p instead of ϵ itself, we can find K and G such that $\mu(G\setminus K)\leq\epsilon^p$. From 4. and 5. there exists $\phi\in C^c_{\mathbf{R}}(\Omega)$ such that:

$$\int|\phi-1_B|^p d\mu\leq\mu(G\setminus K)\leq\epsilon^p$$

from which we conclude that $\|\phi-1_B\|_p\leq\epsilon$.

7. Let $s\in\mathcal{S}_{\mathbf{C}}(\Omega,\mathcal{B}(\Omega))\cap L^p_{\mathbf{C}}(\Omega,\mathcal{B}(\Omega),\mu)$ and $\epsilon>0$. From 3. of exercise (1) there exists an integer $n\geq 1$, together with $\alpha_1,\ldots,\alpha_n\in\mathbf{C}$ and $A_1,\ldots,A_n\in\mathcal{B}(\Omega)$ such that:

$$s=\sum_{i=1}^n\alpha_i 1_{A_i}$$

and $\mu(A_i) < +\infty$ for all $i \in \mathbf{N}_n$. Without loss of generality, we may assume that $\alpha_i \neq 0$ for all i's (if $s = 0$ then $s \in C^c_{\mathbf{C}}(\Omega)$ and finding $\phi \in C^c_{\mathbf{C}}(\Omega)$ such that $\|\phi - s\|_p \leq \epsilon$ is trivial). Applying 6. to $B = A_i$ (recall that $A_i \in \mathcal{B}(\Omega)$ and $\mu(A_i) < +\infty$) and $\epsilon/n|\alpha_i|$ instead of ϵ, there exists $\phi \in C^c_{\mathbf{R}}(\Omega)$ such that $\|\phi_i - 1_{A_i}\|_p \leq \epsilon/n|\alpha_i|$. Since $C^c_{\mathbf{C}}(\Omega)$ is a vector space, the map $\phi = \sum_{i=1}^n \alpha_i\phi_i$ is an element of $C^c_{\mathbf{C}}(\Omega)$ and we have:

$$\begin{aligned} \|\phi - s\|_p &= \left\| \sum_{i=1}^n \alpha_i\phi_i - \sum_{i=1}^n \alpha_i 1_{A_i} \right\|_p \\ &\leq \sum_{i=1}^n |\alpha_i| \cdot \|\phi_i - 1_{A_i}\|_p \\ &\leq \sum_{i=1}^n |\alpha_i| \cdot \left(\frac{\epsilon}{n|\alpha_i|}\right) \\ &= \epsilon \end{aligned}$$

We have found $\phi \in C^c_{\mathbf{C}}(\Omega)$ such that $\|\phi - s\|_p \leq \epsilon$. Note that if $s \in \mathcal{S}_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega))$ then $\alpha_i \in \mathbf{R}$ for all $i \in \mathbf{N}_n$, and $\phi = \sum_{i=1}^n \alpha_i\phi_i$ is in fact an element of $C^c_{\mathbf{R}}(\Omega)$.

8. To show that $C^c_{\mathbf{K}}(\Omega)$ is dense in $L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$, it is sufficient to show that given $f \in L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$ and $\epsilon > 0$, there exists $\phi \in C^c_{\mathbf{K}}(\Omega)$ such that $\|f - \phi\|_p \leq \epsilon$. However, from theorem (67) there exists $s \in \mathcal{S}_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega)) \cap L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$ such that $\|f - s\|_p \leq \epsilon/2$. Applying 7. to s and $\epsilon/2$ instead of ϵ, there exists $\phi \in C^c_{\mathbf{K}}(\Omega)$ such that $\|\phi - s\|_p \leq \epsilon/2$. It follows that we have found $\phi \in C^c_{\mathbf{K}}(\Omega)$ such that $\|f - \phi\|_p \leq \|f - s\|_p + \|\phi - s\|_p \leq \epsilon$. This completes the proof of theorem (78).

Exercise 24

Exercise 25. Let Ω be an open subset of $\mathbf{R}^n$ where $n \geq 1$. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$ and $p \in [1, +\infty[$. For $k \geq 1$, $V_k =]-k, k[^n$ is an open subset of $\mathbf{R}^n$ with compact closure, and $V_k \uparrow \mathbf{R}^n$. From definition (104), $\mathbf{R}^n$ is strongly σ-compact. Furthermore, it is metrizable. It follows from theorem (76) that Ω being an open subset of $\mathbf{R}^n$, is also metrizable and strongly σ-compact. Applying theorem (78), we conclude that $C^c_{\mathbf{K}}(\Omega)$ is dense in $L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$. This completes the proof of theorem (79).

Exercise 25