15. Stieltjes Integration

Definition 112 *$b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation. The* **Stieltjes $\mathbf{L^1}$-spaces** *associated with b are defined as:*

$$L^1_{\mathbf{C}}(b) \stackrel{\triangle}{=} \left\{ f : \mathbf{R}^+ \to \mathbf{C} \text{ measurable}, \int |f|d|b| < +\infty \right\}$$

$$L^{1,loc}_{\mathbf{C}}(b) \stackrel{\triangle}{=} \left\{ f : \mathbf{R}^+ \to \mathbf{C} \text{ measurable}, \int_0^t |f|d|b| < +\infty, \forall t \in \mathbf{R}^+ \right\}$$

where the notation $|f|$ refers to the modulus map $t \to |f(t)|$.

Warning : In these tutorials, $\int_0^t \ldots$ refers to $\int_{[0,t]} \ldots$, i.e. the domain of integration is always $[0,t]$, not $]0,t]$, $[0,t[$, or $]0,t[$.

Exercise 1. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation.

1. Propose a definition for $L^1_{\mathbf{R}}(b)$ and $L^{1,\mathrm{loc}}_{\mathbf{R}}(b)$.
2. Is $L^1_{\mathbf{C}}(b)$ the same thing as $L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), d|b|)$?
3. Is it meaningful to speak of $L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$?
4. Show that $L^1_{\mathbf{C}}(b) = L^1_{\mathbf{C}}(|b|)$ and $L^{1,\mathrm{loc}}_{\mathbf{C}}(b) = L^{1,\mathrm{loc}}_{\mathbf{C}}(|b|)$.
5. Show that $L^1_{\mathbf{C}}(b) \subseteq L^{1,\mathrm{loc}}_{\mathbf{C}}(b)$.

Exercise 2. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. For all $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(a)$, we define $f.a : \mathbf{R}^+ \to \mathbf{C}$ as:

$$f.a(t) \stackrel{\triangle}{=} \int_0^t f da \ , \ \forall t \in \mathbf{R}^+$$

1. Explain why $f.a : \mathbf{R}^+ \to \mathbf{C}$ is a well-defined map.
2. Let $t \in \mathbf{R}^+$, $(t_n)_{n \geq 1}$ be a sequence in $\mathbf{R}^+$ with $t_n \downarrow\downarrow t$. Show:
$$\lim_{n \to +\infty} \int f 1_{[0,t_n]} da = \int f 1_{[0,t]} da$$
3. Show that $f.a$ is right-continuous.
4. Let $t \in \mathbf{R}^+$ and $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[0,t]$. Show:
$$\sum_{i=1}^n |f.a(t_i) - f.a(t_{i-1})| \leq \int_{]0,t]} |f| da$$
5. Show that $f.a$ is a map of finite variation with:
$$|f.a|(t) \leq \int_0^t |f| da \ , \ \forall t \in \mathbf{R}^+$$

Exercise 3. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f \in L^1_{\mathbf{C}}(a)$.

1. Show that $f.a$ is a right-continuous map of bounded variation.
2. Show $d(f.a)([0,t]) = \nu([0,t])$, for all $t \in \mathbf{R}^+$, where $\nu = \int f da$.
3. Prove the following:

Theorem 86 *Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f \in L^1_{\mathbf{C}}(a)$. The map $f.a : \mathbf{R}^+ \to \mathbf{C}$ defined by:*

$$f.a(t) \stackrel{\triangle}{=} \int_0^t f da \ , \ \forall t \in \mathbf{R}^+$$

is a right-continuous map of bounded variation, and its associated complex Stieltjes measure is given by $d(f.a) = \int f da$, i.e.

$$d(f.a)(B) = \int_B f da \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

Exercise 4. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f \in L^{1,\mathrm{loc}}_{\mathbf{R}}(a)$, $f \geq 0$.

1. Show $f.a$ is right-continuous, non-decreasing with $f.a(0) \geq 0$.
2. Show $d(f.a)([0,t]) = \mu([0,t])$, for all $t \in \mathbf{R}^+$, where $\mu = \int f da$.
3. Prove that $d(f.a)([0,T] \cap \cdot\,) = \mu([0,T] \cap \cdot\,)$, for all $T \in \mathbf{R}^+$.
4. Prove with the following:

Theorem 87 *Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f \in L^{1,loc}_{\mathbf{R}}(a)$, $f \geq 0$. The map $f.a : \mathbf{R}^+ \to \mathbf{R}^+$ defined by:*

$$f.a(t) \stackrel{\triangle}{=} \int_0^t f da \ , \ \forall t \in \mathbf{R}^+$$

is right-continuous, non-decreasing with $(f.a)(0) \geq 0$, and its associated Stieltjes measure is given by $d(f.a) = \int f da$, i.e.

$$d(f.a)(B) = \int_B f da \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

Exercise 5. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(a)$ and $T \in \mathbf{R}^+$.

1. Show that $\int |f|1_{[0,T]}da = \int |f|da^{[0,T]} = \int |f|da^T$.
2. Show that $f1_{[0,T]} \in L^1_{\mathbf{C}}(a)$ and $f \in L^1_{\mathbf{C}}(a^T)$.
3. Show that $(f.a)^T = f.(a^T) = (f1_{[0,T]}).a$.
4. Show that for all $B \in \mathcal{B}(\mathbf{R}^+)$:
$$d(f.a)^T(B) = \int_B f da^T = \int_B f1_{[0,T]}da$$
5. Explain why it does not in general make sense to write:
$$d(f.a)^T = d(f.a)([0,T] \cap \cdot\,)$$
6. Show that for all $B \in \mathcal{B}(\mathbf{R}^+)$:
$$|d(f.a)^T|(B) = \int_B |f|1_{[0,T]}da$$
7. Show that $|d(f.a)^T| = d|f.a|([0,T] \cap \cdot\,)$
8. Show that for all $t \in \mathbf{R}^+$
$$|f.a|(t) = (|f|.a)(t) = \int_0^t |f|da$$
9. Show that $f.a$ is of bounded variation if and only if $f \in L^1_{\mathbf{C}}(a)$.
10. Show that $\Delta(f.a)(0) = f(0)\Delta a(0)$.
11. Let $t > 0$, $(t_n)_{n\geq 1}$ be a sequence in $\mathbf{R}^+$ with $t_n \uparrow\uparrow t$. Show:
$$\lim_{n\to+\infty} \int f1_{[0,t_n]}da = \int f1_{[0,t[}da$$
12. Show that $\Delta(f.a)(t) = f(t)\Delta a(t)$ for all $t \in \mathbf{R}^+$.
13. Show that if a is continuous with $a(0) = 0$, then $f.a$ is itself continuous with $(f.a)(0) = 0$.
14. Prove with the following:

Theorem 88 *Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f \in L^{1,loc}_{\mathbf{C}}(a)$. The map $f.a : \mathbf{R}^+ \to \mathbf{C}$ defined by:*
$$f.a(t) \stackrel{\triangle}{=} \int_0^t f da\ ,\ \forall t \in \mathbf{R}^+$$
is right-continuous of finite variation, and we have $|f.a| = |f|.a$, i.e.
$$|f.a|(t) = \int_0^t |f|da\ ,\ \forall t \in \mathbf{R}^+$$

In particular, $f.a$ is of bounded variation if and only if $f \in L^1_{\mathbf{C}}(a)$. Furthermore, we have $\Delta(f.a) = f\Delta a$.

EXERCISE 6. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation.

1. Prove the equivalence between the following:

$$\begin{array}{cl} (i) & d|b| << da \\ (ii) & |db^T| << da \ , \ \forall T \in \mathbf{R}^+ \\ (iii) & db^T << da \ , \ \forall T \in \mathbf{R}^+ \end{array}$$

2. Does it make sense in general to write $db << da$?

Definition 113 *Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. We say that b is* **absolutely continuous** *with respect to a, and we write $b << a$, if and only if, one of the following holds:*

$$\begin{array}{cl} (i) & d|b| << da \\ (ii) & |db^T| << da \ , \ \forall T \in \mathbf{R}^+ \\ (iii) & db^T << da \ , \ \forall T \in \mathbf{R}^+ \end{array}$$

In other words, b is absolutely continuous w.r. to a, if and only if the Stieltjes measure associated with the total variation of b is absolutely continuous w.r. to the Stieltjes measure associated with a.

EXERCISE 7. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation, absolutely continuous w.r. to a, i.e. with $b << a$.

1. Show that for all $T \in \mathbf{R}^+$, there exits $f_T \in L^1_{\mathbf{C}}(a)$ such that:

$$db^T(B) = \int_B f_T da \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

2. Suppose that $T, T' \in \mathbf{R}^+$ and $T \leq T'$. Show that:

$$\int_B f_T da = \int_{B \cap [0,T]} f_{T'} da \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

3. Show that $f_T = f_{T'} 1_{[0,T]}$ da-a.s.

4. Show the existence of a sequence $(f_n)_{n \geq 1}$ in $L^1_{\mathbf{C}}(a)$, such that for all $1 \leq n \leq p$, $f_n = f_p 1_{[0,n]}$ and:

$$\forall n \geq 1 \ , \ db^n(B) = \int_B f_n da \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

5. We define $f : (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ by:

$$\forall t \in \mathbf{R}^+ \ , \ f(t) \stackrel{\triangle}{=} f_n(t) \text{ for any } n \geq 1 : \ t \in [0, n]$$

   Explain why f is unambiguously defined.

6. Show that for all $B \in \mathcal{B}(\mathbf{C})$, $\{f \in B\} = \cup_{n=1}^{+\infty} [0, n] \cap \{f_n \in B\}$.

7. Show that $f : (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is measurable.

8. Show that $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(a)$ and that we have:

$$b(t) = \int_0^t f da \ , \ \forall t \in \mathbf{R}^+$$

9. Prove the following:

Theorem 89 *Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of finite variation. Then, b is absolutely continuous w.r. to a, i.e. $d|b| << da$, if and only if there exists $f \in L^{1,loc}_{\mathbf{C}}(a)$ such that $b = f.a$, i.e.*

$$b(t) = \int_0^t f da \ , \ \forall t \in \mathbf{R}^+$$

If b is $\mathbf{R}$-valued, we can assume that $f \in L^{1,loc}_{\mathbf{R}}(a)$.
If b is non-decreasing with $b(0) \geq 0$, we can assume that $f \geq 0$.

Exercise 8. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f, g \in L^{1,\mathrm{loc}}_{\mathbf{C}}(a)$ be such that $f.a = g.a$, i.e.:

$$\int_0^t f da = \int_0^t g da \ , \ \forall t \in \mathbf{R}^+$$

1. Show that for all $T \in \mathbf{R}^+$ and $B \in \mathcal{B}(\mathbf{R}^+)$:

$$d(f.a)^T(B) = \int_B f 1_{[0,T]} da = \int_B g 1_{[0,T]} da$$

2. Show that for all $T \in \mathbf{R}^+$, $f 1_{[0,T]} = g 1_{[0,T]}$ da-a.s.

3. Show that $f = g$ da-a.s.

Exercise 9. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation.

1. Show the existence of $h \in L^{1,\mathrm{loc}}_{\mathbf{C}}(|b|)$ such that $b = h.|b|$.

2. Show that for all $B \in \mathcal{B}(\mathbf{R}^+)$ and $T \in \mathbf{R}^+$:
$$db^T(B) = \int_B hd|b|^T = \int_B h|db^T|$$
3. Show that $|h| = 1$ $|db^T|$-a.s. for all $T \in \mathbf{R}^+$.
4. Show that for all $T \in \mathbf{R}^+$, $d|b|([0,T] \cap \{|h| \neq 1\}) = 0$.
5. Show that $|h| = 1$ $d|b|$-a.s.
6. Prove the following:

Theorem 90 *Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. There exists $h \in L^{1,loc}_{\mathbf{C}}(|b|)$ such that $|h| = 1$ and $b = h.|b|$, i.e.*
$$b(t) = \int_0^t hd|b| \ , \ \forall t \in \mathbf{R}^+$$

Definition 114 *$b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation. For all $f \in L^1_{\mathbf{C}}(b)$, the* **Stieltjes integral** *of f with respect to b, is defined as:*
$$\int f db \triangleq \int fhd|b|$$
where $h \in L^{1,loc}_{\mathbf{C}}(|b|)$ is such that $|h| = 1$ and $b = h.|b|$.

Warning : the notation $\int fdb$ of definition (114) is controversial and potentially confusing: 'db' is not in general a complex measure on $\mathbf{R}^+$, unless b is of bounded variation.

Exercise 10. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation.

1. Show that if $f \in L^1_{\mathbf{C}}(b)$, then $\int fhd|b|$ is well-defined.
2. Explain why, given $f \in L^1_{\mathbf{C}}(b)$, $\int fdb$ is unambiguously defined.
3. Show that if b is right-continuous, non-decreasing with $b(0) \geq 0$, definition (114) of $\int fdb$ coincides with that of an integral w.r. to the Stieltjes measure db.
4. Show that if b is a right-continuous map of bounded variation, definition (114) of $\int fdb$ coincides with that of an integral w.r. to the complex Stieltjes measure db.

Exercise 11. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of finite variation. For all $f \in L^{1,\text{loc}}_{\mathbf{C}}(b)$, we define $f.b : \mathbf{R}^+ \to \mathbf{C}$ as:
$$f.b(t) \triangleq \int_0^t fdb \triangleq \int f1_{[0,t]}db \ , \ \forall t \in \mathbf{R}^+$$

1. Explain why $f.b : \mathbf{R}^+ \to \mathbf{C}$ is a well-defined map.
2. If b is right-continuous, non-decreasing with $b(0) \geq 0$, show this definition of $f.b$ coincides with that of theorem (88).
3. Show $f.b = (fh).|b|$, where $h \in L^{1,\text{loc}}_{\mathbf{C}}(|b|)$, $|h| = 1$, $b = h.|b|$.
4. Show that $f.b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation, with $|f.b| = |f|.|b|$, i.e.
$$|f.b|(t) = \int_0^t |f|d|b| \ , \ \forall t \in \mathbf{R}^+$$
5. Show that $f.b$ is of bounded variation if and only if $f \in L^1_{\mathbf{C}}(b)$.
6. Show that $\Delta(f.b) = f\Delta b$.
7. Show that if b is continuous with $b(0) = 0$, then $f.b$ is itself continuous with $(f.b)(0) = 0$.
8. Prove the following:

Theorem 91 *Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. For all $f \in L^{1,loc}_{\mathbf{C}}(b)$, the map $f.b : \mathbf{R}^+ \to \mathbf{C}$ defined by:*
$$f.b(t) \stackrel{\triangle}{=} \int_0^t f db \ , \ \forall t \in \mathbf{R}^+$$
is right-continuous of finite variation, and we have $|f.b| = |f|.|b|$, i.e.
$$|f.b|(t) = \int_0^t |f|d|b| \ , \ \forall t \in \mathbf{R}^+$$
In particular, $f.b$ is of bounded variation if and only if $f \in L^1_{\mathbf{C}}(b)$. Furthermore, we have $\Delta(f.b) = f\Delta b$.

Exercise 12. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. Let $f \in L^{1,\text{loc}}_{\mathbf{C}}(b)$ and $T \in \mathbf{R}^+$.

1. Show that $\int |f|1_{[0,T]}d|b| = \int |f|d|b|^{[0,T]} = \int |f|d|b^T|$.
2. Show that $f1_{[0,T]} \in L^1_{\mathbf{C}}(b)$ and $f \in L^1_{\mathbf{C}}(b^T)$.
3. Show $b^T = h.|b^T|$, where $h \in L^{1,\text{loc}}_{\mathbf{C}}(|b|)$, $|h| = 1$, $b = h.|b|$.
4. Show that $(f.b)^T = f.(b^T) = (f1_{[0,T]}).b$
5. Show that $d|f.b|(B) = \int_B |f|d|b|$ for all $B \in \mathcal{B}(\mathbf{R}^+)$.

6. Let $g : \mathbf{R}^+ \to \mathbf{C}$ be a measurable map. Show the equivalence:
$$g \in L^{1,\text{loc}}_{\mathbf{C}}(f.b) \;\Leftrightarrow\; gf \in L^{1,\text{loc}}_{\mathbf{C}}(b)$$
7. Show that $d(f.b)^T(B) = \int_B fhd|b^T|$ for all $B \in \mathcal{B}(\mathbf{R}^+)$.
8. Show that $db^T = \int hd|b^T|$ and conclude that:
$$d(f.b)^T(B) = \int_B fdb^T \ ,\ \forall B \in \mathcal{B}(\mathbf{R}^+)$$
9. Let $g \in L^{1,\text{loc}}_{\mathbf{C}}(f.b)$. Show that $g \in L^1_{\mathbf{C}}((f.b)^T)$ and:
$$\int g1_{[0,t]}d(f.b)^T = \int gf1_{[0,t]}db^T \ ,\ \forall t \in \mathbf{R}^+$$
10. Show that $g.\left((f.b)^T\right) = (gf).(b^T)$.
11. Show that $(g.(f.b))^T = ((gf).b)^T$.
12. Show that $g.(f.b) = (gf).b$
13. Prove the following:

Theorem 92 *Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. For all $f \in L^{1,loc}_{\mathbf{C}}(b)$ and $g : (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ measurable map, we have the equivalence:*
$$g \in L^{1,loc}_{\mathbf{C}}(f.b) \;\Leftrightarrow\; gf \in L^{1,loc}_{\mathbf{C}}(b)$$
and when such condition is satisfied, $g.(f.b) = (fg).b$, i.e.
$$\int_0^t gd(f.b) = \int_0^t gfdb \ ,\ \forall t \in \mathbf{R}^+$$

Exercise 13. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. let $f, g \in L^{1,\text{loc}}_{\mathbf{C}}(b)$ and $\alpha \in \mathbf{C}$. Show that $f + \alpha g \in L^{1,\text{loc}}_{\mathbf{C}}(b)$, and:
$$(f + \alpha g).b = f.b + \alpha(g.b)$$

Exercise 14. Let $b, c : \mathbf{R}^+ \to \mathbf{C}$ be two right-continuous maps of finite variations. Let $f \in L^{1,\text{loc}}_{\mathbf{C}}(b) \cap L^{1,\text{loc}}_{\mathbf{C}}(c)$ and $\alpha \in \mathbf{C}$.

1. Show that for all $T \in \mathbf{R}^+$, $d(b + \alpha c)^T = db^T + \alpha dc^T$.
2. Show that for all $T \in \mathbf{R}^+$, $d|b + \alpha c|^T \leq d|b|^T + |\alpha| d|c|^T$.
3. Show that $d|b + \alpha c| \leq d|b| + |\alpha| d|c|$.
4. Show that $f \in L^{1,\text{loc}}_{\mathbf{C}}(b + \alpha c)$.

5. Show $d(f.(b+\alpha c))^T(B) = \int_B f d(b+\alpha c)^T$ for all $B \in \mathcal{B}(\mathbf{R}^+)$.
6. Show that $d(f.(b+\alpha c))^T = d(f.b)^T + \alpha d(f.c)^T$.
7. Show that $(f.(b+\alpha c))^T = (f.b)^T + \alpha(f.c)^T$
8. Show that $f.(b+\alpha c) = f.b + \alpha(f.c)$.

Exercise 15. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation.

1. Show that $d|b| \leq d|b_1| + d|b_2|$, where $b_1 = Re(b)$ and $b_2 = Im(b)$.
2. Show that $d|b_1| \leq d|b|$ and $d|b_2| \leq d|b|$.
3. Show that $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(b)$, if and only if:
$$f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(|b_1|^+) \cap L^{1,\mathrm{loc}}_{\mathbf{C}}(|b_1|^-) \cap L^{1,\mathrm{loc}}_{\mathbf{C}}(|b_2|^+) \cap L^{1,\mathrm{loc}}_{\mathbf{C}}(|b_2|^-)$$
4. Show that if $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(b)$, for all $t \in \mathbf{R}^+$:
$$\int_0^t f db = \int_0^t f d|b_1|^+ - \int_0^t f d|b_1|^- + i\left(\int_0^t f d|b_2|^+ - \int_0^t f d|b_2|^-\right)$$

Exercise 16. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. We define $c : \mathbf{R}^+ \to [0, +\infty]$ as:
$$c(t) \stackrel{\triangle}{=} \inf\{s \in \mathbf{R}^+ : \ t < a(s)\} \ , \ \forall t \in \mathbf{R}^+$$
where it is understood that $\inf \emptyset = +\infty$. Let $s, t \in \mathbf{R}^+$.

1. Show that $t < a(s) \ \Rightarrow \ c(t) \leq s$.
2. Show that $c(t) < s \ \Rightarrow \ t < a(s)$.
3. Show that $c(t) \leq s \ \Rightarrow \ t < a(s+\epsilon) \ , \ \forall \epsilon > 0$.
4. Show that $c(t) \leq s \ \Rightarrow \ t \leq a(s)$.
5. Show that $c(t) < +\infty \ \Leftrightarrow \ t < a(\infty)$.
6. Show that c is non-decreasing.
7. Show that if $t_0 \in [a(\infty), +\infty[$, c is right-continuous at t_0.
8. Suppose $t_0 \in [0, a(\infty)[$. Given $\epsilon > 0$, show the existence of $s \in \mathbf{R}^+$, such that $c(t_0) \leq s < c(t_0) + \epsilon$ and $t_0 < a(s)$.
9. Show that $t \in [t_0, a(s)[\ \Rightarrow \ c(t_0) \leq c(t) \leq c(t_0) + \epsilon$.
10. Show that c is right-continuous.

11. Show that if $a(\infty) = +\infty$, then c is a map $c : \mathbf{R}^+ \to \mathbf{R}^+$ which is right-continuous, non-decreasing with $c(0) \geq 0$.

12. We define $\bar{a}(s) = \inf\{t \in \mathbf{R}^+ : \ s < c(t)\}$ for all $s \in \mathbf{R}^+$. Show that for all $s, t \in \mathbf{R}^+$, $s < c(t) \ \Rightarrow \ a(s) \leq t$.

13. Show that $a \leq \bar{a}$.

14. Show that for all $s, t \in \mathbf{R}^+$ and $\epsilon > 0$:
$$a(s + \epsilon) \leq t \ \Rightarrow \ s < s + \epsilon \leq c(t)$$

15. Show that for all $s, t \in \mathbf{R}^+$ and $\epsilon > 0$, $a(s + \epsilon) \leq t \ \Rightarrow \ \bar{a}(s) \leq t$.

16. Show that $\bar{a} \leq a$ and conclude that:
$$a(s) = \inf\{t \in \mathbf{R}^+ : \ s < c(t)\} \ , \ \forall s \in \mathbf{R}^+$$

Exercise 17. Let $f : \mathbf{R}^+ \to \bar{\mathbf{R}}$ be a non-decreasing map. Let $\alpha \in \mathbf{R}$. We define:
$$x_0 \stackrel{\triangle}{=} \sup\{x \in \mathbf{R}^+ : \ f(x) \leq \alpha\}$$

1. Explain why $x_0 = -\infty$ if and only if $\{f \leq \alpha\} = \emptyset$.

2. Show that $x_0 = +\infty$ if and only if $\{f \leq \alpha\} = \mathbf{R}^+$.

3. We assume from now on that $x_0 \neq \pm\infty$. Show that $x_0 \in \mathbf{R}^+$.

4. Show that if $f(x_0) \leq \alpha$ then $\{f \leq \alpha\} = [0, x_0]$.

5. Show that if $\alpha < f(x_0)$ then $\{f \leq \alpha\} = [0, x_0[$.

6. Conclude that $f : (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

Exercise 18. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. We define $c : \mathbf{R}^+ \to [0, +\infty]$ as:
$$c(t) \stackrel{\triangle}{=} \inf\{s \in \mathbf{R}^+ : \ t < a(s)\} \ , \ \forall t \in \mathbf{R}^+$$

1. Let $f : \mathbf{R}^+ \to [0, +\infty]$ be non-negative and measurable. Show $(f \circ c)1_{\{c<+\infty\}}$ is well-defined, non-negative and measurable.

2. Let $t, u \in \mathbf{R}^+$, and ds be the Lebesgue measure on $\mathbf{R}^+$. Show:
$$\int_0^{a(t)} (1_{[0,u]} \circ c)1_{\{c<+\infty\}} ds \leq \int 1_{[0,a(t\wedge u)]} 1_{\{c<+\infty\}} ds$$

3. Show that:
$$\int_0^{a(t)} (1_{[0,u]} \circ c)1_{\{c<+\infty\}} ds \leq a(t \wedge u)$$

4. Show that:
$$a(t \wedge u) = \int_0^{a(t)} 1_{[0,a(u)[}ds = \int_0^{a(t)} 1_{[0,a(u)[}1_{\{c<+\infty\}}ds$$

5. Show that:
$$a(t \wedge u) \leq \int_0^{a(t)} (1_{[0,u]} \circ c)1_{\{c<+\infty\}}ds$$

6. Show that:
$$\int_0^t 1_{[0,u]}da = \int_0^{a(t)} (1_{[0,u]} \circ c)1_{\{c<+\infty\}}ds$$

7. Define:
$$\mathcal{D}_t \stackrel{\triangle}{=} \left\{ B \in \mathcal{B}(\mathbf{R}^+) : \int_0^t 1_B da = \int_0^{a(t)} (1_B \circ c)1_{\{c<+\infty\}}ds \right\}$$
   Show that $\mathcal{D}_t$ is a Dynkin system on $\mathbf{R}^+$, and $\mathcal{D}_t = \mathcal{B}(\mathbf{R}^+)$.

8. Show that if $f : \mathbf{R}^+ \to [0, +\infty]$ is non-negative measurable:
$$\int_0^t f da = \int_0^{a(t)} (f \circ c)1_{\{c<+\infty\}}ds \ , \ \forall t \in \mathbf{R}^+$$

9. Let $f : \mathbf{R}^+ \to \mathbf{C}$ be measurable. Show that $(f \circ c)1_{\{c<+\infty\}}$ is itself well-defined and measurable.

10. Show that if $f \in L^{1,\text{loc}}_{\mathbf{C}}(a)$, then for all $t \in \mathbf{R}^+$, we have:
$$(f \circ c)1_{\{c<+\infty\}}1_{[0,a(t)]} \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), ds)$$
   and furthermore:
$$\int_0^t f da = \int_0^{a(t)} (f \circ c)1_{\{c<+\infty\}}ds$$

11. Show that we also have:
$$\int_0^t f da = \int (f \circ c)1_{[0,a(t)[}ds$$

12. Conclude with the following:

Theorem 93 *Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. We define $c : \mathbf{R}^+ \to [0, +\infty]$ as:*
$$c(t) \stackrel{\triangle}{=} \inf\{s \in \mathbf{R}^+ : \ t < a(s)\} \ , \ \forall t \in \mathbf{R}^+$$
Then, for all $f \in L^{1,loc}_{\mathbf{C}}(a)$, we have:
$$\int_0^t f da = \int_0^{a(t)} ((f \circ c)1_{\{c<+\infty\}})(s)ds \ , \ \forall t \in \mathbf{R}^+$$

Solutions to Exercises

Exercise 1.

1. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. In line with definition (112), it is natural to define:

$$L^1_{\mathbf{R}}(b) \stackrel{\triangle}{=} \left\{ f : \mathbf{R}^+ \to \mathbf{R} \text{ measurable}, \int |f| d|b| < +\infty \right\}$$

$$L^{1,\text{loc}}_{\mathbf{R}}(b) \stackrel{\triangle}{=} \left\{ f : \mathbf{R}^+ \to \mathbf{R} \text{ measurable}, \int_0^t |f| d|b| < +\infty, \forall t \in \mathbf{R}^+ \right\}$$

2. Yes, $L^1_{\mathbf{C}}(b)$ and $L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), d|b|)$ are the same thing.

3. No, $L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$ may not be meaningful. The complex Stieltjes measure db is well-defined by definition (110), provided b is right-continuous of bounded variation, not just right-continuous of finite variation.

4. Since $|b|$ is non-decreasing with $|b|(0) \geq 0$, the total variation of $|b|$ is itself, i.e. $|\,|b|\,| = |b|$. Looking back at definition (112), it follows that $L^1_{\mathbf{C}}(b) = L^1_{\mathbf{C}}(|b|)$ and $L^{1,\text{loc}}_{\mathbf{C}}(b) = L^{1,\text{loc}}_{\mathbf{C}}(|b|)$.

5. Let $f \in L^1_{\mathbf{C}}(b)$. Then $f : (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is measurable and for all $t \in \mathbf{R}^+$, we have:

$$\int_0^t |f| d|b| \stackrel{\triangle}{=} \int_{[0,t]} |f| d|b| = \int |f| 1_{[0,t]} d|b| \leq \int |f| d|b| < +\infty$$

so $f \in L^{1,\text{loc}}_{\mathbf{C}}(b)$ and we have proved that $L^1_{\mathbf{C}}(b) \subseteq L^{1,\text{loc}}_{\mathbf{C}}(b)$.

Exercise 1

Exercise 2.

1. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. In particular, a is right-continuous of finite variation, and the space $L^{1,\text{loc}}_{\mathbf{C}}(a)$ is well-defined as per definition (112). Let $f \in L^{1,\text{loc}}_{\mathbf{C}}(a)$. We define $f.a : \mathbf{R}^+ \to \mathbf{C}$ as:

$$f.a(t) \stackrel{\triangle}{=} \int_0^t f da \ , \ \forall t \in \mathbf{R}^+$$

Given $t \in \mathbf{R}^+$, the map $f 1_{[0,t]}$ is measurable and since $|a| = a$ with $f \in L^{1,\text{loc}}_{\mathbf{C}}(a)$, we have:

$$\int |f| 1_{[0,t]} da = \int_0^t |f| d|a| < +\infty$$

So $f 1_{[0,t]}$ is also integrable with respect to the Stieltjes measure da. It follows that the integral $\int_0^t f da$ is well-defined. This being true for all $t \in \mathbf{R}^+$, the map $f.a : \mathbf{R}^+ \to \mathbf{C}$ is well-defined.

2. Let $t \in \mathbf{R}^+$ and $(t_n)_{n\geq 1}$ be a sequence in $\mathbf{R}^+$ such that $t_n \downarrow\downarrow t$, i.e. $t_n \to t$ and $t < t_{n+1} \leq t_n$ for all $n \geq 1$. We have $f1_{[0,t_n]} \to f1_{[0,t]}$ pointwise, and furthermore $|f|1_{[0,t_n]} \leq |f|1_{[0,t_1]}$ with:
$$\int |f|1_{[0,t_1]}da = \int_0^{t_1} |f|da < +\infty$$
From the dominated convergence theorem (23), we obtain:
$$\lim_{n\to+\infty} \int f1_{[0,t_n]}da = \int f1_{[0,t]}da$$

3. From 2. we see that $f.a(t_n) \to f.a(t)$, for all $t \in \mathbf{R}^+$ and $(t_n)_{n\geq 1}$ sequence in $\mathbf{R}^+$ with $t_n \downarrow\downarrow t$. This shows that $f.a$ is right-continuous. To those who may not be convinced by this conclusion, we may offer the following argument (we shall not repeat it very often): the fact that $f.a$ is right-continuous is equivalent to the fact that for all $t \in \mathbf{R}^+$ and for all U open sets in $\mathbf{C}$ with $f.a(t) \in U$, there exists $u \in \mathbf{R}^+$, $t < u$, such that:
$$s \in]t, u[\;\Rightarrow\; f.a(s) \in U \tag{1}$$
If this is not the case, then there exists some $t \in \mathbf{R}^+$ and U open set in $\mathbf{C}$ with $f.a(t) \in U$, such that for all $u \in \mathbf{R}^+$, $t < u$, the implication (1) does not hold. Take $u = t + 1$. Since the implication (1) does not hold, there exists $t_1 \in]t, u[$ such that $f.a(t_1) \notin U$. Take $u = \min(t+1/2, t_1)$. Since the implication (1) does not hold, there exists $t_2 \in]t, u[$ such that $f.a(t_2) \notin U$. Note in particular that $t < t_2 < t + 1/2$ and $t < t_2 \leq t_1$ (even $t_2 < t_1$ but we don't really care). By induction, we may construct a sequence $(t_n)_{n\geq 1}$ such that $t < t_n < t + 1/n$, $t < t_{n+1} \leq t_n$ and $f.a(t_n) \notin U$ for all $n \geq 1$. In particular, we have $t_n \downarrow\downarrow t$ and consequently $f.a(t_n) \to f.a(t)$. But this contradicts the fact that U is open with $f.a(t) \in U$ and $f.a(t_n) \notin U$ for all $n \geq 1$. This contradiction ensures that $f.a$ is right-continuous.

4. Let $t \in \mathbf{R}^+$ and $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[0, t]$, $n \geq 1$. For all $i \in \{1, \ldots, n\}$, we have:
$$\begin{aligned} |f.a(t_i) - f.a(t_{i-1})| &= \left| \int f1_{[0,t_i]}da - \int f1_{[0,t_{i-1}]}da \right| \\ &= \left| \int f1_{]t_{i-1},t_i]}da \right| \\ &\leq \int |f|1_{]t_{i-1},t_i]}da \end{aligned}$$
and consequently:
$$\begin{aligned} \sum_{i=1}^n |f.a(t_i) - f.a(t_{i-1})| &\leq \sum_{i=1}^n \int |f|1_{]t_{i-1},t_i]}da \\ &= \int |f|1_{]t_0,t_n]}da \end{aligned}$$

$$\leq \quad \int |f|1_{]0,t]}da$$

5. It follows from 4. that $\int |f|1_{]0,t]}da$ is an upper-bound of all sums $\sum_{i=1}^n |f.a(t_i)-f.a(t_{i-1})|$ as $t_0 \leq \ldots \leq t_n$ runs through the set of all finite sequences in $[0,t]$, $n \geq 1$. Since $|f.a|(t) - |f.a(0)|$ is the smallest of such upper-bounds, we obtain:
$$|f.a|(t) - |f.a(0)| \leq \int |f|1_{]0,t]}da \tag{2}$$
Furthermore, we have:
$$|f.a(0)| = \left|\int f1_{\{0\}}da\right| \leq \int |f|1_{\{0\}}da \tag{3}$$
From (2), (3) and $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(a)$, we conclude that:
$$|f.a|(t) \leq \int |f|1_{[0,t]}da = \int_0^t |f|da < +\infty$$
So $f.a$ is a map of finite variation.

Exercise 2

Exercise 3.

1. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f \in L^1_{\mathbf{C}}(a)$. In particular, $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(a)$ and from exercise (2) we know that $f.a$ defined by:
$$f.a(t) = \int_0^t f da \ , \ \forall t \in \mathbf{R}^+$$
is well-defined, right-continuous and of finite variation, with:
$$|f.a|(t) \leq \int_0^t |f|da \ , \ \forall t \in \mathbf{R}^+$$
Since $f \in L^1_{\mathbf{C}}(a)$, it follows that:
$$|f.a|(\infty) = \sup_{t\in\mathbf{R}^+} |f.a|(t) \leq \int |f|da < +\infty$$
So $f.a$ is of bounded variation. We have proved that $f.a$ is right-continuous of bounded variation.

2. Let $\nu = \int f da$. Since $f \in L^1_{\mathbf{C}}(a) = L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), da)$, from theorem (63), ν is a complex measure on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$. Since $f.a$ is right-continuous of bounded variation, its complex Stieltjes measure $d(f.a)$ is well-defined as per definition (110). For all $t \in \mathbf{R}^+$, we have:
$$\begin{aligned} d(f.a)([0,t]) &= d(f.a)(\{0\}) + d(f.a)(]0,t]) \\ &= f.a(0) + f.a(t) - f.a(0) \\ &= \int_{[0,t]} f da = \nu([0,t]) \end{aligned}$$

3. In order to prove theorem (86), we need to show that $d(f.a) = \nu$. Define $\mathcal{D} = \{B \in \mathcal{B}(\mathbf{R}^+) \ : \ d(f.a)(B) = \nu(B)\}$ and $\mathcal{C} = \{[0,t] \ : \ t \in \mathbf{R}^+\}$. From 2. we have $\mathcal{C} \subseteq \mathcal{D}$. Since $\mathcal{C}$ is closed under finite intersection and $\mathcal{D}$ is a Dynkin system on $\mathbf{R}^+$, from the Dynkin system theorem (1) we have $\sigma(\mathcal{C}) \subseteq \mathcal{D}$, where $\sigma(\mathcal{C})$ is the σ-algebra on $\mathbf{R}^+$ generated by $\mathcal{C}$. However, one can easily show that that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbf{R}^+)$. We conclude that $\mathcal{B}(\mathbf{R}^+) \subseteq \mathcal{D}$ and finally $d(f.a) = \nu$. This completes the proof of theorem (86). For those who want to say more, here is the following: $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$ is clear. $d(f.a)$ and ν are two complex measures on $\mathbf{R}^+$, so $\mathcal{D}$ is shown to be a Dynkin system as follows:

$$
\begin{aligned}
d(f.a)(\mathbf{R}^+) &= \lim_{n\to+\infty} d(f.a)([0,n]) \\
&= \lim_{n\to+\infty} f.a(n) \\
&= \lim_{n\to+\infty} \int_{[0,n]} f da \\
&= \lim_{n\to+\infty} \nu([0,n]) = \nu(\mathbf{R}^+)
\end{aligned}
$$

where the first and last equality stem from exercise (13) of Tutorial 12. So $\mathbf{R}^+ \in \mathcal{D}$. If $A, B \in \mathcal{D}$, $A \subseteq B$, then $B \setminus A \in \mathcal{D}$ is clear. If $A_n \in \mathcal{D}$, $n \geq 1$ and $A_n \uparrow A$, then from exercise (13) of Tutorial 12, we have:

$$
d(f.a)(A) = \lim_{n\to+\infty} d(f.a)(A_n) = \lim_{n\to+\infty} \nu(A_n) = \nu(A)
$$

So $A \in \mathcal{D}$. Having proved that $\mathcal{D}$ is a Dynkin system, it remains to show that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbf{R}^+)$. Since $\mathcal{C} \subseteq \mathcal{B}(\mathbf{R}^+)$, it is clear that $\sigma(\mathcal{C}) \subseteq \mathcal{B}(\mathbf{R}^+)$. To show the reverse inclusion, we need to show that any open set in $\mathbf{R}^+$ is an element of $\sigma(\mathcal{C})$. However, any non-empty open set in $\mathbf{R}$ can be written as a countable union of closed intervals $[a,b]$ with $a \leq b$. It follows that any non-empty open set in $\mathbf{R}^+$ can be written as a countable union of closed intervals $[a,b]$ with $a, b \in \mathbf{R}^+$, $a \leq b$. Since $\emptyset \in \sigma(\mathcal{C})$, all we need to do is show that for all $a, b \in \mathbf{R}^+$, $a \leq b$, we have $[a,b] \in \sigma(\mathcal{C})$. However, if $a = 0$ then $[a,b] \in \mathcal{C} \subseteq \sigma(\mathcal{C})$. If $a > 0$, then $[a,b] = \cap_{n\geq 1}]t_n, b]$ where $(t_n)_{n\geq 1}$ is an arbitrary sequence in $\mathbf{R}^+$ with $t_n \uparrow\uparrow a$. Since $]t_n, b] = [0,b] \setminus [0,t_n] \in \sigma(\mathcal{C})$, we conclude that $[a,b] \in \sigma(\mathcal{C})$.

Exercise 3

Exercise 4.

1. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f \in L^{1,\text{loc}}_{\mathbf{R}}(a)$, $f \geq 0$. Let $t \in \mathbf{R}^+$ and $(t_n)_{n\geq 1}$ be an arbitrary sequence in $\mathbf{R}^+$ such that $t_n \downarrow\downarrow t$. Then $f 1_{[0,t_n]} \to f 1_{[0,t]}$ pointwise, and for all $n \geq 1$, we have:

$$
|f| 1_{[0,t_n]} = f 1_{[0,t_n]} \leq f 1_{[0,t_1]}
$$

while $\int f1_{[0,t_1]}da < +\infty$ since $f \in L^{1,\text{loc}}_{\mathbf{R}}(a)$, $f \geq 0$. From the dominated convergence theorem (23), we obtain:

$$\lim_{n\to+\infty} \int f1_{[0,t_n]}da = \int f1_{[0,t]}da$$

which shows that $f.a(t_n) \to f.a(t)$. We have proved that $f.a$ is right-continuous. Let $s,t \in \mathbf{R}^+$, $s \leq t$. Then, since $f \geq 0$:

$$f.a(s) = \int f1_{[0,s]}da \leq \int f1_{[0,t]}da = f.a(t)$$

So $f.a$ is non-decreasing. Finally, we have:

$$f.a(0) = \int f1_{\{0\}}da = f(0)da(\{0\}) = f(0)a(0) \geq 0$$

We have proved that $f.a$ is right-continuous, non-decreasing with $f.a(0) \geq 0$. In particular, the Stieltjes measure $d(f.a)$ is well-defined, as per definition (24).

2. From theorem (21), $\mu = \int f da$ is a well defined measure on $\mathbf{R}^+$. For all $t \in \mathbf{R}^+$, we have:

$$\begin{aligned} d(f.a)([0,t]) &= d(f.a)(\{0\}) + d(f.a)(]0,t]) \\ &= f.a(0) + f.a(t) - f.a(0) \\ &= \int_{[0,t]} f da = \mu([0,t]) \end{aligned}$$

3. We claim that $d(f.a)([0,T]\cap\cdot) = \mu([0,T]\cap\cdot)$ for all $T \in \mathbf{R}^+$. Define:

$$\mathcal{D} = \{B \in \mathcal{B}(\mathbf{R}^+) \ : \ d(f.a)([0,T]\cap B) = \mu([0,T]\cap B)\} \tag{4}$$

and furthermore:

$$\mathcal{C} = \{[0,t] \ : \ t \in \mathbf{R}^+\}$$

Then $\mathcal{C}$ is closed under finite intersection and since $[0,T]\cap[0,t] = [0,T\wedge t]$ for all $t \in \mathbf{R}^+$, it is clear from 2. that $\mathcal{C} \subseteq \mathcal{D}$. The two measures involved in (4) being finite measures, $\mathcal{D}$ is easily seen to be a Dynkin system on $\mathbf{R}^+$. From the Dynkin system theorem (1), it follows that $\sigma(\mathcal{C}) \subseteq \mathcal{D}$. Finally, since $\sigma(\mathcal{C}) = \mathcal{B}(\mathbf{R}^+)$ (see exercise (3)), we conclude that $\mathcal{B}(\mathbf{R}^+) \subseteq \mathcal{D}$, which shows that $d(f.a)([0,T]\cap\cdot) = \mu([0,T]\cap\cdot)$. The proof that $\mathcal{D}$ is indeed a Dynkin system goes as follows: the fact that $\mathbf{R}^+ \in \mathcal{D}$ follows from 2. and since $f \in L^{1,\text{loc}}_{\mathbf{R}}(a)$, $f \geq 0$:

$$\mu([0,T]\cap\mathbf{R}^+) = d(f.a)([0,T]\cap\mathbf{R}^+) = \int_0^T f da < +\infty$$

which shows that $\mu([0,T]\cap\cdot)$ and $d(f.a)([0,T]\cap\cdot)$ are indeed finite measures. Hence, if $A,B \in \mathcal{D}$ with $A \subseteq B$ we have:

$$d(f.a)([0,T]\cap(B\setminus A)) = d(f.a)([0,T]\cap B) - d(f.a)([0,T]\cap A)$$

$$= \mu([0,T]\cap B) - \mu([0,T]\cap A)$$
$$= \mu([0,T]\cap (B\setminus A))$$

So $B\setminus A\in\mathcal{D}$. Note that the finiteness of the two measures $d(f.a)([0,T]\cap\cdot)$ and $\mu([0,T]\cap\cdot)$ is very important when writing the above equalities. Finally, if $A_n\in\mathcal{D}$, $n\geq 1$ and $A_n\uparrow A$, then $[0,T]\cap A_n\uparrow[0,T]\cap A$, and from theorem (7), we have:

$$\begin{aligned} d(f.a)([0,T]\cap A) &= \lim_{n\to+\infty} d(f.a)([0,T]\cap A_n) \\ &= \lim_{n\to+\infty}\mu([0,T]\cap A_n) \\ &= \mu([0,T]\cap A) \end{aligned}$$

So $A\in\mathcal{D}$, and $\mathcal{D}$ is indeed a Dynkin system on $\mathbf{R}^+$.

4. It follows from 3. that $d(f.a)([0,n]\cap B)=\mu([0,n]\cap B)$ for all $n\geq 1$ and $B\in\mathcal{B}(\mathbf{R}^+)$. Hence, using theorem (7):

$$\begin{aligned} d(f.a)(B) &= \lim_{n\to+\infty} d(f.a)([0,n]\cap B) \\ &= \lim_{n\to+\infty}\mu([0,n]\cap B) \\ &= \mu(B) \\ &= \int_B f da \end{aligned}$$

This completes the proof of theorem (87).

Exercise 4

Exercise 5.

1. Let $a:\mathbf{R}^+\to\mathbf{R}^+$ be right-continuous, non-decreasing with $a(0)\geq 0$. Let $f\in L^{1,\mathrm{loc}}_{\mathbf{C}}(a)$ and $T\in\mathbf{R}^+$. From exercise (24) (part 6) of Tutorial 14, we have $da^{[0,T]}=da^T$. Hence, using definition (45), we obtain:

$$\int |f|1_{[0,T]}da = \int |f|da^{[0,T]} = \int |f|da^T$$

2. Since $f\in L^{1,\mathrm{loc}}_{\mathbf{C}}(a)$, we have:

$$\int |f|1_{[0,T]}da = \int_0^T |f|da < +\infty$$

and furthermore, using 1.:

$$\int |f|da^T = \int |f|1_{[0,T]}da < +\infty$$

So $f1_{[0,T]}\in L^1_{\mathbf{C}}(a)$ and $f\in L^1_{\mathbf{C}}(a^T)$.

3. Let $t \in \mathbf{R}^+$. Using definition (49), and $da^{[0,T]} = da^T$:
$$\begin{aligned}
(f.a)^T(t) &= f.a(T \wedge t) \\
&= \int_0^{T\wedge t} f da \\
&= \int 1_{[0,T\wedge t]} f da \\
&= \int 1_{[0,t]} 1_{[0,T]} f da \\
&= \int 1_{[0,t]} f da^{[0,T]} \\
&= \int 1_{[0,t]} f da^T \\
&= \int_0^t f da^T = f.(a^T)(t)
\end{aligned}$$
So $(f.a)^T = f.(a^T)$ and furthermore:
$$\begin{aligned}
(f.a)^T(t) &= \int 1_{[0,t]} 1_{[0,T]} f da \\
&= \int_0^t 1_{[0,T]} f da = (f 1_{[0,T]}).a(t)
\end{aligned}$$
We have proved that $(f.a)^T = f.(a^T) = (f1_{[0,T]}).a$.

4. Since a and a^T are both right-continuous, non-decreasing with non-negative initial values, since $f \in L^1_{\mathbf{C}}(a^T)$ and $f1_{[0,T]} \in L^1_{\mathbf{C}}(a)$, from theorem (86), both $f.(a^T)$ and $(f1_{[0,T]}).a$ (and therefore also $(f.a)^T$ from 3.) are right-continuous of bounded variation. Furthermore, still from theorem (86), the complex Stieltjes measures $d(f.(a^T))$ and $d((f1_{[0,T]}).a)$ are respectively equal to $\int f da^T$ and $\int f1_{[0,T]} da$. We conclude from 3. that for all $B \in \mathcal{B}(\mathbf{R}^+)$:
$$d(f.a)^T(B) = \int_B f da^T = \int_B f 1_{[0,T]} da$$

5. In order to write $d(f.a)^T = d(f.a)([0,T]\cap\cdot)$, the expression $d(f.a)$ must be meaningful. This is the case when $f.a$ is right-continuous, non-decreasing with $f.a(0) \geq 0$ (definition (24)), or when $f.a$ is right-continuous of bounded variation (definition (110)). However, We have assumed $f \in L^{1,\text{loc}}_{\mathbf{C}}(a)$ and not $f \in L^1_{\mathbf{C}}(a)$. So we cannot apply theorem (86) to conclude that $f.a$ is right-continuous of bounded variation. We only know from exercise (2) that $f.a$ is right-continuous of finite variation. Furthermore, we have not assumed that $f \in L^{1,\text{loc}}_{\mathbf{C}}(a)$ with $f \geq 0$. So we cannot apply theorem (87) to conclude that $f.a$ is right-continuous, non-decreasing with $f.a(0) \geq 0$. We shall see in 9. that $f.a$ is of bounded variation, if and only if $f \in L^1_{\mathbf{C}}(a)$. Short of this condition being satisfied

(or $f \geq 0$), it is not meaningful to write $d(f.a)$. This explains that a lot of care is being taken in this exercise to consider a^T and $(f.a)^T$.

6. Having proved in 4. that $d(f.a)^T = \int f1_{[0,T]}da$, from theorem (63), we have for all $B \in \mathcal{B}(\mathbf{R}^+)$:

$$|d(f.a)^T|(B) = \int_B |f|1_{[0,T]}da$$

7. From exercise (2), $f.a$ is right-continuous of finite variation. Applying theorem (84), we obtain:

$$|d(f.a)^T| = d|f.a|([0,T] \cap \cdot)$$

8. Let $t \in \mathbf{R}^+$. Applying 6. and 7. to $T = t$, we obtain:

$$\begin{aligned} |f.a|(t) &= d|f.a|([0,t]) \\ &= |d(f.a)^t|(\mathbf{R}^+) \\ &= \int |f|1_{[0,t]}da \\ &= \int_0^t |f|da = (|f|.a)(t) \end{aligned}$$

9. From 8. we have for all $t \in \mathbf{R}^+$:

$$|f.a|(t) = \int_0^t |f|da \leq \int |f|da$$

and consequently $\sup_{t\in\mathbf{R}^+} |f.a|(t) \leq \int |f|da$. However, from the monotone convergence theorem (19):

$$\begin{aligned} \int |f|da &= \lim_{n\to+\infty} \int |f|1_{[0,n]}da \\ &= \lim_{n\to+\infty} |f.a|(n) \\ &\leq \sup_{t\in\mathbf{R}^+} |f.a|(t) \end{aligned}$$

So $\int |f|da = \sup_{t\in\mathbf{R}^+} |f.a|(t)$ and $f.a$ is of bounded variation, if and only if $f \in L^1_{\mathbf{C}}(a)$.

10. Having proved in exercise (2) that $f.a$ is right-continuous of finite variation, from exercise (29) (part 4) of Tutorial 14, $f.a$ is cadlag, and consequently it is meaningful to speak of $\Delta(f.a)$, as per definition (111). We have:

$$\begin{aligned} \Delta(f.a)(0) &= f.a(0) \\ &= \int f1_{\{0\}}da \\ &= f(0)da(\{0\}) \\ &= f(0)a(0) \end{aligned}$$

$$= \quad f(0)\Delta a(0)$$

11. Let $t > 0$ and $(t_n)_{n\geq 1}$ be a sequence in $\mathbf{R}^+$ such that $t_n \uparrow\uparrow t$. Then, $f1_{[0,t_n]} \to f1_{[0,t[}$ pointwise and $|f|1_{[0,t_n]} \leq |f|1_{[0,t[}$ for all $n \geq 1$, with:
$$\int |f|1_{[0,t[}da \leq \int |f|1_{[0,t]}da = \int_0^t |f|da < +\infty$$
From the dominated convergence theorem (23), we obtain:
$$\lim_{n\to+\infty} f1_{[0,t_n]}da = \int f1_{[0,t[}da$$

12. Let $t > 0$ and $(t_n)_{n\geq 1}$ be a sequence in $\mathbf{R}^+$ such that $t_n \uparrow\uparrow t$. Using 11. we obtain:
$$\begin{aligned}
\Delta(f.a)(t) &= f.a(t) - f.a(t-) \\
&= f.a(t) - \lim_{n\to+\infty} f.a(t_n) \\
&= f.a(t) - \lim_{n\to+\infty} \int f1_{[0,t_n]}da \\
&= \int f1_{[0,t]}da - \int f1_{[0,t[}da \\
&= \int f1_{\{t\}}da \\
&= f(t)da(\{t\}) \\
&= f(t)\Delta a(t)
\end{aligned}$$
where the last equality stems from exercise (29) (part 5) of Tutorial 14. Having proved in 10. that $\Delta(f.a)(0) = f(0)\Delta a(0)$ we conclude that $\Delta(f.a)(t) = f(t)\Delta a(t)$ for all $t \in \mathbf{R}^+$.

13. Suppose that a is continuous with $a(0) = 0$. Since a is cadlag, from exercise (29) (part 1) of Tutorial 14, we have $\Delta a(t) = 0$ for all $t \in \mathbf{R}^+$. It follows from 12. that $\Delta(f.a)(t) = 0$ for all $t \in \mathbf{R}^+$. Since $f.a$ is right-continuous of finite variation (exercise (2)), in particular it is cadlag (exercise (29) part 4 of Tutorial 14) and consequently from $\Delta(f.a) = 0$ we conclude that $f.a$ is continuous with $f.a(0) = 0$ (exercise (29) (part 1) of Tutorial 14).

14. Given $a : \mathbf{R}^+ \to \mathbf{R}^+$ right-continuous, non-decreasing with $a(0) \geq 0$, and $f \in L^{1,\text{loc}}_{\mathbf{C}}(a)$, we proved in exercise (2) that $f.a$ is right-continuous of finite variation. We proved in 8. that $|f.a| = |f|.a$ and in 9. that $f.a$ is of bounded variation if and only if $f \in L^1_{\mathbf{C}}(a)$. Finally, we proved in 12. that $\Delta(f.a) = f\Delta a$. This completes the proof of theorem (88).

Exercise 5

Exercise 6.

1. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. We want to prove the equivalence between:

$$\begin{array}{cl} (i) & d|b| << da \\ (ii) & |db^T| << da \ , \ \forall T \in \mathbf{R}^+ \\ (iii) & db^T << da \ , \ \forall T \in \mathbf{R}^+ \end{array}$$

   Suppose (i) holds. Let $T \in \mathbf{R}^+$ and $B \in \mathcal{B}(\mathbf{R}^+)$ be such that $da(B) = 0$. Since $d|b| << da$, from definition (96) we have $d|b|(B) = 0$. In particular $d|b|([0,T] \cap B) = 0$. However, from theorem (84), $d|b|([0,T] \cap \cdot) = |db^T|$ and consequently $|db^T|(B) = 0$. This shows that $|db^T| << da$ and we have proved that $(i) \Rightarrow (ii)$. Conversely, suppose (ii) holds, and let $B \in \mathcal{B}(\mathbf{R}^+)$ be such that $da(B) = 0$. Then, for all $T \in \mathbf{R}^+$ we have $|db^T|(B) = 0$. However from theorem (84), $|db^T| = d|b|([0,T] \cap \cdot)$. Since $[0,n] \cap B \uparrow B$, using theorem (7), we obtain:

$$\begin{aligned} d|b|(B) &= \lim_{n \to +\infty} d|b|([0,n] \cap B) \\ &= \lim_{n \to +\infty} |db^n|(B) = 0 \end{aligned}$$

   This shows that $d|b| << da$ and we have proved that $(ii) \Rightarrow (i)$. So (i) and (ii) are equivalent. From exercise (1) of Tutorial 12, $|db^T| << da$ is equivalent to $db^T << da$. We conclude that (ii) and (iii) are equivalent. So (i), (ii) and (iii) are equivalent.

2. No, in general it does not make sense to write $db << da$, as b being right-continuous of finite variation, it need not be right-continuous, non-decreasing with $b(0) \geq 0$, or right-continuous of bounded variation. So it is not meaningful to speak of 'db'.

Exercise 6

Exercise 7.

1. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. We assume that b is absolutely continuous with respect to a, i.e. $b << a$. Let $T \in \mathbf{R}^+$. From definition (113) we have $db^T << da$. Since $[0,n] \uparrow \mathbf{R}^+$ with $da([0,n]) = a(n) < +\infty$ for all $n \geq 1$, the Stieltjes measure da is σ-finite, while db^T is a well-defined complex measure on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$. Applying the Radon-Nikodym theorem (60), there exists $f_T \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), da)$ such that $db^T = \int f_T da$. However, from definition (112), $L^1_{\mathbf{C}}(a) = L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), da)$. So there exists $f_T \in L^1_{\mathbf{C}}(a)$ such that $db^T = \int f_T da$, i.e.:

$$db^T(B) = \int_B f_T da \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

2. Let $T, T' \in \mathbf{R}^+$, $T \le T'$. From exercise (24) of Tutorial 14, db^T is the unique complex measure on $\mathbf{R}^+$ such that:

$$\begin{array}{ll} (i) & db^T(\{0\}) = b(0) \\ (ii) & \forall s,t \in \mathbf{R}^+, s \le t \ ,\ db^T(]s,t]) = b(T\wedge t) - b(T \wedge s) \end{array}$$

However, we have:

$$db^{T'}([0,T]\cap\{0\}) = db^{T'}(\{0\}) = b(0)$$

and furthermore, given $s,t \in \mathbf{R}^+$, $s \le t$:

$$\begin{aligned} db^{T'}([0,T]\cap]s,t]) &= db^{T'}(]T\wedge s, T\wedge t]) \\ &= b(T'\wedge T\wedge t) - b(T'\wedge T\wedge s) \\ &= b(T\wedge t) - b(T\wedge s) \end{aligned}$$

It follows that the two complex measures $db^{T'}([0,T]\cap\cdot)$ and db^T coincide. Hence, for all $B \in \mathcal{B}(\mathbf{R}^+)$:

$$\begin{aligned} \int_{B\cap[0,T]} f_{T'}da &= db^{T'}([0,T]\cap B) \\ &= db^T(B) \\ &= \int_B f_T da \end{aligned}$$

3. Let $g = f_T - f_{T'}1_{[0,T]} \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), da)$. From 2. we have:

$$\int_B g da = 0 \ ,\ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

Using exercise (7) of Tutorial 12, we conclude that $g = 0$ da-a.s. or equivalently, $f_T = f_{T'}1_{[0,T]}$ da-a.s.

4. Let $n,p \in \mathbf{N}$, $1 \le n \le p$. From 3. we have $f_n = f_p 1_{[0,n]}$ da-a.s. and consequently there exists some $N_{n,p} \in \mathcal{B}(\mathbf{R}^+)$ with $da(N_{n,p}) = 0$ such that $f_n(x) = f_p(x)1_{[0,n]}(x)$ for all $x \in N^c_{n,p}$. Define $N = \cup_{1\le n\le p} N_{n,p}$. Then $N \in \mathcal{B}(\mathbf{R}^+)$ and $da(N) = 0$. Furthermore, for all $1 \le n \le p$ we have:

$$\forall x \in N^c \ ,\ f_n(x) = f_p(x)1_{[0,n]}(x)$$

For all $n \ge 1$, define $g_n = f_n 1_{N^c}$. Then $g_n \in L^1_{\mathbf{C}}(a)$ and for all $1 \le n \le p$ we have $g_n = g_p 1_{[0,n]}$. Furthermore since $da(N) = 0$, for all $B \in \mathcal{B}(\mathbf{R}^+)$, using 1. we obtain:

$$\begin{aligned} db^n(B) &= \int_B f_n da \\ &= \int_B f_n 1_{N^c} da + \int_B f_n 1_N da \\ &= \int_B f_n 1_{N^c} da = \int_B g_n da \end{aligned}$$

Renaming the g_n's as f_n's, we have found a sequence $(f_n)_{n\geq 1}$ in $L^1_{\mathbf{C}}(a)$ such that for all $1 \leq n \leq p$, $f_n = f_p 1_{[0,n]}$ and:

$$\forall n \geq 1\ ,\ db^n(B) = \int_B f_n da\ ,\ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

5. Let $f : \mathbf{R}^+ \to \mathbf{C}$ be defined by $f(t) = f_n(t)$, where $n \geq 1$ is any integer such that $t \in [0, n]$. Suppose $n, p \geq 1$ are two integers such that $t \in [0, n]$ and $t \in [0, p]$. Without loss of generality, we may assume that $n \leq p$. From 4. we have $f_n = f_p 1_{[0,n]}$, and since $t \in [0, n]$, we conclude that $f_n(t) = f_p(t)$. So f is unambiguously defined, i.e. f is well-defined.

6. Let $B \in \mathcal{B}(\mathbf{C})$. Suppose $t \in \{f \in B\}$. Then $t \in \mathbf{R}^+$ and $f(t) \in B$. Let $n \geq 1$ be such that $t \in [0, n]$. Then $f(t) = f_n(t)$ and consequently $f_n(t) \in B$. So $t \in [0, n] \cap \{f_n \in B\}$. This shows that $\{f \in B\} \subseteq \cup_{n\geq 1}[0, n] \cap \{f_n \in B\}$. To show the reverse inclusion, suppose that $t \in [0, n] \cap \{f_n \in B\}$ for some $n \geq 1$. Then $t \in [0, n]$ and $f_n(t) \in B$. But $f_n(t) = f(t)$. So $f(t) \in B$ and we have shown that $\cup_{n\geq 1}[0, n] \cap \{f_n \in B\} \subseteq \{f \in B\}$. Finally, we have proved that:
$$\{f \in B\} = \bigcup_{n=1}^{+\infty} [0, n] \cap \{f_n \in B\} \tag{5}$$

7. Let $B \in \mathcal{B}(\mathbf{C})$. From 4. each f_n is an element of $L^1_{\mathbf{C}}(a)$, and in particular $f_n : (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is a measurable map. Hence, $\{f_n \in B\} \in \mathcal{B}(\mathbf{R}^+)$. It follows from (5) that $\{f \in B\} \in \mathcal{B}(\mathbf{R}^+)$, which shows that $f : (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is measurable.

8. Let $t \in \mathbf{R}^+$ and $n \geq 1$ be such that $t \in [0, n]$. Then:
$$\begin{aligned}
\int_0^t |f| da &= \int |f| 1_{[0,t]} da \\
&\leq \int |f| 1_{[0,n]} da \\
&= \int |f_n| 1_{[0,n]} da \\
&\leq \int |f_n| da < +\infty
\end{aligned}$$
where we have used that fact that $f_n \in L^1_{\mathbf{C}}(a)$. Since f is measurable, we conclude from definition (112) that $f \in L^{1,\text{loc}}_{\mathbf{C}}(a)$. Furthermore, given $t \in \mathbf{R}^+$ and $n \geq 1$ such that $t \in [0, n]$:
$$\begin{aligned}
\int_0^t f da &= \int f 1_{[0,t]} da \\
&= \int f 1_{[0,t]} 1_{[0,n]} da
\end{aligned}$$

$$\begin{aligned} &= \int f_n 1_{[0,t]} 1_{[0,n]} da \\ &= \int_{[0,t]} f_n da \\ &= db^n([0,t]) \\ &= b^n(t) \\ &= b(n \wedge t) = b(t) \end{aligned}$$

which shows that $b = f.a$.

9. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. If b is absolutely continuous with respect to a, i.e. $b << a$, From 8. there exists $f \in L^{1,\text{loc}}_{\mathbf{C}}(a)$ such that $b = f.a$, i.e.:

$$b(t) = \int_0^t f da \ , \ \forall t \in \mathbf{R}^+$$

Conversely, suppose there exists $f \in L^{1,\text{loc}}_{\mathbf{C}}(a)$ such that $b = f.a$. Then, from theorem (88), the total variation map of b is given by $|b| = |f.a| = |f|.a$. It follows from theorem (87) that the Stieltjes measure $d|b|$ is given by:

$$d|b|(B) = d(|f|.a)(B) = \int_B |f| da \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

Hence, if $da(B) = 0$, it is clear that $d|b|(B) = 0$, which shows that $d|b| << da$, i.e. b is absolutely continuous with respect to a. We have proved the equivalence stated in theorem (89). Going back to 5., f was defined from the f_n's as $f(t) = f_n(t)$ for any $n \geq 1$ with $t \in [0, n]$. Each f_n was fundamentally obtained in 1. (before some *cleaning up* in 4.) from an application of the Radon-Nikodym theorem (60). Suppose now that b has values in $\mathbf{R}$. Given $n \geq 1$, we claim that the complex measure db^n is in fact a signed measure (i.e. a complex measure with values in $\mathbf{R}$). Indeed the complex measure $Re(db^n)$ is such that:

$$Re(db^n)(\{0\}) = Re(db^n(\{0\})) = Re(b(0)) = b(0)$$

and furthermore, if $s, t \in \mathbf{R}^+$, $s \leq t$:

$$Re(db^n)(]s,t]) = Re(b(n \wedge t) - b(n \wedge s)) = b(n \wedge t) - b(n \wedge s)$$

and from the uniqueness property proved in exercise (24) of Tutorial 14, we conclude that $Re(db^n) = db^n$, and db^n is indeed a signed measure. From theorem (60), it follows that each f_n may be assumed to be $\mathbf{R}$-valued, and consequently $f \in L^{1,\text{loc}}_{\mathbf{C}}(a)$ may be assumed to lie in $L^{1,\text{loc}}_{\mathbf{R}}(a)$. Suppose now that b is non-decreasing (so with values in $\mathbf{R}$) with $b(0) \geq 0$. Given $n \geq 1$, the complex Stieltjes measure db^n is in fact a finite measure on

$(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$, and from theorem (60), each f_n may be assumed to be non-negative. This shows that f may be assumed to be $\mathbf{R}$-valued with $f \geq 0$. This completes the proof of theorem (89).

Exercise 7

Exercise 8.

1. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f, g \in L^{1,\text{loc}}_{\mathbf{C}}(a)$ be such that $f.a = g.a$, i.e.:

$$\int_0^t f da = \int_0^t g da \ , \ \forall t \in \mathbf{R}^+$$

Let $T \in \mathbf{R}^+$ and $B \in \mathcal{B}(\mathbf{R}^+)$. Using 4. of exercise (5), we have:

$$\begin{aligned} \int_B f 1_{[0,T]} da &= d(f.a)^T(B) \\ &= d(g.a)^T(B) \\ &= \int_B g 1_{[0,T]} da \end{aligned}$$

2. Let $h = (f - g)1_{[0,T]}$. From 2. of exercise (5), $h \in L^1_{\mathbf{C}}(a)$. So h is an element of $L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), da)$ and furthermore from 1.:

$$\int_B h da = 0 \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

Using exercise (7) of Tutorial 12, we conclude that $h = 0$ da-a.s. or equivalently $f 1_{[0,T]} = g 1_{[0,T]}$ da-a.s.

3. Given $n \geq 1$, from 2. we have $f 1_{[0,n]} = g 1_{[0,n]}$ da-a.s.. There exists $N_n \in \mathcal{B}(\mathbf{R}^+)$ with $da(N_n) = 0$ such that:

$$f(x) 1_{[0,n]}(x) = g(x) 1_{[0,n]}(x)$$

for all $x \in N_n^c$. Let $N = \cup_{n \geq 1} N_n$. Then $N \in \mathcal{B}(\mathbf{R}^+)$, $da(N) = 0$ and furthermore for all $x \in N^c$, we have $f(x)1_{[0,n]}(x) = g(x)1_{[0,n]}(x)$ for all $n \geq 1$. So $f(x) = g(x)$ for all $x \in N^c$, and we have proved that $f = g$ da-a.s.

Exercise 8

Exercise 9.

1. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. The total variation map $|b|$ is right-continuous, non-decreasing with $|b|(0) \geq 0$. Applying theorem (89) to b and $|b|$, there exists $h \in L^{1,\text{loc}}_{\mathbf{C}}(|b|)$ such that $b = h.|b|$.

2. Let $T \in \mathbf{R}^+$ and $B \in \mathcal{B}(\mathbf{R}^+)$. From 4. of exercise (5), we have:

$$db^T(B) = d(h.|b|)^T(B) = \int_B h d|b|^T$$

However, from theorem (84), we have $d|b|^T = |db^T|$. Hence:

$$db^T(B) = \int_B hd|b|^T = \int_B h|db^T|$$

3. Let $T \in \mathbf{R}^+$. From theorem (63), the total variation of the complex measure $\int h|db^T|$ is equal to $\int |h||db^T|$. Hence from 2., for all $B \in \mathcal{B}(\mathbf{R}^+)$ we have:
$$\int_B |h||db^T| = |db^T|(B) = \int_B 1|db^T|$$
Using exercise (7) of Tutorial 12, $|h| = 1$ $|db^T|$-a.s.

4. Let $T \in \mathbf{R}^+$. We have proved in 3. that $|h| = 1$, $|db^T|$-a.s.. Hence, there exists $N \in \mathcal{B}(\mathbf{R}^+)$ with $|db^T|(N) = 0$ such that $|h|(x) = 1$ for all $x \in N^c$. It follows that $\{|h| \neq 1\} \subseteq N$ and consequently from theorem (84):
$$\begin{array}{rcl} d|b|([0,T] \cap \{h \neq 1\}) & = & |db^T|(\{h \neq 1\}) \\ & \leq & |db^T|(N) = 0 \end{array}$$

5. From 4. we have $d|b|([0,n] \cap \{|h| \neq 1\}) = 0$ for all $n \geq 1$, and since $[0,n] \cap \{|h| \neq 1\} \uparrow \{|h| \neq 1\}$, from theorem (7):
$$d|b|(\{|h| \neq 1\}) = \lim_{n \to +\infty} d|b|([0,n] \cap \{|h| \neq 1\}) = 0$$
Taking $N = \{|h| \neq 1\}$ we have found $N \in \mathcal{B}(\mathbf{R}^+)$ such that $d|b|(N) = 0$ and $|h|(x) = 1$ for all $x \in N^c$. This shows that $|h| = 1$, $d|b|$-a.s.

6. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. From 1. there exists $h \in L^{1,\text{loc}}_{\mathbf{C}}(|b|)$ such that $b = h.|b|$. However from 5. we have $|h| = 1$, $d|b|$-a.s.. Let $N \in \mathcal{B}(\mathbf{R}^+)$ be such that $d|b|(N) = 0$ and $|h|(x) = 1$ for all $x \in N^c$. Defining:
$$h^* = h1_{N^c} + 1_N$$
Then h^* is measurable, and is $d|b|$-almost surely equal to h. So $h^* \in L^{1,\text{loc}}_{\mathbf{C}}(|b|)$. Furthermore, $|h^*| = 1$ and for all $t \in \mathbf{R}^+$:
$$\begin{array}{rcl} b(t) & = & \int_0^t hd|b| \\ & = & \int h1_{[0,t]}d|b| \\ & = & \int h^*1_{[0,t]}d|b| \\ & = & \int_0^t h^*d|b| \end{array}$$
Renaming h^* by h, we have found $h \in L^{1,\text{loc}}_{\mathbf{C}}(|b|)$ with $|h| = 1$, such that $b = h.|b|$. This completes the proof of theorem (90).

Exercise 9

Exercise 10.

1. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. Let $f \in L^1_{\mathbf{C}}(b)$. Let $h \in L^{1,\mathrm{loc}}_{\mathbf{C}}(|b|)$ be such that $|h| = 1$ and $b = h.|b|$. Then fh is measurable, and since $f \in L^1_{\mathbf{C}}(b)$:
$$\int |fh|d|b| = \int |f|d|b| < +\infty$$
Hence, the integral $\int fhd|b|$ is well-defined.

2. Let $f \in L^1_{\mathbf{C}}(b)$. Suppose h^* is another element of $L^{1,\mathrm{loc}}_{\mathbf{C}}(|b|)$ with $|h^*| = 1$ and $b = h^*.|b|$. Then, for all $t \in \mathbf{R}^+$, we have:
$$b(t) = \int_0^t h^* d|b| = \int_0^t hd|b|$$
From exercise (8) it follows that h and h^* are equal $d|b|$-almost surely. Hence:
$$\int fh^* d|b| = \int fhd|b|$$
which shows that the integral $\int fhd|b|$ does not depend on the particular choice of $h \in L^{1,\mathrm{loc}}_{\mathbf{C}}(|b|)$ with $|h| = 1$ and $b = h.|b|$. It follows that $\int fdb$ as defined in (114) is unambiguously defined.

3. Suppose b is in fact real-valued, right-continuous, non-decreasing with $b(0) \geq 0$. Then db is well-defined as a Stieltjes measure on $\mathbf{R}^+$, as per definition (24). Since $|b| = b$, it is possible to choose $h = 1$ to obtain $h \in L^{1,\mathrm{loc}}_{\mathbf{C}}(|b|)$ with $|h| = 1$ and $b = h.|b|$. Indeed for all $t \in \mathbf{R}^+$, we have:
$$\begin{aligned} h.|b|(t) &= \int_0^t hd|b| \\ &= \int h1_{[0,t]}d|b| \\ &= \int 1_{[0,t]}d|b| \\ &= d|b|([0,t]) = |b|(t) = b(t) \end{aligned}$$
Given $f \in L^1_{\mathbf{C}}(b)$, the integral $\int fdb$ as defined in (114) is equal to $\int fhd|b|$. Since $h = 1$ and $|b| = b$, such integral is equal to $\int fdb$ where db is the Stieltjes measure as defined in (24). Hence, we see that the Stieltjes integral $\int fdb$ as defined in (114) coincides with the integral $\int fdb$ with respect to the Stieltjes measure db as defined in (24).

4. Suppose b is right-continuous of bounded variation. Then db is meaningful as the complex Stieltjes measure, as defined in (110). Given $f \in L^1_{\mathbf{C}}(b)$, the integral $\int fdb$ is meaningful, as per definition (97). However, the notation

$\int f db$ is also used to refer to the Stieltjes integral, as defined in (114). Hence, we need to check that the two definitions do not conflict with one another, i.e. that the two integrals do in fact coincide. Let $h \in L^{1,\mathrm{loc}}_{\mathbf{C}}(|b|)$ be such that $|h| = 1$ with $b = h.|b|$. Since b is of bounded variation, h is in fact an element of $L^1_{\mathbf{C}}(b)$. Indeed:

$$\begin{aligned} \int |h| d|b| &= \int d|b| \\ &= d|b|(\mathbf{R}^+) = |b|(\infty) < +\infty \end{aligned}$$

From theorem (86), $db = d(h.|b|)$ is given by:

$$db(B) = d(h.|b|)(B) = \int_B h d|b| \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

Furthermore from theorem (84), $d|b| = |db|$ and consequently:

$$db(B) = \int_B h|db| \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

It follows that $h \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$ is such that $|h| = 1$ and $db = \int h|db|$. From definition (97), the integral $\int f db$ with respect to the complex measure db is equal to $\int f h|db|$, and since $|db| = d|b|$, such integral is itself equal $\int f h d|b|$, which is exactly $\int f db$ as defined in (114). We conclude that the two integrals $\int f db$ as defined in (97) and (114), do in fact coincide.

Exercise 10

Exercise 11.

1. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. Let $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(b)$. Then, for all $t \in \mathbf{R}^+$:

$$\int |f| 1_{[0,t]} d|b| = \int_0^t |f| d|b| < +\infty$$

So $f1_{[0,t]}$ is an element of $L^1_{\mathbf{C}}(b)$. Hence, the integral:

$$\int_0^t f db \stackrel{\triangle}{=} \int f 1_{[0,t]} db$$

is well-defined by virtue of definition (114). We conclude that the map $f.b : \mathbf{R}^+ \to \mathbf{C}$ is well-defined.

2. Suppose b is right-continuous, non-decreasing with $b(0) \geq 0$. Then $b = |b|$ and we can choose $h = 1$ to obtain $h \in L^{1,\mathrm{loc}}_{\mathbf{C}}(|b|)$ with $|h| = 1$ and $b = h.|b|$. Given $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(b)$ and $t \in \mathbf{R}^+$, from definition (114), we have:

$$f.b(t) \stackrel{\triangle}{=} \int_0^t f db$$

$$\begin{aligned}
&\stackrel{\triangle}{=} \int f1_{[0,t]}db \\
&= \int fh1_{[0,t]}d|b| \\
&= \int f1_{[0,t]}d|b| \\
&= \int f1_{[0,t]}db \\
&= \int_0^t fdb
\end{aligned}$$

where this last integral is that of theorem (88). So $f.b$ as defined in this exercise, coincides with $f.b$ as defined in theorem (88).

3. Let $h \in L^{1,\mathrm{loc}}_{\mathbf{C}}(|b|)$ with $|h| = 1$ and $b = h.|b|$. For all $t \in \mathbf{R}^+$:

$$\begin{aligned}
f.b(t) &\stackrel{\triangle}{=} \int f1_{[0,t]}db \\
&= \int fh1_{[0,t]}d|b| \\
&= \int_0^t fhd|b| = (fh).|b|(t)
\end{aligned}$$

It follows that $f.b = (fh).|b|$.

4. From $f.b = (fh).|b|$ and theorem (88), $f.b$ is right-continuous of finite variation, and furthermore:

$$|f.b| = |(fh).|b|| = |fh|.|b| = |f|.|b|$$

or equivalently:

$$|f.b|(t) = \int_0^t |f|d|b| \ , \ \forall t \in \mathbf{R}^+$$

5. From 4. and the monotone convergence theorem (19), we have:

$$\begin{aligned}
|f.b|(\infty) &= \lim_{n\to+\infty} |f.b|(n) \\
&= \lim_{n\to+\infty} \int |f|1_{[0,n]}d|b| \\
&= \int |f|d|b|
\end{aligned}$$

Hence, given $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(b)$, $|f.b|(\infty) < +\infty$ is equivalent to $\int |f|d|b| < +\infty$ which is itself equivalent to $f \in L^1_{\mathbf{C}}(b)$. So $f.b$ is of bounded variation, if and only if $f \in L^1_{\mathbf{C}}(b)$.

6. From 4. the map $f.b$ is right-continuous of finite variation. It follows from exercise (29) (part 4) of Tutorial 14 that $f.b$ is cadlag. From definition (111), $\Delta(f.b)$ is therefore well-defined. From 3. we have $f.b =$

$(fh).|b|$. Applying theorem (88) to $|b|$ and $fh \in L^{1,\text{loc}}_{\mathbf{C}}(|b|)$, we obtain $\Delta(f.b) = (fh)\Delta|b|$. However, from $b = h.|b|$ and theorem (88), we have $\Delta b = h\Delta|b|$. Hence, we conclude that $\Delta(f.b) = f\Delta b$.

7. Since b and $f.b$ are right-continuous of finite variation, they are both cadlag maps. From exercise (29) (part 1) of Tutorial 14, if b is continuous with $b(0) = 0$, then $\Delta b = 0$ and consequently:

$$\Delta(f.b) = f\Delta b = 0$$

It follows from this same exercise (29) that $f.b$ is continuous with $f.b(0) = 0$.

8. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. Let $f \in L^{1,\text{loc}}_{\mathbf{C}}(b)$. We saw in 1. that $f.b$ is well-defined, and in 4. that it is right-continuous of finite variation with $|f.b| = |f|.|b|$. We saw in 5. that $f.b$ is of bounded variation, if and only if $f \in L^1_{\mathbf{C}}(b)$. Finally, we say in 6. that $\Delta(f.b) = f\Delta b$. This completes the proof of theorem (91)

Exercise 11

Exercise 12.

1. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. Let $f \in L^{1,\text{loc}}_{\mathbf{C}}(b)$ and $T \in \mathbf{R}^+$. From definition (45) we have:

$$\int |f|1_{[0,T]}d|b| = \int |f|d|b|^{[0,T]}$$

Furthermore, from exercise (24) of Tutorial 14:

$$d|b|^{[0,T]} \stackrel{\triangle}{=} d|b|([0,T] \cap \cdot) = d|b|^T = d|b^T|$$

Hence, we conclude that:

$$\int |f|1_{[0,T]}d|b| = \int |f|d|b|^{[0,T]} = \int |f|d|b^T|$$

2. Since $f \in L^{1,\text{loc}}_{\mathbf{C}}(b)$, using 1. we obtain:

$$\int |f|d|b^T| = \int |f|1_{[0,T]}d|b| = \int_0^T |f|d|b| < +\infty$$

It follows that $f \in L^1_{\mathbf{C}}(b^T)$ and $f1_{[0,T]} \in L^1_{\mathbf{C}}(b)$.

3. Let $h \in L^{1,\text{loc}}_{\mathbf{C}}(|b|)$ with $|h| = 1$ and $b = h.|b|$. Let $t \in \mathbf{R}^+$:

$$\begin{aligned} b^T(t) &= b(T \wedge t) \\ &= (h.|b|)(T \wedge t) \\ &= \int h1_{[0,T\wedge t]}d|b| \end{aligned}$$

$$\begin{aligned}
&= \int h1_{[0,t]}1_{[0,T]}d|b| \\
&= \int h1_{[0,t]}d|b|^{[0,T]} \\
&= \int h1_{[0,t]}d|b^T| \\
&= \int_0^t hd|b^T| \\
&= h.|b^T|(t)
\end{aligned}$$

where the fifth equality stems from definition (49) and the sixth from $d|b|^{[0,T]} = d|b^T|$ (exercise (24) of Tutorial 14). We conclude that $b^T = h.|b^T|$.

4. For all $t \in \mathbf{R}^+$, we have:

$$\begin{aligned}
(f.b)^T(t) &= (f.b)(T \wedge t) \\
&= \int f1_{[0,T\wedge t]}db \\
&= \int f1_{[0,T]}1_{[0,t]}db \\
&= \int_0^t f1_{[0,T]}db \\
&= (f1_{[0,T]}).b(t)
\end{aligned}$$

It follows that $(f.b)^T = (f1_{[0,T]}).b$. Furthermore, for all $t \in \mathbf{R}^+$:

$$\begin{aligned}
(f.b)^T(t) &= \int f1_{[0,T]}1_{[0,t]}db \\
&= \int fh1_{[0,T]}1_{[0,t]}d|b| \\
&= \int fh1_{[0,t]}d|b|^{[0,T]} \\
&= \int fh1_{[0,t]}d|b^T| \\
&= \int f1_{[0,t]}db^T \\
&= \int_0^t fdb^T \\
&= f.(b^T)(t)
\end{aligned}$$

where the second equality stems from definition (114), the third from definition (49), the fourth from $d|b|^{[0,T]} = d|b^T|$ and the fifth from definition (114) and the fact proved in 3. that $|h| = 1$ with $b^T = h.|b^T|$. It follows that $(f.b)^T = f.(b^T)$ and we have proved that $(f.b)^T = (f1_{[0,T]}).b = f.(b^T)$.

5. From theorem (91), the total variation map $|f.b|$ is given by:

$$|f.b|(t) = \int_0^t |f|d|b| \ , \ \forall t \in \mathbf{R}^+$$

From theorem (87), the Stieltjes measure $d|f.b|$ is given by:

$$d|f.b|(B) = \int_B |f|d|b| \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

6. Let $g : \mathbf{R}^+ \to \mathbf{C}$ be Borel-measurable. For all $t \in \mathbf{R}^+$:

$$\begin{aligned} \int_0^t |g|d|f.b| &= \int |g|1_{[0,t]}d|f.b| \\ &= \int |g| \cdot |f|1_{[0,t]}d|b| \\ &= \int_0^t |gf|d|b| \end{aligned}$$

where the second equality stems from theorem (21) and the fact proved in 5. that $d|f.b| = \int |f|d|b|$. We conclude from definition (112) that $g \in L^{1,\text{loc}}_{\mathbf{C}}(f.b)$ if and only if $gf \in L^{1,\text{loc}}_{\mathbf{C}}(b)$.

7. Using 4. and exercise (11) (part 3) together with the fact that $|h| = 1$ with $b^T = h.|b^T|$, we obtain:

$$(f.b)^T = f.(b^T) = (fh).|b^T|$$

Since $f \in L^1_{\mathbf{C}}(b^T)$, we have $fh \in L^1_{\mathbf{C}}(|b^T|)$, and applying theorem (86), the complex Stieltjes measure $d(f.b)^T$ is given by:

$$d(f.b)^T(B) = \int_B fhd|b^T| \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

8. Since $|h| = 1$, we have $h \in L^1_{\mathbf{C}}(|b^T|)$. Indeed:

$$\begin{aligned} \int |h|d|b^T| &= d|b^T|(\mathbf{R}^+) \\ &= d|b|^{[0,T]}(\mathbf{R}^+) \\ &= d|b|([0,T] \cap \mathbf{R}^+) \\ &= d|b|([0,T]) \\ &= |b|(T) < +\infty \end{aligned}$$

From $b^T = h.|b^T|$ and theorem (86), the complex Stieltjes measure db^T is given by:

$$db^T(B) = \int_B hd|b^T| \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

i.e. $db^T = \int hd|b^T|$. However, from theorem (84), we have $|db^T| = d|b|^T = d|b^T|$. It follows that $db^T = \int h|db^T|$ and consequently for all $B \in \mathcal{B}(\mathbf{R}^+)$:

$$\begin{aligned}
\int_B f db^T &= \int f 1_B db^T \\
&= \int f h 1_B |db^T| \\
&= \int f h 1_B d|b^T| \\
&= \int_B f h d|b^T| \\
&= d(f.b)^T(B)
\end{aligned}$$

where the second equality stems from definition (97), the third from theorem (84) and the fact that $|b|^T = |b^T|$, and the fifth from the fact proved in 7. that $d(f.b)^T = \int fhd|b^T|$. We conclude that $d(f.b)^T = \int f db^T$.

9. Let $g \in L^{1,\text{loc}}_{\mathbf{C}}(f.b)$. We have:

$$\begin{aligned}
\int |g| d|(f.b)^T| &= \int |g| d|f.b|^T \\
&= \int |g| d|f.b|^{[0,T]} \\
&= \int |g| 1_{[0,T]} d|f.b| \\
&= \int_0^T |g| d|f.b| < +\infty
\end{aligned}$$

where the first and second equalities stem from exercise (24) of Tutorial 14, and the third from definition (45). Hence, we see that $g \in L^1_{\mathbf{C}}((f.b)^T)$. Let $t \in \mathbf{R}^+$. Then $g1_{[0,t]}$ is also an element of $L^1_{\mathbf{C}}((f.b)^T)$. From theorem (84), we have $d|(f.b)^T| = |d(f.b)^T|$ and $g1_{[0,t]}$ is therefore an element of $L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), d(f.b)^T)$. Having proved in 8. that $d(f.b)^T = \int f db^T$, from theorem (65):

$$\int g 1_{[0,t]} d(f.b)^T = \int g f 1_{[0,t]} db^T \tag{6}$$

10. The two integrals in (6) are integrals with respect to complex measures, as defined in (97). However, since $(f.b)^T$ and b^T are both right-continuous of bounded variation, from exercise (10), these integrals coincide with the Stieltjes integrals as defined in (114). Hence, for all $t \in \mathbf{R}^+$, we have:

$$\int g 1_{[0,t]} d(f.b)^T = g.((f.b)^T)(t)$$

and:

$$\int g f 1_{[0,t]} db^T = (gf).(b^T)(t)$$

We conclude from (6) that:

$$g.((f.b)^T) = (gf).(b^T) \tag{7}$$

11. $f.b$ being right-continuous of finite variation and $g \in L^{1,\mathrm{loc}}_{\mathbf{C}}(f.b)$, we can apply 4. to g and $f.b$ to obtain:

$$(g.(f.b))^T = g.((f.b)^T) \tag{8}$$

Furthermore, from 6. we have $gf \in L^{1,\mathrm{loc}}_{\mathbf{C}}(b)$, and from 4.:

$$((gf).b)^T = (gf).(b^T) \tag{9}$$

From (7), (8)and (9) we conclude that:

$$(g.(f.b))^T = ((gf).b)^T \tag{10}$$

12. Let $t \in \mathbf{R}^+$. For all $T \in \mathbf{R}^+$ from (10) we have:

$$(g.(f.b))(T \wedge t) = ((gf).b)(T \wedge t)$$

In particular for $T = t$, $g.(f.b)(t) = (gf).b(t)$. This being true for all $t \in \mathbf{R}^+$, we have proved that $g.(f.b) = (gf).b$.

13. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation and $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(b)$. Let $g : \mathbf{R}^+ \to \mathbf{C}$ be Borel-measurable. The equivalence:

$$g \in L^{1,\mathrm{loc}}_{\mathbf{C}}(f.b) \ \Leftrightarrow \ gf \in L^{1,\mathrm{loc}}_{\mathbf{C}}(b)$$

was proved in 6, and given $g \in L^{1,\mathrm{loc}}_{\mathbf{C}}(f.b)$, we showed in 12. that $g.(f.b) = (gf).b$. This completes the proof of theorem (92).

Exercise 12

Exercise 13. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. Let $f, g \in L^{1,\mathrm{loc}}_{\mathbf{C}}(b)$ and $\alpha \in \mathbf{C}$. For all $t \in \mathbf{R}^+$, we have:

$$\begin{aligned}
\int_0^t |f + \alpha g| d|b| &= \int |f + \alpha g| 1_{[0,t]} d|b| \\
&\leq \int (|f| + |\alpha| \cdot |g|) 1_{[0,t]} d|b| \\
&= \int |f| 1_{[0,t]} d|b| + |\alpha| \int |g| 1_{[0,t]} d|b| \\
&= \int_0^t |f| d|b| + |\alpha| \int_0^t |g| d|b| < +\infty
\end{aligned}$$

So $f + \alpha g \in L^{1,\mathrm{loc}}_{\mathbf{C}}(b)$. Let $h \in L^{1,\mathrm{loc}}_{\mathbf{C}}(|b|)$ be such that $|h| = 1$ and $b = h.|b|$. Then for all $t \in \mathbf{R}^+$:

$$(f + \alpha g).b(t) = \int_0^t (f + \alpha g) db$$

$$\begin{aligned}
&= \int (f+\alpha g)h1_{[0,t]}d|b| \\
&= \int fh1_{[0,t]}d|b| + \alpha\int gh1_{[0,t]}d|b| \\
&= \int_0^t fdb + \alpha\int_0^t gdb \\
&= f.b(t) + \alpha(g.b)(t)
\end{aligned}$$

This being true for all $t \in \mathbf{R}^+$, $(f+\alpha g).b = f.b + \alpha(g.b)$.

Exercise 13

Exercise 14.

1. Let $b, c : \mathbf{R}^+ \to \mathbf{C}$ be two right-continuous maps of finite variation. Let $f \in L^{1,\text{loc}}_{\mathbf{C}}(b) \cap L^{1,\text{loc}}_{\mathbf{C}}(c)$ and $\alpha \in \mathbf{C}$. Let $T \in \mathbf{R}^+$. From exercise (6) of Tutorial 14, $b+\alpha c$ is a map of finite variation, and it is right-continuous. From exercise (24) of Tutorial 14, $d(b+\alpha c)^T$ is the unique complex measure on $\mathbf{R}^+$ with:
$$d(b+\alpha c)^T(\{0\}) = b(0) + \alpha c(0)$$
and for all $s, t \in \mathbf{R}^+$, $s \le t$:
$$d(b+\alpha c)^T(]s,t]) = (b+\alpha c)(T\wedge t) - (b+\alpha c)(T\wedge s)$$
However, db^T and dc^T being two complex measures on $\mathbf{R}^+$, $db^T + \alpha dc^T$ is also a complex measure on $\mathbf{R}^+$, which furthermore, from exercise (24) of Tutorial 14, satisfies:
$$\begin{aligned}
(db^T + \alpha dc^T)(\{0\}) &= db^T(\{0\}) + \alpha dc^T(\{0\}) \\
&= b(0) + \alpha c(0)
\end{aligned}$$
and for all $s, t \in \mathbf{R}^+$, $s \le t$:
$$\begin{aligned}
(db^T + \alpha dc^T)(]s,t]) &= db^T(]s,t]) + \alpha dc^T(]s,t]) \\
&= b(T\wedge t) - b(T\wedge s) \\
&+ \alpha(c(T\wedge t) - c(T\wedge s)) \\
&= (b+\alpha c)(T\wedge t) - (b+\alpha c)(T\wedge s)
\end{aligned}$$
Hence, from the uniqueness property stated above:
$$d(b+\alpha c)^T = db^T + \alpha dc^T$$

2. Using 1., exercise (17) of Tutorial 12 and theorem (84):
$$\begin{aligned}
d|b+\alpha c|^T &= |d(b+\alpha c)^T| \\
&= |db^T + \alpha dc^T| \\
&\le |db^T| + |\alpha dc^T| \\
&= |db^T| + |\alpha|\cdot|dc^T| \\
&= d|b|^T + |\alpha|d|c|^T
\end{aligned}$$

3. Let $B \in \mathcal{B}(\mathbf{R}^+)$. Since $[0,n] \cap B \uparrow B$, using 2. with theorem (7) and exercise (24) of Tutorial 14, we obtain:
$$\begin{aligned} d|b+\alpha c|(B) &= \lim_{n\to+\infty} d|b+\alpha c|([0,n]\cap B) \\ &= \lim_{n\to+\infty} d|b+\alpha c|^n(B) \\ &\leq \lim_{n\to+\infty} (d|b|^n + |\alpha| d|c|^n)(B) \\ &= \lim_{n\to+\infty} (d|b|([0,n]\cap B) + |\alpha| d|c|([0,n]\cap B) \\ &= d|b|(B) + |\alpha| d|c|(B) \\ &= (d|b| + |\alpha| d|c|)(B) \end{aligned}$$
We conclude that $d|b+\alpha c| \leq d|b| + |\alpha| d|c|$.

4. Using 3. with exercise (18) of Tutorial 12, for all $t \in \mathbf{R}^+$:
$$\begin{aligned} \int_0^t |f| d|b+\alpha c| &= \int |f| 1_{[0,t]} d|b+\alpha c| \\ &\leq \int |f| 1_{[0,t]} (d|b| + |\alpha| d|c|) \\ &= \int |f| 1_{[0,t]} d|b| + |\alpha| \int |f| 1_{[0,t]} d|c| \\ &= \int_0^t |f| d|b| + |\alpha| \int_0^t |f| d|c| < +\infty \end{aligned}$$
Hence, we conclude that $f \in L^{1,\text{loc}}_{\mathbf{C}}(b+\alpha c)$.

5. We have proved in 8. of exercise (12) that given $f \in L^{1,\text{loc}}_{\mathbf{C}}(b)$ where b is right-continuous of finite variation, the complex Stieltjes measure $d(f.b)^T$ is given by $d(f.b)^T = \int f db^T$. Applying this result to $b + \alpha c$ (which is indeed right-continuous of finite variation) and $f \in L^{1,\text{loc}}_{\mathbf{C}}(b+\alpha c)$, for all $B \in \mathcal{B}(\mathbf{R}^+)$:
$$d(f.(b+\alpha c))^T(B) = \int_B f d(b+\alpha c)^T$$

6. From 8. of exercise (12) we have $d(f.b)^T = \int f db^T$ and similarly $d(f.c)^T = \int f dc^T$. Hence, using 1. and 5. together with definition (98) and exercise (17) of Tutorial 12, for all $B \in \mathcal{B}(\mathbf{R}^+)$:
$$\begin{aligned} d(f.(b+\alpha c))^T(B) &= \int_B f d(b+\alpha c)^T \\ &= \int f 1_B d(b+\alpha c)^T \\ &= \int f 1_B (db^T + \alpha dc^T) \\ &= \int f 1_B db^T + \alpha \int f 1_B dc^T \end{aligned}$$

$$
\begin{aligned}
&= \int_B f db^T + \alpha \int_B f dc^T \\
&= d(f.b)^T(B) + \alpha d(f.c)^T(B) \\
&= (d(f.b)^T + \alpha d(f.c)^T)(B)
\end{aligned}
$$

7. Evaluating the equality obtained in 6. for $B = [0, t]$:

$$
\begin{aligned}
(f.(b+\alpha c))^T(t) &= d(f.(b+\alpha c))^T([0,t]) \\
&= d(f.b)^T([0,t]) + \alpha d(f.c)^T([0,t]) \\
&= (f.b)^T(t) + \alpha (f.c)^T(t) \\
&= ((f.b)^T + \alpha (f.c)^T)(t)
\end{aligned}
$$

8. Given $t \in \mathbf{R}^+$, it follows from 7.:

$$(f.(b+\alpha c))(T \wedge t) = (f.b)(T \wedge t) + \alpha (f.c)(T \wedge t)$$

   This being true for all $T \in \mathbf{R}^+$, in particular for $T = t$:

$$(f.(b+\alpha c))(t) = (f.b)(t) + \alpha (f.c)(t)$$

   We conclude that $f.(b + \alpha c) = f.b + \alpha (f.c)$.

Exercise 14

Exercise 15.

1. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. Let $b_1 = Re(b)$ and $b_2 = Im(b)$. Then b_1 and b_2 are both right-continuous of finite variation, and $b = b_1 + ib_2$. From 3. of exercise (14), we obtain:

$$d|b| = d|b_1 + ib_2| \leq d|b_1| + d|b_2|$$

2. We want to show that $d|b_1| \leq d|b|$, and $d|b_2| \leq d|b|$. Like on many occasions in this Tutorial and Tutorial 14, we shall resort some *localization technique*, i.e. consider some $T \in \mathbf{R}^+$ together with the complex Stieltjes measure db^T (remember that 'db' does not in general make sense for b right-continuous of finite variation). Since:

$$Re(db^T)(\{0\}) = Re(db^T(\{0\})) = Re(b(0)) = b_1(0)$$

   and furthermore for all $s, t \in \mathbf{R}^+$, $s \leq t$:

$$
\begin{aligned}
Re(db^T)(]s,t]) &= Re(db^T(]s,t])) \\
&= Re(b(T \wedge t) - b(T \wedge s)) \\
&= b_1(T \wedge t) - b_1(T \wedge s)
\end{aligned}
$$

   From the uniqueness property stated in exercise (24) (part 3) of Tutorial 14, we conclude that $Re(db^T) = db_1^T$. However, from exercise (19) of Tutorial 12, we have $|Re(db^T)| \leq |db^T|$ and consequently $|db_1^T| \leq |db^T|$.

Applying theorem (84), we obtain $d|b_1|^T \leq d|b|^T$ and finally, this being true for all $T \in \mathbf{R}^+$, using theorem (7), for all $B \in \mathcal{B}(\mathbf{R}^+)$:

$$\begin{aligned} d|b_1|(B) &= \lim_{n\to+\infty} d|b_1|([0,n]\cap B) \\ &= \lim_{n\to+\infty} d|b_1|^n(B) \\ &\leq \lim_{n\to+\infty} d|b|^n(B) \\ &= \lim_{n\to+\infty} d|b|([0,n]\cap B) \\ &= d|b|(B) \end{aligned}$$

So $d|b_1| \leq d|b|$ and similarly $d|b_2| \leq d|b|$.

3. Since $|b_1| = |b_1|^+ + |b_1|^-$, we have:

$$\begin{aligned} (d|b_1|^+ + d|b_1|^-)(\{0\}) &= d|b_1|^+(\{0\}) + d|b_1|^-(\{0\}) \\ &= |b_1|^+(0) + |b_1|^-(0) = |b_1|(0) \end{aligned}$$

and furthermore, for all $s, t \in \mathbf{R}^+$, $s \leq t$:

$$\begin{aligned} (d|b_1|^+ + d|b_1|^-)(]s,t]) &= d|b_1|^+(]s,t]) + d|b_1|^-(]s,t]) \\ &= |b_1|^+(t) - |b_1|^+(s) + |b_1|^-(t) - |b_1|^-(s) \\ &= |b_1|(t) - |b_1|(s) \end{aligned}$$

From the uniqueness property stated in definition (24), it follows that $d|b_1|^+ + d|b_1|^- = d|b_1|$. Hence, using 2. we obtain $d|b_1|^+ \leq d|b_1| \leq d|b|$ and similarly, $d|b_1|^- \leq d|b|$, $d|b_2|^+ \leq d|b|$ and $d|b_2|^- \leq d|b|$. Now suppose that $f \in L^{1,\text{loc}}_{\mathbf{C}}(b)$. Then, from exercise (18) of Tutorial 12, for all $t \in \mathbf{R}^+$, we have:

$$\begin{aligned} \int_0^t |f| d|b_1|^+ &= \int |f| 1_{[0,t]} d|b_1|^+ \\ &\leq \int |f| 1_{[0,t]} d|b| \\ &= \int_0^t |f| d|b| < +\infty \end{aligned}$$

with similar inequalities involving $|b_1|^-$, $|b_2|^+$ and $|b_2|^-$. Hence:

$$f \in L^{1,\text{loc}}_{\mathbf{C}}(|b_1|^+) \cap L^{1,\text{loc}}_{\mathbf{C}}(|b_1|^-) \cap L^{1,\text{loc}}_{\mathbf{C}}(|b_2|^+) \cap L^{1,\text{loc}}_{\mathbf{C}}(|b_2|^-)$$

Conversely if f belongs to such intersection, using 1. together with exercise (18) of Tutorial 12 we obtain for all $t \in \mathbf{R}^+$:

$$\begin{aligned} \int_0^t |f| d|b| &= \int |f| 1_{[0,t]} d|b| \\ &\leq \int |f| 1_{[0,t]} (d|b_1| + d|b_2|) \end{aligned}$$

$$\begin{aligned}
&= \int |f|1_{[0,t]}d|b_1| + \int |f|1_{[0,t]}d|b_2| \\
&= \int |f|1_{[0,t]}d|b_1|^+ + \int |f|1_{[0,t]}d|b_1|^- \\
&+ \int |f|1_{[0,t]}d|b_2|^+ + \int |f|1_{[0,t]}d|b_2|^- \\
&= \int_0^t |f|d|b_1|^+ + \int_0^t |f|d|b_1|^- \\
&+ \int_0^t |f|d|b_2|^+ + \int_0^t |f|d|b_2|^- < +\infty
\end{aligned}$$

and we conclude that $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(b)$.

4. Let $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(b)$. Then f is an element of all $L^{1,\mathrm{loc}}_{\mathbf{C}}(|b_i|^{\pm})$, and furthermore $b = |b_1|^+ - |b_1|^- + i(|b_2|^+ - |b_2|^-)$ where $|b_1|^+$, $|b_1|^-$, $|b_2|^+$ and $|b_2|^-$ are all right-continuous of finite variation (they are in fact non-decreasing with non-negative initial values). From exercise (14), we obtain:

$$f.b = f.|b_1|^+ - f.|b_1|^- + i(f.|b_2|^+ - f.|b_2|^-)$$

or equivalently, for all $t \in \mathbf{R}^+$:

$$\int_0^t f db = \int_0^t f d|b_1|^+ - \int_0^t f d|b_1|^- + i\left(\int_0^t f d|b_2|^+ - \int_0^t f d|b_2|^-\right)$$

Exercise 15

Exercise 16.

1. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. We define $c : \mathbf{R}^+ \to [0, +\infty]$ as:

$$c(t) \stackrel{\triangle}{=} \inf\{s \in \mathbf{R}^+ :\ t < a(s)\}\ ,\ \forall t \in \mathbf{R}^+$$

with the convention $\inf \emptyset = +\infty$. Let $s, t \in \mathbf{R}^+$, and suppose that $t < a(s)$. Then s is an element of $\{s \in \mathbf{R}^+ : t < a(s)\}$. Since $c(t)$ is a lower bound of this set, we obtain $c(t) \leq s$. We have proved that:

$$t < a(s)\ \Rightarrow\ c(t) \leq s$$

2. Suppose that $c(t) < s$. Since $c(t)$ is the greatest lower-bound of the set $\{u \in \mathbf{R}^+ : t < a(u)\}$, s cannot be such a lower-bound. Hence, there exists $u \in \mathbf{R}^+$ such that $t < a(u)$ and $u < s$. In particular, a being non-decreasing, $a(u) \leq a(s)$. It follows that $t < a(s)$ and we have proved that:

$$c(t) < s\ \Rightarrow\ t < a(s)$$

3. Suppose $c(t) \leq s$ and let $\epsilon > 0$. Then $c(t) < s + \epsilon$ and consequently from 2. we obtain $t < a(s + \epsilon)$. We have proved that:

$$c(t) \leq s\ \Rightarrow\ t < a(s + \epsilon)\ ,\ \forall \epsilon > 0$$

4. Suppose $c(t) \leq s$. Using 3. for all $n \geq 1$, we have $t < a(s+1/n)$. Since a is right-continuous, taking the limit as $n \to +\infty$, we obtain $t \leq a(s)$. We have proved that:
$$c(t) \leq s \;\Rightarrow\; t \leq a(s)$$

5. Suppose $c(t) < +\infty$. Then $\{s \in \mathbf{R}^+ : t < a(s)\}$ is non-empty. Hence, there exists $s \in \mathbf{R}^+$ such that $t < a(s)$. Since $a(s) \leq a(\infty) = \sup_{u\in\mathbf{R}^+} a(u)$, in particular we obtain $t < a(\infty)$. Conversely, suppose that $t < a(\infty)$. Since $a(\infty)$ is the lowest upper-bound of all $a(u)$'s as $u \in \mathbf{R}^+$, t cannot be such an upper-bound. There exists $u \in \mathbf{R}^+$ such that $t < a(u)$, and consequently $c(t) \leq u$. This shows in particular that $c(t) < +\infty$. We have proved the equivalence:
$$c(t) < +\infty \;\Leftrightarrow\; t < a(\infty)$$

6. Let $t, t' \in \mathbf{R}^+$, $t \leq t'$. Suppose $s \in \mathbf{R}^+$ is such that $t' < a(s)$. In particular $t < a(s)$, and consequently $c(t) \leq s$. It follows that $c(t)$ is a lower-bound of the set $\{s \in \mathbf{R}^+ : t' < a(s)\}$. Since $c(t')$ is the greatest of such lower-bounds, we obtain $c(t) \leq c(t')$. This shows that c is non-decreasing.

7. Let $t_0 \in [a(\infty), +\infty[$. Then in particular $a(\infty) \leq t_0$ and from 5. we obtain $c(t_0) = +\infty$. From 6. the map $c : \mathbf{R}^+ \to [0, +\infty]$ is non-decreasing. Hence, for all $t \in \mathbf{R}^+$, $t_0 \leq t$, we have $c(t_0) \leq c(t)$. It follows that $c(t) = +\infty$ for all $t \in \mathbf{R}^+$, $t_0 \leq t$. In particular, $\lim_{t\downarrow t_0} c(t) = +\infty = c(t_0)$. This shows that c is right-continuous at t_0.

8. Let $t_0 \in [0, a(\infty)[$ and $\epsilon > 0$. Since $t_0 < a(\infty)$, from 5. we obtain $c(t_0) < +\infty$. Hence, $c(t_0) < c(t_0) + \epsilon$. Since $c(t_0)$ is the greatest lower-bound of the set$\{s \in \mathbf{R}^+ : t_0 < a(s)\}$, $c(t_0) + \epsilon$ cannot be such a lower-bound. There exists $s \in \mathbf{R}^+$ such that $t_0 < a(s)$ and $s < c(t_0) + \epsilon$. From $t_0 < a(s)$ we obtain $c(t_0) \leq s$. We have found $s \in \mathbf{R}^+$ such that $c(t_0) \leq s < c(t_0) + \epsilon$ and $t_0 < a(s)$.

9. Suppose $t \in [t_0, a(s)[$. From 6. the map c is non-decreasing, and consequently $c(t_0) \leq c(t)$. From $t < a(s)$ we have $c(t) \leq s$, and since $s < c(t_0) + \epsilon$, we obtain $c(t) < c(t_0) + \epsilon$. In particular, $c(t) \leq c(t_0) + \epsilon$. We have proved that:
$$t \in [t_0, a(s)[\;\Rightarrow\; c(t_0) \leq c(t) \leq c(t_0) + \epsilon$$

10. For all $t_0 \in [a(\infty), +\infty[$, we have seen in 7. that c is right-continuous at t_0. Suppose $t_0 \in [0, a(\infty)[$. Given $\epsilon > 0$, we have shown the existence of $s \in \mathbf{R}^+$ such that if $u = a(s)$, then $t_0 < u$ and furthermore:
$$t \in [t_0, u[\;\Rightarrow\; c(t_0) \leq c(t) \leq c(t_0) + \epsilon$$
This shows that $\lim_{t\downarrow t_0} c(t) = c(t_0)$, and c is right-continuous at t_0. Finally, c is right-continuous at t_0 for all $t_0 \in \mathbf{R}^+$. So c is right-continuous.

11. Suppose $a(\infty) = +\infty$. Then for all $t \in \mathbf{R}^+$, we have $t < a(\infty)$. From 5. it follows that $c(t) < +\infty$, and $c : \mathbf{R}^+ \to [0, +\infty]$ is in fact a map with values in $\mathbf{R}^+$. We have shown in 10. that c is right-continuous. We have shown in 6. that c is non-decreasing. If one needs to prove that $c(0) \geq 0$, recall that 0 is a lower-bound of the set $\{s \in \mathbf{R}^+ : 0 < a(s)\}$, and that $c(0)$ is the greatest of such lower-bounds. We have proved that $c : \mathbf{R}^+ \to \mathbf{R}^+$ is right-continuous, non-decreasing with $c(0) \geq 0$.

12. We define $\bar{a} : \mathbf{R}^+ \to [0 + \infty]$ as:

$$\bar{a}(s) = \inf\{t \in \mathbf{R}^+ : \ s < c(t)\} \ , \ \forall s \in \mathbf{R}^+$$

Let $s, t \in \mathbf{R}^+$ and suppose that $s < c(t)$. Then $c(t) \leq s$ is not true. From 1. it follows that $t < a(s)$ is not true, i.e. $a(s) \leq t$.

13. From 12. $a(s)$ is a lower-bound of $\{t \in \mathbf{R}^+ : s < c(t)\}$. Since $\bar{a}(s)$ is the greatest of such lower-bounds, we obtain $a(s) \leq \bar{a}(s)$. This being true for all $s \in \mathbf{R}^+$, we have $a \leq \bar{a}$.

14. Let $s, t \in \mathbf{R}^+$ and $\epsilon > 0$. Suppose that $a(s + \epsilon) \leq t$. Then $t < a(s + \epsilon)$ is not true. From 2. it follows that $c(t) < s + \epsilon$ is not true or equivalently $s + \epsilon \leq c(t)$. Since $s \in \mathbf{R}^+$, we have $s < s + \epsilon \leq c(t)$.

15. Let $s, t \in \mathbf{R}^+$ and $\epsilon > 0$. Suppose $a(s + \epsilon) \leq t$. It follows from 14. that $s < c(t)$. So t is an element of $\{u \in \mathbf{R}^+ : s < c(u)\}$, and since $\bar{a}(s)$ is a lower-bound of this set, we obtain $\bar{a}(s) \leq t$.

16. Let $s \in \mathbf{R}^+$ and suppose that $a(s) < \bar{a}(s)$. Let t be an arbitrary element of $]a(s), \bar{a}(s)[$. Then $a(s) < t$, and from the right-continuity of a, there exists $\epsilon > 0$ such that $a(s + \epsilon) < t$. In particular $a(s + \epsilon) \leq t$ and it follows from 15. that $\bar{a}(s) \leq t$. This contradicts the fact that $t < \bar{a}(s)$. We conclude that $\bar{a}(s) \leq a(s)$. This being true for all $s \in \mathbf{R}^+$, $\bar{a} \leq a$. Having proved in 13. that $a \leq \bar{a}$, we obtain $a = \bar{a}$ or equivalently:

$$a(s) = \inf\{t \in \mathbf{R}^+ : \ s < c(t)\} \ , \ \forall s \in \mathbf{R}^+$$

Exercise 16

Exercise 17.

1. Let $f : \mathbf{R}^+ \to \bar{\mathbf{R}}$ be a non-decreasing map. Let $\alpha \in \mathbf{R}$. We define:

$$x_0 \stackrel{\triangle}{=} \sup\{x \in \mathbf{R}^+ : \ f(x) \leq \alpha\}$$

Suppose $x_0 = -\infty$. Then $\{f \leq \alpha\}$ has to be the empty set. Otherwise, there would exist $x \in \mathbf{R}^+$ with $f(x) \leq \alpha$, and we would have $x \leq x_0$, contradicting $x_0 = -\infty$. Conversely, suppose $\{f \leq \alpha\}$ is the empty set. Then $-\infty$ is an upper-bound of $\{f \leq \alpha\}$ and it is clearly the lowest. So $x_0 = -\infty$. We have proved that $x_0 = -\infty$ if and only if $\{f \leq \alpha\} = \emptyset$.

2. Suppose $x_0 = +\infty$. Let $x \in \mathbf{R}^+$. Then $x < x_0$ and therefore, x cannot be an upper-bound of the set $\{u \in \mathbf{R}^+ : f(u) \leq \alpha\}$. There exits $u \in \mathbf{R}^+$ with $f(u) \leq \alpha$ and $x < u$. In particular, since f is non-decreasing, $f(x) \leq f(u)$. So $f(x) \leq \alpha$. This being true for all $x \in \mathbf{R}^+$, we obtain $\{f \leq \alpha\} = \mathbf{R}^+$. Conversely, suppose $\{f \leq \alpha\} = \mathbf{R}^+$. Then $x_0 = \sup \mathbf{R}^+ = +\infty$. We have proved that $x_0 = +\infty$ if and only if $\{f \leq \alpha\} = \mathbf{R}^+$.

3. We assume that $-\infty < x_0 < +\infty$. So $x_0 \in \mathbf{R}$. However from 1., the set $\{f \leq \alpha\}$ is not empty. There exists $x \in \mathbf{R}^+$ such that $f(x) \leq \alpha$. Since x_0 is an upper-bound of $\{f \leq \alpha\}$, we obtain $x \leq x_0$. In particular, $x_0 \geq 0$. So $x_0 \in \mathbf{R}^+$.

4. Suppose that $f(x_0) \leq \alpha$. If $x \in \mathbf{R}^+$ and $f(x) \leq \alpha$, then $x \leq x_0$. So $\{f \leq \alpha\} \subseteq [0, x_0]$. To show the reverse inclusion, suppose that $x \in [0, x_0]$. If $x = x_0$, by assumption we have $f(x) \leq \alpha$. We assume that $x \in [0, x_0[$. Since x_0 is the lowest of all upper-bounds of $\{f \leq \alpha\}$, x cannot be such an upper-bound. There exists $u \in \mathbf{R}^+$ with $f(u) \leq \alpha$ and $x < u$. In particular, f being non-decreasing, $f(x) \leq f(u)$. So $f(x) \leq \alpha$. We have proved that $[0, x_0] \subseteq \{f \leq \alpha\}$ and finally $\{f \leq \alpha\} = [0, x_0]$.

5. Suppose that $\alpha < f(x_0)$. If $x \in \mathbf{R}^+$ and $f(x) \leq \alpha$, then $x \neq x_0$. Furthermore, we have $x \leq x_0$ and consequently $x \in [0, x_0[$. So $\{f \leq \alpha\} \subseteq [0, x_0[$. An identical reasoning as in 4. shows that $[0, x_0[\subseteq \{f \leq \alpha\}$. We conclude that $\{f \leq \alpha\} = [0, x_0[$.

6. From theorem (15), to show that $f : \mathbf{R}^+ \to \bar{\mathbf{R}}$ is measurable, it is equivalent to show that $\{f \leq \alpha\} \in \mathcal{B}(\mathbf{R}^+)$ for all $\alpha \in \mathbf{R}$. If $x_0 = -\infty$ or $x_0 = +\infty$ then it is clear from 1. and 2. that $\{f \leq \alpha\} \in \mathcal{B}(\mathbf{R}^+)$. If $x_0 \in \mathbf{R}$, then $\{f \leq \alpha\}$ is equal $[0, x_0]$ or $[0, x_0[$, depending on whether $f(x_0) \leq \alpha$ or $\alpha < f(x_0)$. In any case, we have $\{f \leq \alpha\} \in \mathcal{B}(\mathbf{R}^+)$. We have proved that f is measurable. The purpose of this exercise is to show that any non-decreasing map $f : \mathbf{R}^+ \to \bar{\mathbf{R}}$ is Borel-measurable.

Exercise 17

Exercise 18.

1. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $c : \mathbf{R}^+ \to [0, +\infty]$ be defined as:

$$c(t) = \inf\{s \in \mathbf{R}^+ : t < a(s)\} \ , \ \forall t \in \mathbf{R}^+$$

Let $f : \mathbf{R}^+ \to [0, +\infty]$ be non-negative and measurable. The map $f \circ c$ may not be well-defined, since it is possible that $c(t) = +\infty$ for some $t \in \mathbf{R}^+$. The notation $(f \circ c)1_{\{c<+\infty\}}$ may therefore seem controversial, as it formally looks like a product of two well-defined mappings with values in $[0, +\infty]$, when in fact it isn't. As one may have guessed, $(f \circ c)1_{\{c<+\infty\}}$ refers to the mapping defined by:

$$\forall t \in \mathbf{R}^+ \ , \ ((f \circ c)1_{\{c<+\infty\}})(t) = \left\{ \begin{array}{lll} f(c(t)) & , & \text{if } c(t) < +\infty \\ 0 & , & \text{if } c(t) = +\infty \end{array} \right.$$

and this is certainly well-defined and non-negative. To show that this mapping is also measurable, note that $c : \mathbf{R}^+ \to [0, +\infty]$ is measurable, as follows from exercise (17), and the fact that c is non-decreasing, which we have proved in exercise (16). Now, it is impossible to argue that $f \circ c$ (being the composition of two measurable maps) is measurable, and that $(f \circ c)1_{\{c<+\infty\}}$ is therefore measurable as the product of two measurable maps. As we have already indicated, $f \circ c$ is not even well-defined, so a little more care is required: let $B \in \mathcal{B}(\bar{\mathbf{R}})$. We have:

$$\begin{aligned}
\{(f \circ c)1_{\{c<+\infty\}} \in B\} &= \{t \in \mathbf{R}^+ : c(t) < +\infty, f(c(t)) \in B\} \\
&\uplus \{c = +\infty\} \cap \{0 \in B\} \\
&= \{t \in \mathbf{R}^+ : c(t) < +\infty, c(t) \in f^{-1}(B)\} \\
&\uplus \{c = +\infty\} \cap \{0 \in B\} \\
&= \{c < +\infty\} \cap \{c \in f^{-1}(B)\} \\
&\uplus \{c = +\infty\} \cap \{0 \in B\}
\end{aligned}$$

where $\{0 \in B\}$ is just a convenient notation to indicate $\mathbf{R}^+$ or $\emptyset$, depending respectively on whether $0 \in B$ or $0 \notin B$. Since f is measurable and $B \in \mathcal{B}(\bar{\mathbf{R}})$, we have $f^{-1}(B) \in \mathcal{B}(\mathbf{R}^+) \subseteq \mathcal{B}(\bar{\mathbf{R}})$. Since c is measurable, $\{c \in f^{-1}(B)\}$, $\{c < +\infty\}$ and $\{c = +\infty\}$ are all elements of $\mathcal{B}(\mathbf{R}^+)$. We conclude that $\{(f \circ c)1_{\{c<+\infty\}} \in B\} \in \mathcal{B}(\mathbf{R}^+)$, and we have proved that $(f \circ c)1_{\{c<+\infty\}}$ is well-defined, non-negative and measurable.

2. Since $[0, u] \in \mathcal{B}(\mathbf{R}^+)$, the map $1_{[0,u]}$ is non-negative and measurable. It follows from 1. that $(1_{[0,u]} \circ c)1_{\{c<+\infty\}}$ is also non-negative and measurable, and consequently the integral:

$$I_1 = \int_0^{a(t)} (1_{[0,u]} \circ c)1_{\{c<+\infty\}} ds$$

is well-defined. Likewise, $[0, a(t \wedge u)]$ is a Borel set, and since c is measurable, $\{c < +\infty\}$ is also a Borel set of $\mathbf{R}^+$. Hence, the integral:

$$I_2 = \int 1_{[0,a(t\wedge u)]} 1_{\{c<+\infty\}} ds$$

is also well-defined. To show that $I_1 \leq I_2$, let $s \in \mathbf{R}^+$. Then:

$$\begin{aligned}
1_{[0,a(t)]}(s)1_{[0,u]}(c(s)) &= 1_{[0,a(t)]}(s)1_{\{c(s)\leq u\}} \\
&\leq 1_{[0,a(t)]}(s)1_{\{s\leq a(u)\}} \\
&= 1_{[0,a(t)]}(s)1_{[0,a(u)]}(s) \\
&= 1_{[0,a(t)\wedge a(u)]}(s) \\
&= 1_{[0,a(t\wedge u)]}(s)
\end{aligned}$$

where the inequality stems from the fact proven in 4. of exercise (16) that $c(s) \leq u \Rightarrow s \leq a(u)$, and the last equality from the fact that a is non-decreasing. We have proved that $I_1 = \int f(s)ds$ and $I_2 = \int g(s)ds$ where f, g are non-negative and measurable with $f \leq g$. This shows that $I_1 \leq I_2$.

3. From 2. and the fact that:
$$\int 1_{[0,a(t\wedge u)]}1_{\{c<+\infty\}}ds \leq \int 1_{[0,a(t\wedge u)]}ds = a(t\wedge u)$$
we conclude that:
$$\int_0^{a(t)} (1_{[0,u]}\circ c)1_{\{c<+\infty\}}ds \leq a(t\wedge u)$$

4. We have:
$$\begin{aligned} a(t\wedge u) &= ds([0,a(t\wedge u)]) \\ &= \int 1_{[0,a(t\wedge u)]}ds \\ &= \int 1_{[0,a(t)\wedge a(u)]}ds \\ &= \int 1_{[0,a(t)]}1_{[0,a(u)]}ds \\ &= \int 1_{[0,a(t)]}1_{[0,a(u)[}ds \\ &= \int_0^{a(t)} 1_{[0,a(u)[}ds \\ &= \int_0^{a(t)} 1_{[0,a(u)[}1_{\{c<+\infty\}}ds \end{aligned}$$
where the fifth equality stems from the fact the Lebesgue measure has no mass at $a(u)$ (i.e. $ds(\{a(u)\})=0$), and the last equality from the fact proven in 1. of exercise (16) that:
$$s<a(u) \;\Rightarrow\; c(s)\leq u<+\infty \tag{11}$$

5. From (11) we obtain $1_{[0,a(u)[}\leq 1_{[0,u]}\circ c$. Hence, from 4.:
$$\begin{aligned} a(t\wedge u) &= \int_0^{a(t)} 1_{[0,a(u)[}1_{\{c<+\infty\}}ds \\ &\leq \int_0^{a(t)} (1_{[0,u]}\circ c)1_{\{c<+\infty\}}ds \end{aligned}$$

6. It follows from 3. and 5. that:
$$a(t\wedge u) = \int_0^{a(t)} (1_{[0,u]}\circ c)1_{\{c<+\infty\}}ds$$
and consequently, we have:
$$\begin{aligned} \int_0^t 1_{[0,u]}da &= \int 1_{[0,t]}1_{[0,u]}da \\ &= \int 1_{[0,t\wedge u]}da \end{aligned}$$

$$\begin{aligned} &= a(t \wedge u) \\ &= \int_0^{a(t)} (1_{[0,u]} \circ c) 1_{\{c<+\infty\}} \end{aligned}$$

7. Let $\mathcal{D}_t$ be the set of all $B \in \mathcal{B}(\mathbf{R}^+)$ with:

$$\int_0^t 1_B da = \int_0^{a(t)} (1_B \circ c) 1_{\{c<+\infty\}} ds$$

We shall first prove that $\mathcal{D}_t$ is a Dynkin system on $\mathbf{R}^+$. Suppose $(B_n)_{n\geq 1}$ is a sequence of elements of $\mathcal{D}_t$ such that $B_n \uparrow B$ (i.e. $B_n \subseteq B_{n+1}$ for all $n \geq 1$, and $B = \cup_{n\geq 1} B_n$). From the monotone convergence theorem (19), we obtain:

$$\begin{aligned} \int_0^t 1_B da &= \lim_{n\to+\infty} \int_0^t 1_{B_n} da \\ &= \lim_{n\to+\infty} \int 1_{[0,a(t)]} (1_{B_n} \circ c) 1_{\{c<+\infty\}} ds \\ &= \int 1_{[0,a(t)]} (1_B \circ c) 1_{\{c<+\infty\}} ds \\ &= \int_0^{a(t)} (1_B \circ c) 1_{\{c<+\infty\}} ds \end{aligned}$$

and consequently $B \in \mathcal{D}_t$. Having proved in 6. that $[0, n] \in \mathcal{D}_t$ for all $n \geq 1$, from $[0, n] \uparrow \mathbf{R}^+$ we obtain $\mathbf{R}^+ \in \mathcal{D}_t$. Suppose $A, B \in \mathcal{D}_t$ with $A \subseteq B$. Then $B = (B \setminus A) \uplus A$ and consequently:

$$\int_0^t 1_B da = \int_0^t 1_{B\setminus A} da + \int_0^t 1_A da$$

Each integral involved being finite, we have equivalently:

$$\int_0^t 1_{B\setminus A} da = \int_0^t 1_B da - \int_0^t 1_A da$$

Similarly:

$$\begin{aligned} \int_0^{a(t)} (1_{B\setminus A} \circ c) 1_{\{c<+\infty\}} ds &= \int_0^{a(t)} (1_B \circ c) 1_{\{c<+\infty\}} ds \\ &- \int_0^{a(t)} (1_A \circ c) 1_{\{c<+\infty\}} da \end{aligned}$$

and from $A, B \in \mathcal{D}_t$ we conclude that:

$$\int_0^t 1_{B\setminus A} da = \int_0^{a(t)} (1_{B\setminus A} \circ c) 1_{\{c<+\infty\}} ds$$

i.e. $B \setminus A \in \mathcal{D}_t$. We have proved that $\mathcal{D}_t$ is a Dynkin system on $\mathbf{R}^+$. To show that $\mathcal{D}_t = \mathcal{B}(\mathbf{R}^+)$, define $\mathcal{C} = \{[0, u] : u \in \mathbf{R}^+\}$. Then $\mathcal{C}$ is a π-system on $\mathbf{R}^+$ (i.e. it is closed under (non-empty) finite intersection), and we have proved in 6. that $\mathcal{C} \subseteq \mathcal{D}_t$. From the Dynkin system theorem (1), we obtain $\mathcal{B}(\mathbf{R}^+) = \sigma(\mathcal{C}) \subseteq \mathcal{D}_t$. So $\mathcal{D}_t = \mathcal{B}(\mathbf{R}^+)$. If anyone still needs a proof that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbf{R}^+)$, please note that any open set in $\mathbf{R}$ is a countable union of intervals of the form $]a, b]$, and it is therefore sufficient to show that $]a, b] \cap \mathbf{R}^+ \in \sigma(\mathcal{C})$ for all $a \leq b$. But this follows from the fact that $]a, b] \cap \mathbf{R}^+$ is either of the form $[0, b] \in \mathcal{C}$, or of the form $]a, b] = [0, b] \setminus [0, a] \in \sigma(\mathcal{C})$.

8. Given $f : \mathbf{R}^+ \to [0, +\infty]$ non-negative and measurable and $t \in \mathbf{R}^+$, we wish to establish the formula:

$$\int_0^t f da = \int_0^{a(t)} (f \circ c) 1_{\{c<+\infty\}} ds \tag{12}$$

From 7. if f is of the form $f = 1_B$ with $B \in \mathcal{B}(\mathbf{R}^+)$, then (12) is true. By linearity, if f is a simple function on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$, then (12) is also true for f. Suppose f is an arbitrary non-negative and measurable map. From theorem (18) there exists a sequence $(s_n)_{n\geq 1}$ of simple functions such that $s_n \uparrow f$. Having established (12) for each s_n, $n \geq 1$, we have:

$$\int_0^t s_n da = \int_0^{a(t)} (s_n \circ c) 1_{\{c<+\infty\}} ds$$

From the monotone convergence theorem (19), taking the limit as $n \to +\infty$, we conclude that equation (12) is true for f.

9. Let $f : \mathbf{R}^+ \to \mathbf{C}$ be measurable. Let $u = Re(f)$ and $v = Im(f)$. Let $u_1 = u^+$, $u_2 = u^-$, $u_3 = v^+$ and $u_4 = v^-$. Then each $u_i : \mathbf{R}^+ \to \mathbf{R}^+$ is non-negative and measurable, and from 1. each $(u_i \circ c) 1_{\{c<+\infty\}}$ is non-negative and measurable. Furthermore, $(f \circ c) 1_{\{c<+\infty\}}$ is clearly well-defined, and we have:

$$\begin{aligned} (f \circ c) 1_{\{c<+\infty\}} &= (u_1 \circ c) 1_{\{c<+\infty\}} \\ &- (u_2 \circ c) 1_{\{c<+\infty\}} \\ &+ i(u_3 \circ c) 1_{\{c<+\infty\}} \\ &- i(u_4 \circ c) 1_{\{c<+\infty\}} \end{aligned}$$

We conclude that $(f \circ c) 1_{\{c<+\infty\}}$ is measurable.

10. Let $f \in L^{1,\text{loc}}_{\mathbf{C}}(a)$ and $t \in \mathbf{R}^+$. Applying formula (12) to $|f|$:

$$\begin{aligned} \int |(f \circ c) 1_{\{c<+\infty\}} 1_{[0,a(t)]}| ds &= \int (|f| \circ c) 1_{\{c<+\infty\}} 1_{[0,a(t)]} ds \\ &= \int_0^{a(t)} (|f| \circ c) 1_{\{c<+\infty\}} ds \\ &= \int_0^t |f| da < +\infty \end{aligned}$$

It follows that $(f \circ c)1_{\{c<+\infty\}}1_{[0,a(t)]} \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), ds)$. Furthermore, Since $u_1, u_2 \leq |u| \leq |f|$ and $u_3, u_4 \leq |v| \leq |f|$, each $(u_i \circ c)1_{\{c<+\infty\}}1_{[0,a(t)]}$ is itself and element of L^1. From the linearity of the integral, we obtain:

$$\begin{aligned} \int_0^{a(t)} (f \circ c)1_{\{c<+\infty\}} ds &= \int_0^{a(t)} (u_1 \circ c)1_{\{c<+\infty\}} ds \\ &- \int_0^{a(t)} (u_2 \circ c)1_{\{c<+\infty\}} ds \\ &+ i \int_0^{a(t)} (u_3 \circ c)1_{\{c<+\infty\}} ds \\ &- i \int_0^{a(t)} (u_4 \circ c)1_{\{c<+\infty\}} ds \end{aligned}$$

However, applying 8. to each u_i:

$$\int_0^t u_i da = \int_0^{a(t)} (u_i \circ c)1_{\{c<+\infty\}} ds$$

and consequently:

$$\begin{aligned} \int_0^{a(t)} (f \circ c)1_{\{c<+\infty\}} ds &= \int_0^t u_1 da - \int_0^t u_2 da \\ &+ i \int_0^t u_3 da - i \int_0^t u_4 da \\ &= \int_0^t f da \end{aligned}$$

11. Similarly to the case of 1., the map $(f \circ c)1_{[0,a(t)[}$ is not strictly speaking the product of $f \circ c$ with $1_{[0,a(t)[}$, for the simple reason that $f \circ c$ may not be well-defined. However, it does not take much to guess that the notation $(f \circ c)1_{[0,a(t)[}$ refers to the map defined by:

$$((f \circ c)1_{[0,a(t)[})(s) = \left\{ \begin{array}{lll} f(c(s)) & , & \text{if } s < a(t) \\ 0 & , & \text{if } a(t) \leq s \end{array} \right.$$

Since $s < a(t) \Rightarrow c(s) \leq t < +\infty$, such a map is well-defined, and furthermore:

$$(f \circ c)1_{[0,a(t)[} = ((f \circ c)1_{\{c<+\infty\}})1_{[0,a(t)[}$$

Hence, from 10. we have:

$$\begin{aligned} \int_0^t f da &= \int_0^{a(t)} (f \circ c)1_{\{c<+\infty\}} da \\ &= \int (f \circ c)1_{\{c<+\infty\}}1_{[0,a(t)]} ds \\ &= \int (f \circ c)1_{\{c<+\infty\}}1_{[0,a(t)[} ds \\ &= \int (f \circ c)1_{[0,a(t)[} ds \end{aligned}$$

12. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Given $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(a)$, we have proved in 10. that:

$$\int_0^t f da = \int_0^{a(t)} (f \circ c) 1_{\{c<+\infty\}} ds$$

This completes the proof of theorem (93).

Exercise 18