2. Caratheodory's Extension

In the following, Ω is a set. Whenever a union of sets is denoted $\uplus$ as opposed to $\cup$, it indicates that the sets involved are pairwise disjoint.

Definition 6 *A* **semi-ring** *on* Ω *is a subset* $\mathcal{S}$ *of the power set* $\mathcal{P}(\Omega)$ *with the following properties:*

$$(i) \qquad \emptyset \in \mathcal{S}$$

$$(ii) \qquad A, B \in \mathcal{S} \;\Rightarrow\; A \cap B \in \mathcal{S}$$

$$(iii) \qquad A, B \in \mathcal{S} \;\Rightarrow\; \exists n \geq 0,\; \exists A_i \in \mathcal{S} :\; A \setminus B = \biguplus_{i=1}^{n} A_i$$

The last property (iii) says that whenever $A, B \in \mathcal{S}$, there is $n \geq 0$ and $A_1, \ldots, A_n$ in $\mathcal{S}$ which are pairwise disjoint, such that $A \setminus B = A_1 \uplus \ldots \uplus A_n$. If $n = 0$, it is understood that the corresponding union is equal to $\emptyset$, (in which case $A \subseteq B$).

Definition 7 *A* **ring** *on* Ω *is a subset* $\mathcal{R}$ *of the power set* $\mathcal{P}(\Omega)$ *with the following properties:*

$$
\begin{array}{rl}
(i) & \emptyset \in \mathcal{R} \\
(ii) & A, B \in \mathcal{R} \ \Rightarrow \ A \cup B \in \mathcal{R} \\
(iii) & A, B \in \mathcal{R} \ \Rightarrow \ A \setminus B \in \mathcal{R}
\end{array}
$$

Exercise 1. Show that $A \cap B = A \setminus (A \setminus B)$ and therefore that a ring is closed under pairwise intersection.

Exercise 2. Show that a ring on Ω is also a semi-ring on Ω.

Exercise 3. Suppose that a set Ω can be decomposed as $\Omega = A_1 \uplus A_2 \uplus A_3$ where A_1, A_2 and A_3 are distinct from $\emptyset$ and Ω. Define $\mathcal{S}_1 \stackrel{\triangle}{=} \{\emptyset, A_1, A_2, A_3, \Omega\}$ and $\mathcal{S}_2 \stackrel{\triangle}{=} \{\emptyset, A_1, A_2 \uplus A_3, \Omega\}$. Show that $\mathcal{S}_1$ and $\mathcal{S}_2$ are semi-rings on Ω, but that $\mathcal{S}_1 \cap \mathcal{S}_2$ fails to be a semi-ring on Ω.

Exercise 4. Let $(\mathcal{R}_i)_{i \in I}$ be an arbitrary family of rings on Ω, with $I \neq \emptyset$. Show that $\mathcal{R} \stackrel{\triangle}{=} \cap_{i \in I} \mathcal{R}_i$ is also a ring on Ω.

Exercise 5. Let $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. Define:

$$R(\mathcal{A}) \stackrel{\triangle}{=} \{\mathcal{R} \text{ ring on } \Omega : \ \mathcal{A} \subseteq \mathcal{R}\}$$

Show that $\mathcal{P}(\Omega)$ is a ring on Ω, and that $R(\mathcal{A})$ is not empty. Define:

$$\mathcal{R}(\mathcal{A}) \stackrel{\triangle}{=} \bigcap_{\mathcal{R} \in R(\mathcal{A})} \mathcal{R}$$

Show that $\mathcal{R}(\mathcal{A})$ is a ring on Ω such that $\mathcal{A} \subseteq \mathcal{R}(\mathcal{A})$, and that it is the smallest ring on Ω with such property, (i.e. if $\mathcal{R}$ is a ring on Ω and $\mathcal{A} \subseteq \mathcal{R}$ then $\mathcal{R}(\mathcal{A}) \subseteq \mathcal{R}$).

Definition 8 *Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. We call* **ring generated** *by $\mathcal{A}$, the ring on Ω, denoted $\mathcal{R}(\mathcal{A})$, equal to the intersection of all rings on Ω, which contain $\mathcal{A}$.*

Exercise 6.Let $\mathcal{S}$ be a semi-ring on Ω. Define the set $\mathcal{R}$ of all finite unions of pairwise disjoint elements of $\mathcal{S}$, i.e.

$$\mathcal{R} \stackrel{\triangle}{=} \{A : \ A = \uplus_{i=1}^{n} A_i \text{ for some } n \geq 0, A_i \in \mathcal{S}\}$$

(where if $n = 0$, the corresponding union is empty, i.e. $\emptyset \in \mathcal{R}$). Let $A = \uplus_{i=1}^n A_i$ and $B = \uplus_{j=1}^p B_j \in \mathcal{R}$:

1. Show that $A \cap B = \uplus_{i,j}(A_i \cap B_j)$ and that $\mathcal{R}$ is closed under pairwise intersection.

2. Show that if $p \geq 1$ then $A \setminus B = \cap_{j=1}^p (\uplus_{i=1}^n (A_i \setminus B_j))$.

3. Show that $\mathcal{R}$ is closed under pairwise difference.

4. Show that $A \cup B = (A \setminus B) \uplus B$ and conclude that $\mathcal{R}$ is a ring on Ω.

5. Show that $\mathcal{R}(\mathcal{S}) = \mathcal{R}$.

Exercise 7. Everything being as before, define:

$$\mathcal{R}' \stackrel{\triangle}{=} \{A : \ A = \cup_{i=1}^n A_i \text{ for some } n \geq 0, A_i \in \mathcal{S}\}$$

(We do not require the sets involved in the union to be pairwise disjoint). Using the fact that $\mathcal{R}$ is closed under finite union, show that $\mathcal{R}' \subseteq \mathcal{R}$, and conclude that $\mathcal{R}' = \mathcal{R} = \mathcal{R}(\mathcal{S})$.

Definition 9 *Let* $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ *with* $\emptyset \in \mathcal{A}$. *We call* **measure** *on* $\mathcal{A}$, *any map* $\mu : \mathcal{A} \to [0, +\infty]$ *with the following properties:*

$$(i) \qquad \mu(\emptyset) = 0$$

$$(ii) \qquad A \in \mathcal{A}, A_n \in \mathcal{A} \text{ and } A = \biguplus_{n=1}^{+\infty} A_n \;\Rightarrow\; \mu(A) = \sum_{n=1}^{+\infty} \mu(A_n)$$

The $\uplus$ indicates that we assume the A_n's to be pairwise disjoint in the l.h.s. of (ii). It is customary to say in view of condition (ii) that a measure is *countably additive*.

Exercise 8. If $\mathcal{A}$ is a σ-algebra on Ω explain why property (ii) can be replaced by:

$$(ii)' \; A_n \in \mathcal{A} \text{ and } A = \biguplus_{n=1}^{+\infty} A_n \;\Rightarrow\; \mu(A) = \sum_{n=1}^{+\infty} \mu(A_n)$$

Exercise 9. Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ with $\emptyset \in \mathcal{A}$ and $\mu : \mathcal{A} \to [0, +\infty]$ be a measure on $\mathcal{A}$.

1. Show that if $A_1, \ldots, A_n \in \mathcal{A}$ are pairwise disjoint and the union $A = \uplus_{i=1}^n A_i$ lies in $\mathcal{A}$, then $\mu(A) = \mu(A_1) + \ldots + \mu(A_n)$.
2. Show that if $A, B \in \mathcal{A}, A \subseteq B$ and $B \setminus A \in \mathcal{A}$ then $\mu(A) \leq \mu(B)$.

Exercise 10. Let $\mathcal{S}$ be a semi-ring on Ω, and $\mu : \mathcal{S} \to [0, +\infty]$ be a measure on $\mathcal{S}$. Suppose that there exists an extension of μ on $\mathcal{R}(\mathcal{S})$, i.e. a measure $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$ such that $\bar{\mu}_{|\mathcal{S}} = \mu$.

1. Let A be an element of $\mathcal{R}(\mathcal{S})$ with representation $A = \uplus_{i=1}^n A_i$ as a finite union of pairwise disjoint elements of $\mathcal{S}$. Show that $\bar{\mu}(A) = \sum_{i=1}^n \mu(A_i)$
2. Show that if $\bar{\mu}' : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$ is another measure with $\bar{\mu}'_{|\mathcal{S}} = \mu$, i.e. another extension of μ on $\mathcal{R}(\mathcal{S})$, then $\bar{\mu}' = \bar{\mu}$.

Exercise 11. Let $\mathcal{S}$ be a semi-ring on Ω and $\mu : \mathcal{S} \to [0, +\infty]$ be a measure. Let A be an element of $\mathcal{R}(\mathcal{S})$ with two representations:

$$A = \biguplus_{i=1}^{n} A_i = \biguplus_{j=1}^{p} B_j$$

as a finite union of pairwise disjoint elements of $\mathcal{S}$.

1. For $i = 1, \ldots, n$, show that $\mu(A_i) = \sum_{j=1}^{p} \mu(A_i \cap B_j)$
2. Show that $\sum_{i=1}^{n} \mu(A_i) = \sum_{j=1}^{p} \mu(B_j)$
3. Explain why we can define a map $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$ as:

$$\bar{\mu}(A) \stackrel{\triangle}{=} \sum_{i=1}^{n} \mu(A_i)$$

4. Show that $\bar{\mu}(\emptyset) = 0$.

Exercise 12. Everything being as before, suppose that $(A_n)_{n\geq 1}$ is a sequence of pairwise disjoint elements of $\mathcal{R}(\mathcal{S})$, each A_n having the representation:

$$A_n = \biguplus_{k=1}^{p_n} A_n^k \ , \ n \geq 1$$

as a finite union of disjoint elements of $\mathcal{S}$. Suppose moreover that $A = \uplus_{n=1}^{+\infty} A_n$ is an element of $\mathcal{R}(\mathcal{S})$ with representation $A = \uplus_{j=1}^{p} B_j$, as a finite union of pairwise disjoint elements of $\mathcal{S}$.

1. Show that for $j = 1, \ldots, p$, $B_j = \cup_{n=1}^{+\infty} \cup_{k=1}^{p_n} (A_n^k \cap B_j)$ and explain why B_j is of the form $B_j = \uplus_{m=1}^{+\infty} C_m$ for some sequence $(C_m)_{m\geq 1}$ of pairwise disjoint elements of $\mathcal{S}$.

2. Show that $\mu(B_j) = \sum_{n=1}^{+\infty} \sum_{k=1}^{p_n} \mu(A_n^k \cap B_j)$

3. Show that for $n \geq 1$ and $k = 1, \ldots, p_n$, $A_n^k = \uplus_{j=1}^{p} (A_n^k \cap B_j)$

4. Show that $\mu(A_n^k) = \sum_{j=1}^{p} \mu(A_n^k \cap B_j)$

5. Recall the definition of $\bar{\mu}$ of exercise (11) and show that it is a measure on $\mathcal{R}(\mathcal{S})$.

EXERCISE 13. Prove the following theorem:

Theorem 2 *Let $\mathcal{S}$ be a semi-ring on Ω. Let $\mu : \mathcal{S} \to [0, +\infty]$ be a measure on $\mathcal{S}$. There exists a unique measure $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$ such that $\bar{\mu}_{|\mathcal{S}} = \mu$.*

Definition 10 *We define an* **outer-measure** *on* Ω *as being any map* $\mu^* : \mathcal{P}(\Omega) \to [0, +\infty]$ *with the following properties:*

$$\begin{array}{ll} (i) & \mu^*(\emptyset) = 0 \\ (ii) & A \subseteq B \;\Rightarrow\; \mu^*(A) \leq \mu^*(B) \\ (iii) & \mu^*\left(\bigcup_{n=1}^{+\infty} A_n\right) \leq \sum_{n=1}^{+\infty} \mu^*(A_n) \end{array}$$

EXERCISE 14. Show that $\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$, where μ^* is an outer-measure on Ω and $A, B \subseteq \Omega$.

Definition 11 *Let* μ^* *be an outer-measure on* Ω. *We define:*

$$\Sigma(\mu^*) \stackrel{\triangle}{=} \{A \subseteq \Omega : \; \mu^*(T) = \mu^*(T \cap A) + \mu^*(T \cap A^c) \;, \; \forall T \subseteq \Omega\}$$

We call $\Sigma(\mu^*)$ *the* **σ-algebra associated** *with the outer-measure* μ^*.

Note that the fact that $\Sigma(\mu^*)$ is indeed a σ-algebra on Ω, remains to be proved. This will be your task in the following exercises.

Exercise 15. Let μ^* be an outer-measure on Ω. Let $\Sigma = \Sigma(\mu^*)$ be the σ-algebra associated with μ^*. Let $A, B \in \Sigma$ and $T \subseteq \Omega$

1. Show that $\Omega \in \Sigma$ and $A^c \in \Sigma$.
2. Show that $\mu^*(T \cap A) = \mu^*(T \cap A \cap B) + \mu^*(T \cap A \cap B^c)$
3. Show that $T \cap A^c = T \cap (A \cap B)^c \cap A^c$
4. Show that $T \cap A \cap B^c = T \cap (A \cap B)^c \cap A$
5. Show that $\mu^*(T \cap A^c) + \mu^*(T \cap A \cap B^c) = \mu^*(T \cap (A \cap B)^c)$
6. Adding $\mu^*(T \cap (A \cap B))$ on both sides 5., conclude that $A \cap B \in \Sigma$.
7. Show that $A \cup B$ and $A \setminus B$ belong to Σ.

Exercise 16. Everything being as before, let $A_n \in \Sigma, n \geq 1$. Define $B_1 = A_1$ and $B_{n+1} = A_{n+1} \setminus (A_1 \cup \ldots \cup A_n)$. Show that the B_n's are pairwise disjoint elements of Σ and that $\cup_{n=1}^{+\infty} A_n = \uplus_{n=1}^{+\infty} B_n$.

Exercise 17. Everything being as before, show that if $B, C \in \Sigma$ and $B \cap C = \emptyset$, then $\mu^*(T \cap (B \uplus C)) = \mu^*(T \cap B) + \mu^*(T \cap C)$ for any $T \subseteq \Omega$.

Exercise 18. Everything being as before, let $(B_n)_{n\geq 1}$ be a sequence of pairwise disjoint elements of Σ, and let $B \stackrel{\triangle}{=} \uplus_{n=1}^{+\infty} B_n$. Let $N \geq 1$.

1. Explain why $\uplus_{n=1}^{N} B_n \in \Sigma$
2. Show that $\mu^*(T \cap (\uplus_{n=1}^{N} B_n)) = \sum_{n=1}^{N} \mu^*(T \cap B_n)$
3. Show that $\mu^*(T \cap B^c) \leq \mu^*(T \cap (\uplus_{n=1}^{N} B_n)^c)$
4. Show that $\mu^*(T \cap B^c) + \sum_{n=1}^{+\infty} \mu^*(T \cap B_n) \leq \mu^*(T)$, and:
5. $\mu^*(T) \leq \mu^*(T \cap B^c) + \mu^*(T \cap B) \leq \mu^*(T \cap B^c) + \sum_{n=1}^{+\infty} \mu^*(T \cap B_n)$
6. Show that $B \in \Sigma$ and $\mu^*(B) = \sum_{n=1}^{+\infty} \mu^*(B_n)$.
7. Show that Σ is a σ-algebra on Ω, and $\mu^*_{|\Sigma}$ is a measure on Σ.

Theorem 3 *Let $\mu^* : \mathcal{P}(\Omega) \to [0, +\infty]$ be an outer-measure on Ω. Then $\Sigma(\mu^*)$, the so-called σ-*algebra associated *with μ^*, is indeed a σ-algebra on Ω and $\mu^*_{|\Sigma(\mu^*)}$, is a measure on $\Sigma(\mu^*)$.*

Exercise 19. Let $\mathcal{R}$ be a ring on Ω and $\mu : \mathcal{R} \to [0, +\infty]$ be a measure on $\mathcal{R}$. For all $T \subseteq \Omega$, define:

$$\mu^*(T) \stackrel{\triangle}{=} \inf \left\{ \sum_{n=1}^{+\infty} \mu(A_n) \ , \ (A_n) \text{ is an } \mathcal{R}\text{-cover of } T \right\}$$

where an $\mathcal{R}$-cover of T is defined as any sequence $(A_n)_{n\geq 1}$ of elements of $\mathcal{R}$ such that $T \subseteq \cup_{n=1}^{+\infty} A_n$. By convention $\inf \emptyset \stackrel{\triangle}{=} +\infty$.

1. Show that $\mu^*(\emptyset) = 0$.

2. Show that if $A \subseteq B$ then $\mu^*(A) \leq \mu^*(B)$.

3. Let $(A_n)_{n\geq 1}$ be a sequence of subsets of Ω, with $\mu^*(A_n) < +\infty$ for all $n \geq 1$. Given $\epsilon > 0$, show that for all $n \geq 1$, there exists an $\mathcal{R}$-cover $(A_n^p)^{p\geq 1}$ of A_n such that:

$$\sum_{p=1}^{+\infty} \mu(A_n^p) < \mu^*(A_n) + \epsilon/2^n$$

   Why is it important to assume $\mu^*(A_n) < +\infty$.

4. Show that there exists an $\mathcal{R}$-cover (R_k) of $\cup_{n=1}^{+\infty} A_n$ such that:

$$\sum_{k=1}^{+\infty} \mu(R_k) = \sum_{n=1}^{+\infty} \sum_{p=1}^{+\infty} \mu(A_n^p)$$

5. Show that $\mu^*(\cup_{n=1}^{+\infty} A_n) \leq \epsilon + \sum_{n=1}^{+\infty} \mu^*(A_n)$

6. Show that μ^* is an outer-measure on Ω.

Exercise 20. Everything being as before, Let $A \in \mathcal{R}$. Let $(A_n)_{n\geq 1}$ be an $\mathcal{R}$-cover of A and put $B_1 = A_1 \cap A$, and:

$$B_{n+1} \stackrel{\triangle}{=} (A_{n+1} \cap A) \setminus ((A_1 \cap A) \cup \ldots \cup (A_n \cap A))$$

1. Show that $\mu^*(A) \leq \mu(A)$.
2. Show that $(B_n)_{n\geq 1}$ is a sequence of pairwise disjoint elements of $\mathcal{R}$ such that $A = \uplus_{n=1}^{+\infty} B_n$.
3. Show that $\mu(A) \leq \mu^*(A)$ and conclude that $\mu^*_{|\mathcal{R}} = \mu$.

Exercise 21. Everything being as before, Let $A \in \mathcal{R}$ and $T \subseteq \Omega$.

1. Show that $\mu^*(T) \leq \mu^*(T \cap A) + \mu^*(T \cap A^c)$.
2. Let (T_n) be an $\mathcal{R}$-cover of T. Show that $(T_n \cap A)$ and $(T_n \cap A^c)$ are $\mathcal{R}$-covers of $T \cap A$ and $T \cap A^c$ respectively.
3. Show that $\mu^*(T \cap A) + \mu^*(T \cap A^c) \leq \mu^*(T)$.

4. Show that $\mathcal{R} \subseteq \Sigma(\mu^*)$.

5. Conclude that $\sigma(\mathcal{R}) \subseteq \Sigma(\mu^*)$.

EXERCISE 22. Prove the following theorem:

Theorem 4 (Caratheodory's extension) *Let $\mathcal{R}$ be a ring on Ω and $\mu : \mathcal{R} \to [0, +\infty]$ be a measure on $\mathcal{R}$. There exists a measure $\mu' : \sigma(\mathcal{R}) \to [0, +\infty]$ such that $\mu'_{|\mathcal{R}} = \mu$.*

EXERCISE 23. Let $\mathcal{S}$ be a semi-ring on Ω. Show that $\sigma(\mathcal{R}(\mathcal{S})) = \sigma(\mathcal{S})$.

EXERCISE 24. Prove the following theorem:

Theorem 5 *Let $\mathcal{S}$ be a semi-ring on Ω and $\mu : \mathcal{S} \to [0, +\infty]$ be a measure on $\mathcal{S}$. There exists a measure $\mu' : \sigma(\mathcal{S}) \to [0, +\infty]$ such that $\mu'_{|\mathcal{S}} = \mu$.*

Solutions to Exercises

Exercise 1.

- Let $x \in A \cap B$. Then $x \in B$. So $x \notin A \setminus B$. It follows that $x \in A \setminus (A \setminus B)$, and $A \cap B \subseteq A \setminus (A \setminus B)$. Let $x \in A \setminus (A \setminus B)$. Then $x \in A$ and $x \notin A \setminus B$. But $x \notin A \setminus B$ implies that either $x \notin A$ or $x \in B$. Hence, $x \in B$. finally, $x \in A \cap B$ and $A \setminus (A \setminus B) \subseteq A \cap B$. We have proved that $A \cap B = A \setminus (A \setminus B)$

- Let $\mathcal{R}$ be a ring and $A, B \in \mathcal{R}$. From (iii) of definition (7), $A \setminus B \in \mathcal{R}$. Hence, $A \setminus (A \setminus B) \in \mathcal{R}$. It follows from the previous point that $A \cap B \in \mathcal{R}$. We have proved that a ring is *closed under pairwise intersection.*

Exercise 1

Exercise 2. Let $\mathcal{R}$ be ring on Ω. Then (i) of definition (6) is immediately satisfied for $\mathcal{R}$. From exercise (1), we know that $\mathcal{R}$ is closed under finite intersection. So (ii) of definition (6) is satisfied for $\mathcal{R}$. Let $A, B \in \mathcal{R}$. From (iii) of definition (7), $A \setminus B \in \mathcal{R}$. Therefore, if we take $n = 1$ and $A_1 = A \setminus B \in \mathcal{R}$, we see that $A \setminus B = \uplus_{i=1}^n A_i$ and (iii) of definition (6) is satisfied for $\mathcal{R}$. Finally, having checked (i), (ii) and (iii) of definition (6), we conclude that $\mathcal{R}$ is a semi-ring on Ω. Any ring on Ω is therefore also a semi-ring on Ω.

Exercise 2

Exercise 3.

- $\emptyset \in \mathcal{S}_1$ so (i) of definition (6) is satisfied for $\mathcal{S}_1$. If $A, B \in \mathcal{S}_1$, then $A \cap B$ is equal to the empty set (remember that A_1, A_2 and A_3 are disjoint), unless A (resp. B) is Ω itself, or $A = B \neq \emptyset$, in which case $A \cap B$ is equal to B (resp. A). In any case, $A \cap B \in \mathcal{S}_1$ and condition (ii) of definition (6) is satisfied for $\mathcal{S}_1$. If $A, B \in \mathcal{S}_1$, since $\mathcal{S}_1$ has 5 elements, $A \setminus B$ is one of 25 cases to consider. It is equal to $\emptyset$, ($\emptyset \setminus \emptyset$, $\emptyset \setminus A_i$, $\emptyset \setminus \Omega$, $A_i \setminus \Omega$, $A_i \setminus A_i$, $\Omega \setminus \Omega$) in 12 of those cases. It is equal to A itself ($A_i \setminus \emptyset$, $A_i \setminus A_j, j \neq i$, $\Omega \setminus \emptyset$) in 10 of those cases. The last three cases are $\Omega \setminus A_1 = A_2 \uplus A_3$, $\Omega \setminus A_2 = A_1 \uplus A_3$ and $\Omega \setminus A_3 = A_1 \uplus A_2$. Hence, we see that condition (iii) of definition (6) is satisfied for $\mathcal{S}_1$. We have proved that $\mathcal{S}_1$ is indeed a semi-ring on Ω.

- If we put $B_1 = A_1$ and $B_2 = A_2 \uplus A_3$, then $\Omega = B_1 \uplus B_2$ where B_1, B_2 are distinct from $\emptyset$ and Ω. Moreover, $\mathcal{S}_2 = \{\emptyset, B_1, B_2, \Omega\}$, and proving that $\mathcal{S}_2$ is a semi-ring on Ω is identical to the previous point, but is just a little bit easier...

- $S_1 \cap \mathcal{S}_2 = \{\emptyset, A_1, \Omega\}$ (remember that all A_i's are not empty and pairwise disjoint, so $A_3 \neq A_2 \uplus A_3$ and $A_2 \neq A_2 \uplus A_3$). Suppose that $\mathcal{S}_1 \cap \mathcal{S}_2$is a semi-ring on Ω. Then from (iii) of definition (6), there exists $n \geq 0$ and B_1, $B_2, \ldots, B_n$ in $\mathcal{S}_1 \cap \mathcal{S}_2$ such that:

$$\Omega \setminus A_1 = B_1 \uplus \ldots \uplus B_n$$

Since A_1 is assumed to be distinct from Ω, $\Omega \setminus A_1 \neq \emptyset$. It follows that $n \geq 1$ and at least one of the B_i's is not empty. If $B_i = \Omega$ then $\Omega \setminus A_1 = \Omega$ and this would be a contradiction since A_1 is assumed to be not empty. If $B_i = A_1$ then $\Omega \setminus A_1 \supseteq A_1$ would also be a contradiction. Hence, the initial assumption of $\mathcal{S}_1 \cap \mathcal{S}_2$ being a semi-ring on Ω is absurd. $\mathcal{S}_1 \cap \mathcal{S}_2$ fails to be a semi-ring on Ω. The purpose of this exercise is to show that contrary to Dynkin systems, σ-algebras and rings (as we shall see in the next exercise), taking intersections of semi-rings does not necessarily create another semi-ring. Hence, no attempt will be made to define the notion of *generated semi-ring*...

Exercise 3

Exercise 4. Each $\mathcal{R}_i$ being a ring on Ω, $\emptyset \in \mathcal{R}_i$. This being true for all $i \in I$, $\emptyset \in \cap_{i\in I}\mathcal{R}_i = \mathcal{R}$, and condition (i) of definition (7) is satisfied for $\mathcal{R}$. Let $A, B \in \mathcal{R}$. Then for all $i \in I$, A, B belong to $\mathcal{R}_i$. It follows that $A \setminus B$ and $A \cup B$ belong to $\mathcal{R}_i$. This being true for all $i \in I$, both $A \setminus B$ and $A \cup B$ lie in $\cap_{i\in I}\mathcal{R}_i$, and conditions (ii) and (iii) of definition (7) are satisfied for $\mathcal{R}$. Having checked (i), (ii) and (iii) of definition (7), we conclude that $\mathcal{R}$ is indeed a ring on Ω. The purpose of this exercise is to show that an arbitrary (non-empty) intersection of rings on Ω, is still a ring on Ω.

Exercise 4

Exercise 5.

- $\emptyset$ being a subset of Ω, $\emptyset \in \mathcal{P}(\Omega)$ and condition (i) of definition (7) is satisfied for $\mathcal{P}(\Omega)$. Given two subsets A, B of Ω, $A \setminus B$ and $A \cup B$ are still subsets of Ω, i.e. $A \setminus B \in \mathcal{P}(\Omega)$ and $A \cup B \in \mathcal{P}(\Omega)$. Hence, conditions (ii) and (iii) of definition (7) are satisfied for $\mathcal{P}(\Omega)$. It follows that $\mathcal{P}(\Omega)$ is a ring on Ω.

- By assumption, $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. Moreover, $\mathcal{P}(\Omega)$ is a ring on Ω. Therefore, $\mathcal{P}(\Omega) \in R(\mathcal{A})$. In particular, $R(\mathcal{A})$ is not empty.

- $\mathcal{R}(\mathcal{A})$ is a non-empty intersection of rings on Ω. From exercise (4), it is therefore a ring on Ω.

- For all $\mathcal{R} \in R(\mathcal{A})$, $\mathcal{A} \subseteq \mathcal{R}$. Hence:

$$\mathcal{A} \subseteq \bigcap_{\mathcal{R} \in R(\mathcal{A})} \mathcal{R} \stackrel{\triangle}{=} \mathcal{R}(\mathcal{A})$$

- Suppose $\mathcal{R}$ is another ring on Ω, with $\mathcal{A} \subseteq \mathcal{R}$. Then, by definition of the set $R(\mathcal{A})$, $\mathcal{R} \in R(\mathcal{A})$. It follows that:

$$\mathcal{R}(\mathcal{A}) \stackrel{\triangle}{=} \bigcap_{\mathcal{R}' \in R(\mathcal{A})} \mathcal{R}' \subseteq \mathcal{R}$$

So $\mathcal{R}(\mathcal{A})$ is indeed the *smallest ring* on Ω which contains $\mathcal{A}$.

Exercise 5

Exercise 6.

1. If $x \in A_i \cap B_j$ for some $i = 1, \ldots, n$ and $j = 1, \ldots, p$, then $x \in A \cap B$. Conversely if $x \in A \cap B$, then $n \geq 1$, $p \geq 1$, and there exist $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, p\}$ such that $x \in A_i \cap B_j$. So $A \cap B = \cup_{i,j} A_i \cap B_j$. Suppose (i, j) and (i', j') are such that $(A_i \cap B_j) \cap (A_{i'} \cap B_{j'}) \neq \emptyset$. In particular, $A_i \cap A_{i'} \neq \emptyset$. Since the A_i's are pairwise disjoint, we have $i = i'$ and similarly $j = j'$. Hence, we see that the $(A_i \cap B_j)_{i,j}$'s are pairwise disjoint, and finally $A \cap B = \uplus_{i,j} A_i \cap B_j$. From (ii) of definition (6), all the $A_i \cap B_j$'s lie in the semi-ring $\mathcal{S}$, and we see that $A \cap B$ is also an element of $\mathcal{R}$. We have proved that $\mathcal{R}$ is *closed under finite intersection*.

2. Since the A_i's are pairwise disjoint, for all $j \in \{1, \ldots, p\}$ being given, the $A_i \setminus B_j$ $i = 1, \ldots, n$, are also pairwise disjoint. Hence, the union $\cup_{i=1}^n A_i \setminus B_j$ can legitimately be written as $\uplus_{i=1}^n A_i \setminus B_j$. let $x \in A \setminus B$. Then $x \notin B$. Thus, for all $j = 1, \ldots, p$, $x \notin B_j$. But $x \in A$. So there exists $i \in \{1, \ldots, n\}$ such that $x \in A_i$.

It follows that for all $j \in \{1,\ldots,p\}$, $x \in A_i \setminus B_j$ for some $i \in \{1,\ldots,n\}$. So $x \in \cap_{j=1}^p \uplus_{i=1}^n (A_i \setminus B_j)$. Conversely, suppose that $x \in \cap_{j=1}^p \uplus_{i=1}^n (A_i \setminus B_j)$. Then for all $j \in \{1,\ldots,p\}$, there exists $i_j \in \{1,\ldots,n\}$ such that $x \in A_{i_j} \setminus B_j$. Since we have assumed $p \geq 1$, in particular $x \in A_{i_1} \subseteq A$, and for all $j \in \{1,\ldots,p\}$, $x \notin B_j$, so $x \notin B$. It follows that $x \in A \setminus B$. We have proved that:

$$A \setminus B = \cap_{j=1}^p \uplus_{i=1}^n (A_i \setminus B_j)$$

3. If $p = 0$, then $B = \emptyset$ and $A \setminus B = A \in \mathcal{R}$. We assume that $p \geq 1$. From the previous point, we know that $A \setminus B = \cap_{j=1}^p C_j$ where C_j is defined as $C_j = \uplus_{i=1}^n A_i \setminus B_j$. But each A_i and B_j is an element of the semi-ring $\mathcal{S}$. From (iii) of definition (6), each $A_i \setminus B_j$ can be written as a finite union of pairwise disjoint elements of $\mathcal{S}$. It follows that C_j itself can be written as a finite union of pairwise disjoint elements of $\mathcal{S}$. Hence, we see that for all $j \in \{1,\ldots,p\}$, C_j is an element of $\mathcal{R}$. From 1. we know that $\mathcal{R}$ is closed under finite intersection. We conclude that

$A \setminus B = \cap_{j=1}^{p} C_j \in \mathcal{R}$. We have proved that $\mathcal{R}$ is closed under pairwise difference.

4. Let $x \in A \cup B$. then $x \in A$ or $x \in B$. If $x \in B$ then $x \in A \setminus B \uplus B$. If $x \notin B$ then $x \in A \setminus B$. In any case, $x \in A \setminus B \uplus B$, and $A \cup B \subseteq A \setminus B \uplus B$. Conversely, $A \setminus B \subseteq A$, so $A \setminus B \uplus B \subseteq A \cup B$. Now, if $A, B \in \mathcal{R}$, from the previous point, $A \setminus B \in \mathcal{R}$. It follows that $A \setminus B$ can be written as a finite union of pairwise disjoint elements of $\mathcal{S}$. But B itself (being an element of $\mathcal{R}$), can be written as a finite union of pairwise disjoint elements of $\mathcal{S}$. It follows that $A \setminus B \uplus B$ is also a finite union of pairwise disjoint elements of $\mathcal{S}$, hence an element of $\mathcal{R}$. From $A \cup B = A \setminus B \uplus B$, we conclude that $A \cup B$ is an element of $\mathcal{R}$. We have proved that $\mathcal{R}$ is *closed under finite union*. Finally, (i), (ii), (iii) of definition (7) being satisfied for $\mathcal{R}$, $\mathcal{R}$ is indeed a ring on Ω.

5. Let $A \in \mathcal{S}$. A can obviously be written as a finite union of pairwise disjoint elements of $\mathcal{S}$. (Take $n = 1$, $A_1 = A \in \mathcal{S}$ and $A = \uplus_{i=1}^{n} A_i$). Hence, $A \in \mathcal{R}$ and $\mathcal{S} \subseteq \mathcal{R}$. Consequently, from

exercise (5) and the fact that $\mathcal{R}$ is a ring on Ω, $\mathcal{R}(\mathcal{S}) \subseteq \mathcal{R}$. Conversely, let $A \in \mathcal{R}$. Then $A = \uplus_{i=1}^n A_i$ for some $n \geq 0$ and $A_i \in \mathcal{S}$. Since $\mathcal{S} \subseteq \mathcal{R}(\mathcal{S})$ (see exercise (5)), each A_i lies in $\mathcal{R}(\mathcal{S})$. But from (ii) of definition (7), $\mathcal{R}(\mathcal{S})$ being a ring is closed under finite union. Hence, $A \in \mathcal{R}(\mathcal{S})$ and we have $\mathcal{R} \subseteq \mathcal{R}(\mathcal{S})$. We have proved that $\mathcal{R}(\mathcal{S}) = \mathcal{R}$. The purpose of this exercise is to show that the ring $\mathcal{R}(\mathcal{S})$ generated by a semi-ring $\mathcal{S}$ on Ω, is equal to the set of all finite unions of pairwise disjoint elements of $\mathcal{S}$.

Exercise 6

Exercise 7. Any finite union of pairwise disjoint elements of $\mathcal{S}$, is in particular a finite union of elements of $\mathcal{S}$ ... So $\mathcal{R} \subseteq \mathcal{R}'$. Let $A \in \mathcal{R}'$. There exists $n \geq 0$ and $A_i \in \mathcal{S}$ for $i = 1, \ldots, n$ such that $A = \cup_{i=1}^n A_i$. If $n = 0$, then $A = \emptyset \in \mathcal{R}$. If $n \geq 1$, since $\mathcal{S} \subseteq \mathcal{R} = \mathcal{R}(\mathcal{S})$, all A_i's are elements of $\mathcal{R}$. $\mathcal{R}$ being closed under finite union (it is a ring on Ω), A is itself an element of $\mathcal{R}$. Hence $\mathcal{R}' \subseteq \mathcal{R}$. We have proved that $\mathcal{R} = \mathcal{R}' = \mathcal{R}(\mathcal{S})$. The purpose of this exercise is to show that the generated ring $\mathcal{R}(\mathcal{S})$ of a semi-ring $\mathcal{S}$ on Ω, is also equal to the set of all finite unions of (not necessarily pairwise disjoint) elements of $\mathcal{S}$.

Exercise 7

Exercise 8. If $\mathcal{A}$ is a σ-algebra on Ω, then $A_n \in \mathcal{A}$ and $A = \uplus_{n=1}^{+\infty} A_n$ automatically implies that $A \in \mathcal{A}$. Hence, the l.h.s of (ii) and $(ii)'$ are equivalent, whenever $\mathcal{A}$ is a σ-algebra on Ω.

Exercise 8

Exercise 9.

1. Define the sequence $(B_n)_{n\geq 1}$ of elements of $\mathcal{A}$, by $B_i = A_i$ for all $i = 1, \ldots, n$ and $B_k = \emptyset$ for all $k > n$. Then $A = \uplus_{k=1}^{\infty} B_k$, and since $A \in \mathcal{A}$, from (ii) of definition (9), we have:

$$\mu(A) = \sum_{k=1}^{+\infty} \mu(B_k)$$

But from (i) of definition (9), $\mu(B_k) = 0$ for all $k > n$. Hence:

$$\mu(A) = \mu(A_1) + \ldots + \mu(A_n)$$

In view of this property, it is customary to say that a measure is *finitely additive*.

2. Suppose $A, B \in \mathcal{A}$ with $A \subseteq B$ and $B \setminus A \in \mathcal{A}$. Then, we have $B = A \cup B = A \uplus (B \setminus A)$. From the previous point we conclude:

$$\mu(A) \leq \mu(A) + \mu(B \setminus A) = \mu(B)$$

Exercise 9

Exercise 10.

1. If $A = \emptyset$, then either $n = 0$ or $A_i = \emptyset$ for all $i = 1, \ldots, n$. In any case, $\bar{\mu}(A) = \sum_{i=1}^n \mu(A_i)$ is true. If $A \neq \emptyset$, then $n \geq 1$. Since $\mathcal{S} \subseteq \mathcal{R}(\mathcal{S})$, all sets involved in $A = \uplus_{i=1}^n A_i$ are elements of $\mathcal{R}(\mathcal{S})$. Since $\bar{\mu}$ is a measure on $\mathcal{R}(\mathcal{S})$, from exercise (9) we have $\bar{\mu}(A) = \sum_{i=1}^n \bar{\mu}(A_i)$. By assumption, $\bar{\mu}_{|\mathcal{S}} = \mu$ and $A_i \in \mathcal{S}$ for all $i = 1, \ldots, n$. Hence, $\bar{\mu}(A_i) = \mu(A_i)$ for all $i = 1, \ldots, n$. It follows that $\bar{\mu}(A) = \sum_{i=1}^n \mu(A_i)$.

2. Let $A \in \mathcal{R}(\mathcal{S})$. Then A has a representation $A = \uplus_{i=1}^n A_i$ as a finite union of pairwise disjoint elements of $\mathcal{S}$. From the previous point, $\bar{\mu}(A) = \sum_{i=1}^n \mu(A_i)$. If $\bar{\mu}'$ is another measure on $\mathcal{R}(\mathcal{S})$ with $\bar{\mu}'_{|\mathcal{S}} = \mu$, then similarly we have $\bar{\mu}'(A) = \sum_{i=1}^n \mu(A_i)$. So $\bar{\mu}(A) = \bar{\mu}'(A)$. This being true for all $A \in \mathcal{R}(\mathcal{S})$, $\bar{\mu} = \bar{\mu}'$. The purpose of this exercise is to show that if a measure μ on a semi-ring $\mathcal{S}$ can be extended to its generated ring $\mathcal{R}(\mathcal{S})$, then such extension is unique.

Exercise 10

Exercise 11.

1. If $p = 0$, then $A = \emptyset$. Then either $n = 0$ and there is nothing to prove, or $n \geq 1$ with all A_i's equal to the empty set. In any case, $\mu(A_i) = \sum_{j=1}^p \mu(A_i \cap B_j)$ is true. Hence we can assume that $p \geq 1$. Since $A_i \subseteq A$:
$$A_i = A_i \cap A = \biguplus_{j=1}^{p} A_i \cap B_j \tag{1}$$
Since $\mathcal{S}$ is a semi-ring, it is closed under finite intersection (definition (6)), hence all sets involved in (1) are elements of $\mathcal{S}$. From exercise (9), and the fact that μ is a measure on $\mathcal{S}$, we conclude that $\mu(A_i) = \sum_{j=1}^p \mu(A_i \cap B_j)$.

2. Similarly to the previous point, for all $j = 1, \ldots, p$ we have $\mu(B_j) = \sum_{i=1}^n \mu(A_i \cap B_j)$. It follows that:
$$\sum_{i=1}^{n} \mu(A_i) = \sum_{i=1}^{n} \sum_{j=1}^{p} \mu(A_i \cap B_j) = \sum_{j=1}^{p} \sum_{i=1}^{n} \mu(A_i \cap B_j) = \sum_{j=1}^{p} \mu(B_j)$$

3. Suppose we want to define a map $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$ with:

$$\bar{\mu}(A) \stackrel{\triangle}{=} \sum_{i=1}^{n} \mu(A_i) \tag{2}$$

where $A = \uplus_{i=1}^{n} A_i$ is a representation of A as a finite union of pairwise disjoint elements of $\mathcal{S}$. The problem is that such representation may not be unique. However, if $A = \uplus_{j=1}^{p} B_j$ is another representation of A in terms of finite union of pairwise disjoint elements of $\mathcal{S}$, then from 2., $\sum_{i=1}^{n} \mu(A_i) = \sum_{j=1}^{p} \mu(B_j)$. It follows that whichever representation is considered, the sum involved in (2) will still be the same. In other words, definition (2) is unambiguous, and therefore legitimate.

4. $\emptyset$ has a representation with $n = 0$, or $n = 1$ with $A_1 = \emptyset$, or $n = 2$ with $A_1 = A_2 = \emptyset \ldots$ Whichever representation we choose for $\emptyset$, definition (2) leads to $\bar{\mu}(\emptyset) = 0$.

Exercise 11

Exercise 12.

1. For all $j = 1, \ldots, p$, since $B_j \subseteq A$, we have:

$$B_j = A \cap B_j = \bigcup_{n=1}^{+\infty} (A_n \cap B_j) = \bigcup_{n=1}^{+\infty} \bigcup_{k=1}^{p_n} (A_n^k \cap B_j)$$

   Consider the set $I = \{(n, k) : n \geq 1, 1 \leq k \leq p_n\}$. Being a countable union of finite sets, I is a countable set. Hence, there exists a one-to-one map $\phi : \{m : m \geq 1\} \to I$. Given $m \geq 1$, define $C_m = A_n^k \cap B_j$ where $(n, k) = \phi(m)$. Then we have $B_j = \cup_{m=1}^{+\infty} C_m$. Since all A_n^k's and B_j itself are elements of the semi-ring $\mathcal{S}$, all C_m's are elements of $\mathcal{S}$. Suppose $C_m \cap C_{m'} \neq \emptyset$ for some $m, m' \geq 1$. Then in particular, $A_n^k \cap A_{n'}^{k'} \neq \emptyset$, where we have put $(n, k) = \phi(m)$ and $(n', k') = \phi(m')$. Since $A_n^k \subseteq A_n$ and $A_{n'}^{k'} \subseteq A_{n'}$, it follows that $A_n \cap A_{n'} \neq \emptyset$, and the A_n's being pairwise disjoint, we see that $n = n'$. Thus, $A_n^k \cap A_n^{k'} \neq \emptyset$. But the A_n^k's for $k = 1, \ldots, p_n$ are also pairwise disjoint. We conclude that $k = k'$ and $\phi(m) = (n, k) = (n', k') = \phi(m')$. Since

ϕ is one-to-one, $m = m'$, and we have proved that $(C_m)_{m\geq 1}$ is a sequence of pairwise disjoint elements of $\mathcal{S}$.

2. In the previous point, we saw that $B_j = \uplus_{m=1}^{+\infty} C_m$. Since all sets involved are elements of $\mathcal{S}$ and μ is a measure on $\mathcal{S}$, from (ii) of definition (9), we have:

$$\mu(B_j) = \sum_{m=1}^{+\infty} \mu(C_m) = \sum_{(n,k)\in I} \mu(A_n^k \cap B_j) = \sum_{n=1}^{+\infty} \sum_{k=1}^{p_n} \mu(A_n^k \cap B_j) \quad (3)$$

3. For $n \geq 1$ and $k \in \{1, \ldots, p_n\}$, we have $A_n^k \subseteq A_n \subseteq A$. Hence:

$$A_n^k = A_n^k \cap A = \biguplus_{j=1}^{p} (A_n^k \cap B_j)$$

4. From the previous point, using exercise (9), we obtain:

$$\mu(A_n^k) = \sum_{j=1}^{p} \mu(A_n^k \cap B_j) \quad (4)$$

5. In exercise (11), we saw that the map $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$ is such that $\bar{\mu}(\emptyset) = 0$. Hence (i) of definition (9) is satisfied for $\bar{\mu}$. Moreover, by definition, $\bar{\mu}(A) = \sum_{j=1}^{p} \mu(B_j)$. Using equation (3), we have:

$$\bar{\mu}(A) = \sum_{j=1}^{p} \sum_{n=1}^{+\infty} \sum_{k=1}^{p_n} \mu(A_n^k \cap B_j) = \sum_{n=1}^{+\infty} \sum_{k=1}^{p_n} \sum_{j=1}^{p} \mu(A_n^k \cap B_j)$$

Using equation (4), it follows that:

$$\bar{\mu}(A) = \sum_{n=1}^{+\infty} \sum_{k=1}^{p_n} \mu(A_n^k)$$

But, for all $n \geq 1$, $\bar{\mu}(A_n) = \sum_{k=1}^{p_n} \mu(A_n^k)$, by definition of $\bar{\mu}$. Hence:

$$\bar{\mu}(A) = \sum_{n=1}^{+\infty} \bar{\mu}(A_n)$$

It follows that (ii) of definition (9) is satisfied for $\bar{\mu}$. Finally, $\bar{\mu}$ is a measure on the ring $\mathcal{R}(\mathcal{S})$.

Exercise 12

Exercise 13.

- Uniqueness is a consequence of exercise (10)

- Take $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$ as defined in exercise (11). We proved in exercise (12) that $\bar{\mu}$ is indeed a measure on the ring $\mathcal{R}(\mathcal{S})$. Moreover, given $A \in \mathcal{S}$, if we take $n = 1$ and $A_1 = A$, then $A = \uplus_{i=1}^n A_i$ is a representation of A as a finite union of pairwise disjoint elements of $\mathcal{S}$. By definition of $\bar{\mu}$ (see exercise (11)), it follows that $\bar{\mu}(A) = \mu(A)$. This being true for all $A \in \mathcal{S}$, we have $\bar{\mu}_{|\mathcal{S}} = \mu$. This shows the existence of $\bar{\mu}$, and theorem (2) is proved.

Exercise 13

Exercise 14. Let $(A_n)_{n\geq 1}$ be the sequence of subsets of Ω defined by $A_1 = A$, $A_2 = B$ and $A_n = \emptyset$ for all $n \geq 3$. Using (i) and (iii) of definition (10), we obtain:

$$\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$$

Exercise 14

Exercise 15.

1. μ^* being an outer measure on Ω, by (i) of definition (10), we have $\mu^*(\emptyset) = 0$. It follows that given an arbitrary $T \subseteq \Omega$, $\mu^*(T) = \mu^*(T \cap \Omega) + \mu^*(T \cap \Omega^c)$ is obviously true. Hence, from definition (11), $\Omega \in \Sigma(\mu^*) = \Sigma$. The fact that $A^c \in \Sigma$ is an immediate consequence of definition (11).

2. Since $B \in \Sigma$, using definition (11) with $T \cap A$ in place of T, we obtain:

$$\mu^*(T \cap A) = \mu^*(T \cap A \cap B) + \mu^*(T \cap A \cap B^c)$$

3. Since $A \cap B \subseteq A$, we have $A^c \subseteq (A \cap B)^c$, and consequently:

$$T \cap A^c \subseteq T \cap (A \cap B)^c$$

It follows that:

$$T \cap A^c = (T \cap (A \cap B)^c) \cap T \cap A^c = T \cap (A \cap B)^c \cap A^c$$

4. From $(A \cap B)^c \cap A = (A^c \cup B^c) \cap A = A \cap B^c$, we obtain:

$$T \cap (A \cap B)^c \cap A = T \cap A \cap B^c$$

5. Using 3. and 4., we see that the sum $\mu^*(T \cap A^c) + \mu^*(T \cap A \cap B^c)$ can be expressed as:

$$\mu^*(T \cap (A \cap B)^c \cap A^c) + \mu^*(T \cap (A \cap B)^c \cap A)$$

Since $A \in \Sigma$, using definition (11) with $T \cap (A \cap B)^c$ in place of T, we obtain:

$$\mu^*(T \cap A^c) + \mu^*(T \cap A \cap B^c) = \mu^*(T \cap (A \cap B)^c) \qquad (5)$$

6. Adding $\mu^*(T \cap (A \cap B))$ on both sides of equation (5), it appears that the sum:

$$\mu^*(T \cap A^c) + \mu^*(T \cap A \cap B^c) + \mu^*(T \cap A \cap B)$$

is equal to:

$$\mu^*(T \cap (A \cap B)^c) + \mu^*(T \cap (A \cap B))$$

Since $B \in \Sigma$, using definition (11) with $T \cap A$ in place of T, we obtain:

$$\mu^*(T \cap A^c) + \mu^*(T \cap A) = \mu^*(T \cap (A \cap B)^c) + \mu^*(T \cap (A \cap B))$$

and finally, since $A \in \Sigma$:

$$\mu^*(T) = \mu^*(T \cap (A \cap B)^c) + \mu^*(T \cap (A \cap B))$$

This being true for all $T \subseteq \Omega$, it follows that $A \cap B \in \Sigma$. We have proved that $\Sigma = \Sigma(\mu^*)$ is closed under finite intersection.

7. From $A \cup B = (A^c \cap B^c)^c$ and the fact that Σ is closed under complementation and finite intersection, we have $A \cup B \in \Sigma$. Similarly, $A \setminus B = A \cap B^c \in \Sigma$. The purpose of this exercise is to show that the so-called σ-algebra $\Sigma(\mu^*)$ *associated* with an outer measure μ^*, is closed under finite intersection and union, and closed under complementation and difference.

Exercise 15

Exercise 16.

- Suppose $n \geq 1$, $p \geq 1$ and $B_n \cap B_p \neq \emptyset$. Without loss of generality, we can assume that $n \leq p$. Suppose $n < p$ and $x \in B_n \cap B_p$. Since $x \in B_n$, we have $x \in A_n$. However, since $x \in B_p$, $x \notin A_1 \cup \ldots \cup A_{p-1}$. In particular, $x \notin A_n$. This is a contradiction. It follows that if $B_n \cap B_p \neq \emptyset$ then $n = p$, and $(B_n)_{n\geq 1}$ is a sequence of pairwise disjoint subsets of Ω.

- From exercise (15), all B_n's are in fact elements of Σ.

- Since for all $n \geq 1$, $B_n \subseteq A_n$, we have: $\uplus_{n=1}^{+\infty} B_n \subseteq \cup_{n=1}^{+\infty} A_n$. Conversely, suppose $x \in \cup_{n=1}^{+\infty} A_n$. Then, there exists $n \geq 1$ such that $x \in A_n$. Consider the set:

  $$I(x) \stackrel{\triangle}{=} \{n \geq 1, x \in A_n\}$$

  This set is a non-empty subset of $\mathbf{N}^*$ (the set of all positive integers). It follows that $I(x)$ has a smallest element p. If $p = 1$, then $x \in A_1 = B_1$. If $p > 1$, then $x \in A_p \setminus (A_1 \cup \ldots \cup A_{p-1}) = B_p$.

In any case, $x \in B_p \subseteq \uplus_{n=1}^{+\infty} B_n$. Consequently, it follows that $\cup_{n=1}^{+\infty} A_n \subseteq \uplus_{n=1}^{+\infty} B_n$.

- We have proved that $(B_n)_{n\geq 1}$ is a sequence of pairwise disjoint elements of Σ, such that:

$$\bigcup_{n=1}^{+\infty} A_n = \biguplus_{n=1}^{+\infty} B_n$$

Exercise 16

Exercise 17. Let $B, C \in \Sigma$ be such that $B \cap C = \emptyset$. Since $B \in \Sigma$, using definition (11) with $T \cap (B \uplus C)$ in place of T, we have:

$$\mu^*(T \cap (B \uplus C)) = \mu^*(T \cap (B \uplus C) \cap B) + \mu^*(T \cap (B \uplus C) \cap B^c)$$

From $B \cap C = \emptyset$ and in particular $C \subseteq B^c$, we obtain:

$$\mu^*(T \cap (B \uplus C)) = \mu^*(T \cap B) + \mu^*(T \cap C)$$

Note that it was not necessary to use the fact that *both* B and C were elements of Σ.

Exercise 17

Exercise 18.

1. $\uplus_{n=1}^{N} B_n \in \Sigma$ is an immediate consequence of exercise (15).

2. Using exercise (17) with a simple induction argument, we obtain:

$$\mu^*(T \cap (\uplus_{n=1}^{N} B_n)) = \sum_{n=1}^{N} \mu^*(T \cap B_n)$$

3. Since $\uplus_{n=1}^{N} B_n \subseteq B$, we have $T \cap B^c \subseteq T \cap (\uplus_{n=1}^{N} B_n)^c$. Using (ii) of definition (10), we obtain:

$$\mu^*(T \cap B^c) \leq \mu^*(T \cap (\uplus_{n=1}^{N} B_n)^c)$$

4. Using 2. and 3., if we put $C_N = \uplus_{n=1}^{N} B_n$, we have:

$$\mu^*(T \cap B^c) + \sum_{n=1}^{N} \mu^*(T \cap B_n) \leq \mu^*(T \cap (C_N)^c) + \mu^*(T \cap C_N)$$

However from 1., $C_N \in \Sigma$. Using definition (11), we obtain:

$$\mu^*(T \cap B^c) + \sum_{n=1}^{N} \mu^*(T \cap B_n) \leq \mu^*(T)$$

Taking the limit as $N \to +\infty$, we conclude:

$$\mu^*(T \cap B^c) + \sum_{n=1}^{+\infty} \mu^*(T \cap B_n) \leq \mu^*(T)$$

5. Since $T = (T \cap B^c) \cup (T \cap B)$, using exercise (14):

$$\mu^*(T) \leq \mu^*(T \cap B^c) + \mu^*(T \cap B)$$

However, $T \cap B = \cup_{n=1}^{+\infty} T \cap B_n$. Using (iii) of definition (10), we have:

$$\mu^*(T \cap B) \leq \sum_{n=1}^{+\infty} \mu^*(T \cap B_n)$$

It follows that:

$$\mu^*(T) \leq \mu^*(T \cap B^c) + \mu^*(T \cap B) \leq \mu^*(T \cap B^c) + \sum_{n=1}^{+\infty} \mu^*(T \cap B_n)$$

6. From 4. and 5., we see that $\mu^*(T) = \mu^*(T \cap B^c) + \mu^*(T \cap B)$. This being true for all $T \subseteq \Omega$, it follows that $B = \uplus_{n=1}^{+\infty} B_n \in \Sigma$. Also, from 4. and 5., we have:

$$\mu^*(T) = \mu^*(T \cap B^c) + \sum_{n=1}^{+\infty} \mu^*(T \cap B_n)$$

In particular, taking $T = B$, using the fact that $\mu^*(\emptyset) = 0$, we obtain:

$$\mu^*(B) = \sum_{n=1}^{+\infty} \mu^*(B_n)$$

7. We saw in exercise (15) that Σ contains Ω, and is closed under complementation. If $(A_n)_{n\geq 1}$ is a sequence of elements of

Σ, then from exercise (16), there exists a sequence $(B_n)_{n\geq 1}$ of pairwise disjoint elements of Σ, with $B = \uplus_{n=1}^{+\infty} B_n = \cup_{n=1}^{+\infty} A_n$. In 6., we saw that such B is an element of Σ. It follows that $\cup_{n=1}^{+\infty} A_n \in \Sigma$, and Σ is closed under countable union. Hence, we have proved that Σ is a σ-algebra on Ω. μ^* being an outer measure on Ω, $\mu^*(\emptyset) = 0$. So (i) of definition (9) is satisfied for $\mu^*_{|\Sigma}$. If $(B_n)_{n\geq 1}$ is a sequence of pairwise disjoint elements of Σ, and $B = \uplus_{n=1}^{+\infty} B_n$, we saw in 6. that:

$$\mu^*(B) = \sum_{n=1}^{+\infty} \mu^*(B_n)$$

It follows that (ii) of definition (9) is satisfied for $\mu^*_{|\Sigma}$. Finally, $\mu^*_{|\Sigma}$ is indeed a measure on Σ. The purpose of the exercise is to prove theorem (3).

Exercise 18

Exercise 19.

1. $\mathcal{R}$ being a ring on Ω, $\emptyset \in \mathcal{R}$. If we define a sequence $(A_n)_{n\geq 1}$, with $A_n = \emptyset$ for all $n \geq 1$, then $(A_n)_{n\geq 1}$ is an $\mathcal{R}$-cover of the empty set. It follows that:

$$\mu^*(\emptyset) \leq \sum_{n=1}^{+\infty} \mu(A_n) = 0$$

Moreover, $\mu^*(\emptyset)$ being the infimum over a set of non-negative numbers, we have $\mu^*(\emptyset) \geq 0$. Finally $\mu^*(\emptyset) = 0$.

2. Let $A \subseteq B \subseteq \Omega$. Let $(B_n)_{n\geq 1}$ be an $\mathcal{R}$-cover of B. Then in particular, $(B_n)_{n\geq 1}$ is an $\mathcal{R}$-cover of A. It follows that:

$$\mu^*(A) \leq \sum_{n=1}^{+\infty} \mu(B_n) \tag{6}$$

Hence, $\mu^*(A)$ is a lower bound of all sums involved in (6), as $(B_n)_{n\geq 1}$ ranges over all $\mathcal{R}$-covers of B. $\mu^*(B)$ being the infimum of those sums, it is the greatest of such lower bounds, from which we conclude that $\mu^*(A) \leq \mu^*(B)$.

3. Since $\mu^*(A_n) < +\infty$, we have $\mu^*(A_n) < \mu^*(A_n) + \epsilon/2^n$. It follows that $\mu^*(A_n) + \epsilon/2^n$ cannot be a lower bound of all sums $\sum_{p=1}^{+\infty} \mu(B_p)$, as $(B_p)_{p\geq 1}$ ranges over all $\mathcal{R}$-covers of A_n. Hence, there exists an $\mathcal{R}$-cover $(A_n^p)^{p\geq 1}$ of A_n such that:

$$\sum_{p=1}^{+\infty} \mu(A_n^p) < \mu^*(A_n) + \frac{\epsilon}{2^n}$$

It is important to assume $\mu^*(A_n) < +\infty$, since otherwise the inequality $\mu^*(A_n) \leq \mu^*(A_n) + \epsilon/2^n$ may not be a strict inequality, and the above reasoning would fail.

4. $\mathbf{N}^*$ being the set of positive integers, $\mathbf{N}^* \times \mathbf{N}^*$ is a countable set. There exists a one-to-one map $\phi : \mathbf{N}^* \to \mathbf{N}^* \times \mathbf{N}^*$. Given $k \geq 1$, define $R_k = A_n^p$, where $(n,p) = \phi(k)$. Then $(R_k)_{k\geq 1}$ is a

sequence of elements of $\mathcal{R}$ such that:

$$\bigcup_{n=1}^{+\infty} A_n \subseteq \bigcup_{n=1}^{+\infty}\bigcup_{p=1}^{+\infty} A_n^p = \bigcup_{k=1}^{+\infty} R_k$$

In other words, $(R_k)_{k\geq 1}$ is an $\mathcal{R}$-cover of $\cup_{n=1}^{+\infty} A_n$. Moreover:

$$\sum_{k=1}^{+\infty} \mu(R_k) = \sum_{(n,p)\in \mathbf{N}^*\times\mathbf{N}^*} \mu(A_n^p) = \sum_{n=1}^{+\infty}\sum_{p=1}^{+\infty} \mu(A_n^p)$$

5. It follows from 4. that:

$$\mu^*(\cup_{n=1}^{+\infty} A_n) \leq \sum_{k=1}^{+\infty} \mu(R_k) = \sum_{n=1}^{+\infty}\sum_{p=1}^{+\infty} \mu(A_n^p)$$

Hence, using 3.:

$$\mu^*(\cup_{n=1}^{+\infty} A_n) \leq \sum_{n=1}^{+\infty}(\mu^*(A_n) + \frac{\epsilon}{2^n})$$

and finally:

$$\mu^*(\cup_{n=1}^{+\infty} A_n) \leq \epsilon + \sum_{n=1}^{+\infty} \mu^*(A_n) \quad (7)$$

6. From 1. and 2., we see that (i) and (ii) of definition (10) are satisfied for μ^*. Let $(A_n)_{n\geq 1}$ be a sequence of subsets of Ω. If $\mu^*(A_n) = +\infty$ for some $n \geq 1$, then:

$$\mu^*(\cup_{n=1}^{+\infty} A_n) \leq \sum_{n=1}^{+\infty} \mu^*(A_n) \quad (8)$$

is obviously true. If $\mu^*(A_n) < +\infty$ for all $n \geq 1$, then given $\epsilon > 0$ from 5., inequality (7) holds. Since ϵ is arbitrary, it follows that inequality (8) still holds. Hence, (iii) of definition (10) is satisfied for μ^*. Finally, μ^* is an outer-measure on Ω.

Exercise 19

Exercise 20.

1. Since $A \in \mathcal{R}$, the sequence $(R_n)_{n\geq 1}$ defined by $R_1 = A$ and $R_n = \emptyset$ for all $n \geq 2$, is an $\mathcal{R}$-cover of A. Hence:

$$\mu^*(A) \leq \sum_{n=1}^{+\infty} \mu(R_n) = \mu(A)$$

2. Suppose $n \geq 1$, $p \geq 1$ and $B_n \cap B_p \neq \emptyset$. Without loss of generality, we can assume that $n \leq p$. Suppose $n < p$ and $x \in B_n \cap B_p$. Since $x \in B_n$, we have $x \in A_n \cap A$. However, since $x \in B_p$, $x \notin (A_1 \cap A) \cup \ldots \cup (A_{p-1} \cap A)$. In particular, $x \notin A_n \cap A$. This is a contradiction. It follows that if $B_n \cap B_p \neq \emptyset$ then $n = p$, and $(B_n)_{n\geq 1}$ is a sequence of pairwise disjoint subsets of Ω. From exercise (1), we know that a ring is closed under finite intersection. From (ii) and (iii) of definition (7), it is also closed under finite union and difference. It follows that all B_n's are in fact elements of $\mathcal{R}$. Since for all $n \geq 1$, $B_n \subseteq A_n \cap A$, we

have:

$$\biguplus_{n=1}^{+\infty} B_n \subseteq \bigcup_{n=1}^{+\infty} A_n \cap A = A \cap \bigcup_{n=1}^{+\infty} A_n = A$$

Conversely, suppose $x \in A \subseteq \cup_{n=1}^{+\infty} A_n$. Then, there exists $n \geq 1$ such that $x \in A_n \cap A$. Consider the set:

$$I(x) \stackrel{\triangle}{=} \{n \geq 1, x \in A_n \cap A\}$$

This set is a non-empty subset of $\mathbf{N}^*$ (the set of all positive integers). It follows that $I(x)$ has a smallest element p. If $p = 1$, then $x \in A_1 \cap A = B_1$. If $p > 1$, then by definition of p, we have $x \in (A_p \cap A) \setminus ((A_1 \cap A) \cup \ldots \cup (A_{p-1} \cap A)) = B_p$. In any case, $x \in B_p \subseteq \uplus_{n=1}^{+\infty} B_n$. Consequently, it follows that $A \subseteq \uplus_{n=1}^{+\infty} B_n$. We have proved that $(B_n)_{n\geq 1}$ is a sequence of pairwise disjoint elements of $\mathcal{R}$, such that: $A = \biguplus_{n=1}^{+\infty} B_n$

3. μ being a measure on $\mathcal{R}$, from 2. we obtain:

$$\mu(A) = \sum_{n=1}^{+\infty} \mu(B_n)$$

Since for all $n \geq 1$, we have $B_n \subseteq A_n$, it follows from exercise (9) that $\mu(B_n) \leq \mu(A_n)$. Hence:

$$\mu(A) \leq \sum_{n=1}^{+\infty} \mu(A_n) \tag{9}$$

The $\mathcal{R}$-cover $(A_n)_{n\geq 1}$ of A being arbitrary, we see that $\mu(A)$ is a lower bound of all sums involved in (9), as $(A_n)_{n\geq 1}$ ranges across all $\mathcal{R}$-covers of A. $\mu^*(A)$ being the greatest of such lower bounds, it follows that $\mu(A) \leq \mu^*(A)$. Using 1., we conclude that $\mu(A) = \mu^*(A)$. This being true for all $A \in \mathcal{R}$, we have proved that $\mu^*_{|\mathcal{R}} = \mu$.

Exercise 20

Exercise 21.

1. We saw in exercise (19) that μ^* is an outer measure on Ω. From exercise (14), and the fact that $T = (T \cap A) \cup (T \cap A^c)$, we obtain:
$$\mu^*(T) \leq \mu^*(T \cap A) + \mu^*(T \cap A^c)$$

2. If $(T_n)_{n\geq 1}$ is an $\mathcal{R}$-cover of T, then in particular $T_n \in \mathcal{R}$ for all $n \geq 1$. Since $A \in \mathcal{R}$, it follows from exercise (1) that $T_n \cap A \in \mathcal{R}$, and from (iii) of definition (7) that $T_n \cap A^c = T_n \setminus A \in \mathcal{R}$, for all $n \geq 1$. Moreover, from $T \subseteq \cup_{n=1}^{+\infty} T_n$, we have:
$$T \cap A \subseteq \bigcup_{n=1}^{+\infty} T_n \cap A$$
$$T \cap A^c \subseteq \bigcup_{n=1}^{+\infty} T_n \cap A^c$$
We conclude that $(T_n \cap A)_{n\geq 1}$ and $(T_n \cap A^c)_{n\geq 1}$ are $\mathcal{R}$-covers of $T \cap A$ and $T \cap A^c$ respectively.

3. It follows from 2. that:

$$\mu^*(T \cap A) \leq \sum_{n=1}^{+\infty} \mu(T_n \cap A)$$

$$\mu^*(T \cap A^c) \leq \sum_{n=1}^{+\infty} \mu(T_n \cap A^c)$$

However, μ being a measure on $\mathcal{R}$, from exercise (9), we have:

$$\mu(T_n) = \mu(T_n \cap A) + \mu(T_n \cap A^c)$$

for all $n \geq 1$. It follows that:

$$\mu^*(T \cap A) + \mu^*(T \cap A^c) \leq \sum_{n=1}^{+\infty} \mu(T_n)$$

This being true for all $\mathcal{R}$-covers $(T_n)_{n\geq 1}$ of T, we finally have:

$$\mu^*(T \cap A) + \mu^*(T \cap A^c) \leq \mu^*(T)$$

4. Given $A \in \mathcal{R}$, we see from 1. and 3. that for all $T \subseteq \Omega$:

$$\mu^*(T) = \mu^*(T \cap A) + \mu^*(T \cap A^c)$$

Hence, from definition (11), it follows that A is an element of $\Sigma(\mu^*)$, (the σ-algebra associated with the outer measure μ^*). This being true for all $A \in \mathcal{R}$, we have proved that $\mathcal{R} \subseteq \Sigma(\mu^*)$.

5. The σ-algebra $\sigma(\mathcal{R})$ generated by $\mathcal{R}$, is the smallest σ-algebra on Ω containing $\mathcal{R}$. Thus, it follows immediately from 4. that $\sigma(\mathcal{R}) \subseteq \Sigma(\mu^*)$.

Exercise 21

Exercise 22.

- Let $\mu' : \sigma(\mathcal{R}) \to [0, +\infty]$ be defined by $\mu' = \mu^*_{|\sigma(\mathcal{R})}$, where μ^* is the outer measure on Ω defined in exercise (19). We saw in exercise (20) that $\mu^*_{|\mathcal{R}} = \mu$. Hence, since $\mathcal{R} \subseteq \sigma(\mathcal{R})$, we have $\mu'_{|\mathcal{R}} = \mu^*_{|\mathcal{R}} = \mu$.

- From theorem (3), we know that $\mu^*_{|\Sigma(\mu^*)}$ is a measure on $\Sigma(\mu^*)$. However, $\sigma(\mathcal{R}) \subseteq \Sigma(\mu^*)$ (exercise (21)). It is an immediate consequence of definition (9), that if we restrict the measure $\mu^*_{|\Sigma(\mu^*)}$ to the smaller σ-algebra $\sigma(\mathcal{R})$, the resulting map is a measure defined on $\sigma(\mathcal{R})$. But the restriction of $\mu^*_{|\Sigma(\mu^*)}$ to $\sigma(\mathcal{R})$ is nothing but μ'. It follows that μ' is indeed a measure on $\sigma(\mathcal{R})$. This proves theorem (4).

Exercise 22

Exercise 23. Let $\mathcal{S}$ be a semi-ring on Ω. Since $\mathcal{S} \subseteq \mathcal{R}(\mathcal{S}) \subseteq \sigma(\mathcal{R}(\mathcal{S}))$, we have $\sigma(\mathcal{S}) \subseteq \sigma(\mathcal{R}(\mathcal{S}))$. However, $\mathcal{S} \subseteq \sigma(\mathcal{S})$. Moreover, from exercise (7), $\mathcal{R}(\mathcal{S})$ is the set of all finite unions of elements of $\mathcal{S}$. Since the σ-algebra $\sigma(\mathcal{S})$ is in particular closed under finite union, it follows that $\mathcal{R}(\mathcal{S}) \subseteq \sigma(\mathcal{S})$ and consequently $\sigma(\mathcal{R}(\mathcal{S})) \subseteq \sigma(\mathcal{S})$. Finally, we have proved that $\sigma(\mathcal{R}(\mathcal{S})) = \sigma(\mathcal{S})$.

Exercise 23

Exercise 24. From theorem (2), the measure $\mu : \mathcal{S} \to [0, +\infty]$ can be extended to the ring $\mathcal{R}(\mathcal{S})$ generated by $\mathcal{S}$. In other words, there exists a measure $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$ such that $\bar{\mu}_{|\mathcal{S}} = \mu$. From theorem (4), the measure $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$ can be extended the σ-algebra $\sigma(\mathcal{R}(\mathcal{S}))$ generated by $\mathcal{R}(\mathcal{S})$. In other words, there exists a measure $\mu' : \sigma(\mathcal{R}(\mathcal{S})) \to [0, +\infty]$, such that ${\mu'}_{|\mathcal{R}(\mathcal{S})} = \bar{\mu}$. However, from exercise (23), $\sigma(\mathcal{R}(\mathcal{S})) = \sigma(\mathcal{S})$. Moreover, since $\mathcal{S} \subseteq \mathcal{R}(\mathcal{S})$, we have ${\mu'}_{|\mathcal{S}} = \bar{\mu}_{|\mathcal{S}} = \mu$. It follows that μ' is a measure on $\sigma(\mathcal{S})$ such that ${\mu'}_{|\mathcal{S}} = \mu$. This proves theorem (5).

Exercise 24