11. Complex Measures

In the following, $(\Omega, \mathcal{F})$ denotes an arbitrary measurable space.

Definition 90 *Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers. We say that $(a_n)_{n\geq 1}$ has the* **permutation property** *if and only if, for all bijections $\sigma : \mathbf{N}^* \to \mathbf{N}^*$, the series $\sum_{k=1}^{+\infty} a_{\sigma(k)}$ converges in $\mathbf{C}$*[1]

Exercise 1. Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers.

1. Show that if $(a_n)_{n\geq 1}$ has the permutation property, then the same is true of $(Re(a_n))_{n\geq 1}$ and $(Im(a_n))_{n\geq 1}$.

2. Suppose $a_n \in \mathbf{R}$ for all $n \geq 1$. Show that if $\sum_{k=1}^{+\infty} a_k$ converges:

$$\sum_{k=1}^{+\infty} |a_k| = +\infty \ \Rightarrow \ \sum_{k=1}^{+\infty} a_k^+ = \sum_{k=1}^{+\infty} a_k^- = +\infty$$

[1] which excludes $\pm\infty$ as limit.

Exercise 2. Let $(a_n)_{n\geq 1}$ be a sequence in $\mathbf{R}$, such that the series $\sum_{k=1}^{+\infty} a_k$ converges, and $\sum_{k=1}^{+\infty} |a_k| = +\infty$. Let $A > 0$. We define:

$$N^+ \stackrel{\triangle}{=} \{k \geq 1 \ : \ a_k \geq 0\} \ \ , \ \ N^- \stackrel{\triangle}{=} \{k \geq 1 \ : \ a_k < 0\}$$

1. Show that N^+ and N^- are infinite.

2. Let $\phi^+ : \mathbf{N}^* \to N^+$ and $\phi^- : \mathbf{N}^* \to N^-$ be two bijections. Show the existence of $k_1 \geq 1$ such that:

$$\sum_{k=1}^{k_1} a_{\phi^+(k)} \geq A$$

3. Show the existence of an increasing sequence $(k_p)_{p\geq 1}$ such that:

$$\sum_{k=k_{p-1}+1}^{k_p} a_{\phi^+(k)} \geq A$$

for all $p \geq 1$, where $k_0 = 0$.

4. Consider the permutation $\sigma : \mathbf{N}^* \to \mathbf{N}^*$ defined informally by:

$$(\phi^-(1), \underbrace{\phi^+(1), \ldots, \phi^+(k_1)}, \phi^-(2), \underbrace{\phi^+(k_1+1), \ldots, \phi^+(k_2)}, \ldots)$$

representing $(\sigma(1), \sigma(2), \ldots)$. More specifically, define $k_0^* = 0$ and $k_p^* = k_p + p$ for all $p \geq 1$. For all $n \in \mathbf{N}^*$ and $p \geq 1$ with: [2]

$$k_{p-1}^* < n \leq k_p^* \tag{1}$$

we define:

$$\sigma(n) = \left\{ \begin{array}{lll} \phi^-(p) & \text{if} & n = k_{p-1}^* + 1 \\ \phi^+(n-p) & \text{if} & n > k_{p-1}^* + 1 \end{array} \right. \tag{2}$$

Show that $\sigma : \mathbf{N}^* \to \mathbf{N}^*$ is indeed a bijection.

[2]Given an integer $n \geq 1$, there exists a unique $p \geq 1$ such that (1) holds.

5. Show that if $\sum_{k=1}^{+\infty} a_{\sigma(k)}$ converges, there is $N \geq 1$, such that:

$$n \geq N \ , \ p \geq 1 \ \Rightarrow \ \left| \sum_{k=n+1}^{n+p} a_{\sigma(k)} \right| < A$$

6. Explain why $(a_n)_{n\geq 1}$ cannot have the permutation property.

7. Prove the following theorem:

Theorem 56 *Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers such that for all bijections $\sigma : \mathbf{N}^* \to \mathbf{N}^*$, the series $\sum_{k=1}^{+\infty} a_{\sigma(k)}$ converges. Then, the series $\sum_{k=1}^{+\infty} a_k$ converges absolutely, i.e.*

$$\sum_{k=1}^{+\infty} |a_k| < +\infty$$

Definition 91 *Let $(\Omega, \mathcal{F})$ be a measurable space and $E \in \mathcal{F}$. We call* **measurable partition** *of E, any sequence $(E_n)_{n\geq 1}$ of pairwise disjoint elements of $\mathcal{F}$, such that $E = \uplus_{n\geq 1} E_n$.*

Definition 92 *We call* **complex measure** *on a measurable space $(\Omega, \mathcal{F})$ any map $\mu : \mathcal{F} \to \mathbf{C}$, such that for all $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ measurable partition of E, the series $\sum_{n=1}^{+\infty} \mu(E_n)$ converges to $\mu(E)$. The set of all complex measures on $(\Omega, \mathcal{F})$ is denoted $M^1(\Omega, \mathcal{F})$.*

Definition 93 *We call* **signed measure** *on a measurable space $(\Omega, \mathcal{F})$, any complex measure on $(\Omega, \mathcal{F})$ with values in $\mathbf{R}$.*[3]

Exercise 3.

1. Show that a measure on $(\Omega, \mathcal{F})$ may not be a complex measure.
2. Show that for all $\mu \in M^1(\Omega, \mathcal{F})$, $\mu(\emptyset) = 0$.

[3] In these tutorials, signed measure may not have values in $\{-\infty, +\infty\}$.

3. Show that a finite measure on $(\Omega, \mathcal{F})$ is a complex measure with values in $\mathbf{R}^+$, and conversely.

4. Let $\mu \in M^1(\Omega, \mathcal{F})$. Let $E \in \mathcal{F}$ and $(E_n)_{n \geq 1}$ be a measurable partition of E. Show that:

$$\sum_{n=1}^{+\infty} |\mu(E_n)| < +\infty$$

5. Let μ be a measure on $(\Omega, \mathcal{F})$ and $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Define:

$$\forall E \in \mathcal{F} \ , \ \nu(E) \stackrel{\triangle}{=} \int_E f d\mu$$

Show that ν is a complex measure on $(\Omega, \mathcal{F})$.

Definition 94 *Let μ be a complex measure on a measurable space $(\Omega, \mathcal{F})$. We call* **total variation** *of μ, the map $|\mu| : \mathcal{F} \to [0, +\infty]$, defined by:*

$$\forall E \in \mathcal{F} \ , \ |\mu|(E) \stackrel{\triangle}{=} \sup \sum_{n=1}^{+\infty} |\mu(E_n)|$$

where the 'sup' is taken over all measurable partitions $(E_n)_{n\geq 1}$ of E.

Exercise 4. Let μ be a complex measure on $(\Omega, \mathcal{F})$.

1. Show that for all $E \in \mathcal{F}$, $|\mu(E)| \leq |\mu|(E)$.
2. Show that $|\mu|(\emptyset) = 0$.

Exercise 5. Let μ be a complex measure on $(\Omega, \mathcal{F})$. Let $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E.

1. Show that there exists $(t_n)_{n\geq 1}$ in $\mathbf{R}$, with $t_n < |\mu|(E_n)$ for all n.

2. Show that for all $n \geq 1$, there exists a measurable partition $(E_n^p)_{p\geq 1}$ of E_n such that:

$$t_n < \sum_{p=1}^{+\infty} |\mu(E_n^p)|$$

3. Show that $(E_n^p)_{n,p\geq 1}$ is a measurable partition of E.

4. Show that for all $N \geq 1$, we have $\sum_{n=1}^{N} t_n \leq |\mu|(E)$.

5. Show that for all $N \geq 1$, we have:

$$\sum_{n=1}^{N} |\mu|(E_n) \leq |\mu|(E)$$

6. Suppose that $(A_p)_{p\geq 1}$ is another arbitrary measurable partition

of E. Show that for all $p \geq 1$:

$$|\mu(A_p)| \leq \sum_{n=1}^{+\infty} |\mu(A_p \cap E_n)|$$

7. Show that for all $n \geq 1$:

$$\sum_{p=1}^{+\infty} |\mu(A_p \cap E_n)| \leq |\mu|(E_n)$$

8. Show that:

$$\sum_{p=1}^{+\infty} |\mu(A_p)| \leq \sum_{n=1}^{+\infty} |\mu|(E_n)$$

9. Show that $|\mu| : \mathcal{F} \to [0, +\infty]$ is a measure on $(\Omega, \mathcal{F})$.

Exercise 6. Let $a, b \in \mathbf{R}, a < b$. Let $F \in C^1([a,b];\mathbf{R})$, and define:

$$\forall x \in [a,b]\ ,\ H(x) \stackrel{\triangle}{=} \int_a^x F'(t)dt$$

1. Show that $H \in C^1([a,b];\mathbf{R})$ and $H' = F'$.
2. Show that:
$$F(b) - F(a) = \int_a^b F'(t)dt$$
3. Show that:
$$\frac{1}{2\pi}\int_{-\pi/2}^{+\pi/2} \cos\theta d\theta = \frac{1}{\pi}$$
4. Let $u \in \mathbf{R}^n$ and $\tau_u : \mathbf{R}^n \to \mathbf{R}^n$ be the translation $\tau_u(x) = x + u$. Show that the Lebesgue measure dx on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ is invariant by translation τ_u, i.e. $dx(\{\tau_u \in B\}) = dx(B)$ for all $B \in \mathcal{B}(\mathbf{R}^n)$.

5. Show that for all $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, and $u \in \mathbf{R}^n$:

$$\int_{\mathbf{R}^n} f(x+u)dx = \int_{\mathbf{R}^n} f(x)dx$$

6. Show that for all $\alpha \in \mathbf{R}$, we have:

$$\int_{-\pi}^{+\pi} \cos^+(\alpha - \theta)d\theta = \int_{-\pi-\alpha}^{+\pi-\alpha} \cos^+ \theta d\theta$$

7. Let $\alpha \in \mathbf{R}$ and $k \in \mathbf{Z}$ such that $k \leq \alpha/2\pi < k+1$. Show:

$$-\pi - \alpha \leq -2k\pi - \pi < \pi - \alpha \leq -2k\pi + \pi$$

8. Show that:

$$\int_{-\pi-\alpha}^{-2k\pi-\pi} \cos^+ \theta d\theta = \int_{\pi-\alpha}^{-2k\pi+\pi} \cos^+ \theta d\theta$$

9. Show that:

$$\int_{-\pi-\alpha}^{+\pi-\alpha} \cos^+ \theta d\theta = \int_{-2k\pi-\pi}^{-2k\pi+\pi} \cos^+ \theta d\theta = \int_{-\pi}^{+\pi} \cos^+ \theta d\theta$$

10. Show that for all $\alpha \in \mathbf{R}$:

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} \cos^+(\alpha - \theta) d\theta = \frac{1}{\pi}$$

Exercise 7. Let $z_1, \ldots, z_N$ be N complex numbers. Let $\alpha_k \in \mathbf{R}$ be such that $z_k = |z_k|e^{i\alpha_k}$, for all $k = 1, \ldots, N$. For all $\theta \in [-\pi, +\pi]$, we define $S(\theta) = \{k = 1, \ldots, N \ : \ \cos(\alpha_k - \theta) > 0\}$.

1. Show that for all $\theta \in [-\pi, +\pi]$, we have:

$$\left| \sum_{k \in S(\theta)} z_k \right| = \left| \sum_{k \in S(\theta)} z_k e^{-i\theta} \right| \geq \sum_{k \in S(\theta)} |z_k| \cos(\alpha_k - \theta)$$

2. Define $\phi : [-\pi, +\pi] \to \mathbf{R}$ by $\phi(\theta) = \sum_{k=1}^{N} |z_k| \cos^{+}(\alpha_k - \theta)$. Show the existence of $\theta_0 \in [-\pi, +\pi]$ such that:

$$\phi(\theta_0) = \sup_{\theta \in [-\pi, +\pi]} \phi(\theta)$$

3. Show that:

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} \phi(\theta) d\theta = \frac{1}{\pi} \sum_{k=1}^{N} |z_k|$$

4. Conclude that:

$$\frac{1}{\pi} \sum_{k=1}^{N} |z_k| \leq \left| \sum_{k \in S(\theta_0)} z_k \right|$$

Exercise 8. Let $\mu \in M^1(\Omega, \mathcal{F})$. Suppose that $|\mu|(E) = +\infty$ for some $E \in \mathcal{F}$. Define $t = \pi(1 + |\mu(E)|) \in \mathbf{R}^+$.

1. Show that there is a measurable partition $(E_n)_{n\geq 1}$ of E, with:

$$t < \sum_{n=1}^{+\infty} |\mu(E_n)|$$

2. Show the existence of $N \geq 1$ such that:

$$t < \sum_{n=1}^{N} |\mu(E_n)|$$

3. Show the existence of $S \subseteq \{1, \ldots, N\}$ such that:

$$\sum_{n=1}^{N} |\mu(E_n)| \leq \pi \left| \sum_{n\in S} \mu(E_n) \right|$$

4. Show that $|\mu(A)| > t/\pi$, where $A = \uplus_{n\in S} E_n$.

5. Let $B = E \setminus A$. Show that $|\mu(B)| \geq |\mu(A)| - |\mu(E)|$.

6. Show that $E = A \uplus B$ with $|\mu(A)| > 1$ and $|\mu(B)| > 1$.

7. Show that $|\mu|(A) = +\infty$ or $|\mu|(B) = +\infty$.

Exercise 9. Let $\mu \in M^1(\Omega, \mathcal{F})$. Suppose that $|\mu|(\Omega) = +\infty$.

1. Show the existence of $A_1, B_1 \in \mathcal{F}$, such that $\Omega = A_1 \uplus B_1$, $|\mu(A_1)| > 1$ and $|\mu|(B_1) = +\infty$.

2. Show the existence of a sequence $(A_n)_{n\geq 1}$ of pairwise disjoint elements of $\mathcal{F}$, such that $|\mu(A_n)| > 1$ for all $n \geq 1$.

3. Show that the series $\sum_{n=1}^{+\infty} \mu(A_n)$ does not converge to $\mu(A)$ where $A = \uplus_{n=1}^{+\infty} A_n$.

4. Conclude that $|\mu|(\Omega) < +\infty$.

Theorem 57 *Let μ be a complex measure on a measurable space $(\Omega, \mathcal{F})$. Then, its total variation $|\mu|$ is a finite measure on $(\Omega, \mathcal{F})$.*

Exercise 10. Show that $M^1(\Omega, \mathcal{F})$ is a $\mathbf{C}$-vector space, with:

$$\begin{aligned} (\lambda + \mu)(E) &\stackrel{\triangle}{=} \lambda(E) + \mu(E) \\ (\alpha\lambda)(E) &\stackrel{\triangle}{=} \alpha.\lambda(E) \end{aligned}$$

where $\lambda, \mu \in M^1(\Omega, \mathcal{F})$, $\alpha \in \mathbf{C}$, and $E \in \mathcal{F}$.

Definition 95 *Let $\mathcal{H}$ be a $\mathbf{K}$-vector space, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. We call* **norm** *on $\mathcal{H}$, any map $N : \mathcal{H} \to \mathbf{R}^+$, with the following properties:*

$$\begin{array}{rl} (i) & \forall x \in \mathcal{H}\ ,\ (N(x) = 0 \ \Leftrightarrow\ x = 0) \\ (ii) & \forall x \in \mathcal{H}, \forall \alpha \in \mathbf{K}\ ,\ N(\alpha x) = |\alpha| N(x) \\ (iii) & \forall x, y \in \mathcal{H}\ ,\ N(x + y) \leq N(x) + N(y) \end{array}$$

Exercise 11.

1. Explain why $\|.\|_p$ may not be a norm on $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$.
2. Show that $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ is a norm, when $\langle \cdot, \cdot \rangle$ is an inner-product.
3. Show that $\|\mu\| \stackrel{\triangle}{=} |\mu|(\Omega)$ defines a norm on $M^1(\Omega, \mathcal{F})$.

Exercise 12. Let $\mu \in M^1(\Omega, \mathcal{F})$ be a signed measure. Show that:

$$\begin{aligned} \mu^+ &\stackrel{\triangle}{=} \frac{1}{2}(|\mu| + \mu) \\ \mu^- &\stackrel{\triangle}{=} \frac{1}{2}(|\mu| - \mu) \end{aligned}$$

are finite measures such that:

$$\mu = \mu^+ - \mu^- \quad , \quad |\mu| = \mu^+ + \mu^-$$

Exercise 13. Let $\mu \in M^1(\Omega, \mathcal{F})$ and $l : \mathbf{R}^2 \to \mathbf{R}$ be a linear map.

1. Show that l is continuous.
2. Show that $l \circ \mu$ is a signed measure on $(\Omega, \mathcal{F})$. [4]
3. Show that all $\mu \in M^1(\Omega, \mathcal{F})$ can be decomposed as:

$$\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$$

where $\mu_1, \mu_2, \mu_3, \mu_4$ are finite measures.

[4] $l \circ \mu$ refers strictly speaking to $l(Re(\mu), Im(\mu))$.

Solutions to Exercises

Exercise 1.

1. Suppose $(a_n)_{n\geq 1}$ has the permutation property, and let $\sigma : \mathbf{N}^* \to \mathbf{N}^*$ be an arbitrary bijection. Then, the series $\sum_{k=1}^{+\infty} a_{\sigma(k)}$ converges to some $l \in \mathbf{C}$. However, for all $n \geq 1$, we have:

$$\left|\sum_{k=1}^{n} Re(a_{\sigma(k)}) - Re(l)\right| \leq \left|\sum_{k=1}^{n} a_{\sigma(k)} - l\right|$$

   It follows that the series $\sum_{k=1}^{+\infty} Re(a_{\sigma(k)})$ converges to $Re(l)$, and similarly the series $\sum_{k=1}^{+\infty} Im(a_{\sigma(k)})$ converges to $Im(l)$. We conclude that $(Re(a_n))_{n\geq 1}$ and $(Im(a_n))_{n\geq 1}$ have the permutation property.

2. Suppose that $a_n \in \mathbf{R}$ for all $n \geq 1$, and the series $\sum_{k=1}^{+\infty} a_k$ converges. Since $a_k^+ = (|a_k| + a_k)/2$, the series $\sum_{k=1}^{+\infty} a_k^+$ and $\sum_{k=1}^{+\infty} |a_k|$ are either both convergent, or both divergent. In

particular:

$$\sum_{k=1}^{+\infty} |a_k| = +\infty \ \Rightarrow \ \sum_{k=1}^{+\infty} a_k^+ = +\infty$$

Similarly, from $a_k^- = (|a_k| - a_k)/2$, we have:

$$\sum_{k=1}^{+\infty} |a_k| = +\infty \ \Rightarrow \ \sum_{k=1}^{+\infty} a_k^- = +\infty$$

Exercise 1

Exercise 2.

1. Suppose N^+ is finite. Then $N^+ \subseteq \{1, \ldots, n_0\}$ for some $n_0 \geq 1$. It follows that $a_n < 0$ for $n > n_0$, and in particular we have $a_n = -|a_n|$ for $n > n_0$. This contradicts the fact that $\sum_{k=1}^{+\infty} a_k$ is a convergent series, whereas $\sum_{k=1}^{+\infty} |a_k|$ is a divergent series. We conclude that N^+ is an infinite set. Similarly, if N^- is finite, then $a_n = |a_n|$ for n large enough, leading to a contradiction. We have proved that both N^+ and N^- are infinite.

2. Since $\sum_{k=1}^{+\infty} a_k$ converges and $\sum_{k=1}^{+\infty} |a_k| = +\infty$, from ex. (1):

$$+\infty = \sum_{k=1}^{+\infty} a_k^+ = \sum_{k \in N^+} a_k = \sum_{k=1}^{+\infty} a_{\phi^+(k)}$$

where we have used the fact that $\phi^+ : N^* \to N^+$ is a bijection.

It follows that there exists $k_1 \geq 1$ such that:

$$\sum_{k=1}^{k_1} a_{\phi^+(k)} \geq A$$

3. Let $n \geq 1$ and suppose we have $k_1 < \ldots < k_n$ such that:

$$\sum_{k=k_{p-1}+1}^{k_p} a_{\phi^+(k)} \geq A \tag{3}$$

for all $p = 1, \ldots, n$. Since $\sum_{k=k_n+1}^{+\infty} a_{\phi^+(k)} = +\infty$, there exists $k_{n+1} > k_n$ such that:

$$\sum_{k=k_n+1}^{k_{n+1}} a_{\phi^+(k)} \geq A$$

By induction (having found k_1 from 2.), we construct an increasing sequence $(k_p)_{p\geq 1}$ such that (3) holds for all $p \geq 1$.

4. To show that $\sigma : \mathbf{N}^* \to \mathbf{N}^*$ is a bijection, we need to show that it is both injective and surjective. To show that σ is injective, consider $n, m \in \mathbf{N}^*$ such that $\sigma(n) = \sigma(m)$. Let $p, q \in \mathbf{N}^*$ be such that $k^*_{p-1} < n \leq k^*_p$ and $k^*_{q-1} < m \leq k^*_q$.

   Case 1: suppose $n = k^*_{p-1} + 1$ and $m = k^*_{q-1} + 1$. From (2), we have $\sigma(n) = \phi^-(p)$ and $\sigma(m) = \phi^-(q)$, and therefore $\phi^-(p) = \phi^-(q)$. Since $\phi^- : \mathbf{N}^* \to N^-$ is injective , we have $p = q$ and consequently $n = k^*_{p-1} + 1 = k^*_{q-1} + 1 = m$.

   Case 2: suppose $n = k^*_{p-1} + 1$ and $m > k^*_{q-1} + 1$. From (2), we have $\sigma(n) = \phi^-(p) \in N^-$ and $\sigma(m) = \phi^+(m - q) \in N^+$. Since $N^- \cap N^+ = \emptyset$, we conclude that this case cannot occur, having assumed $\sigma(n) = \sigma(m)$.

   Case 3: suppose $n > k^*_{p-1} + 1$ and $m = k^*_{q-1} + 1$. Similarly, this case cannot possibly occur, having assumed $\sigma(n) = \sigma(m)$.

   Case 4: suppose $n > k^*_{p-1} + 1$ and $m > k^*_{q-1} + 1$. From (2), we have $\sigma(n) = \phi^+(n - p)$ and $\sigma(m) = \phi^+(m - q)$, and therefore $\phi^+(n - p) = \phi^+(m - q)$. Since $\phi^+ : \mathbf{N}^* \to N^+$ is injective, it follows that:

$$n - p = m - q \tag{4}$$

Now, if we assume that $p < q$, then $n \leq k_p^* \leq k_{q-1}^* < m - 1$ and therefore:

$$m - 1 - n > k_{q-1}^* - k_p^* = q - 1 - p + k_{q-1} - k_p \geq q - 1 - p$$

and so $m - n > q - p$, contradicting (4). Similarly, assuming $q < p$ leads to a contradiction, from which we conclude that $p = q$. From (4), it follows that $n = m$.

Having assumed that $\sigma(n) = \sigma(m)$, we have proved that necessarily $n = m$. This shows that σ is injective. To show that σ is surjective, given $N \in \mathbf{N}^*$ we need to show the existence of $n \in \mathbf{N}^*$ such that $\sigma(n) = N$.

Case 1: suppose $a_N < 0$. Then $N \in N^-$. Since $\phi^- : \mathbf{N}^* \to N^-$ is surjective, there exists $p \in \mathbf{N}^*$ such that $N = \phi^-(p)$. Take $n = k_{p-1}^* + 1$. From (2), we have $\sigma(n) = \phi^-(p) = N$. Hence, we have found $n \in \mathbf{N}^*$ such that $\sigma(n) = N$.

Case 2: suppose $a_N \geq 0$. Then $N \in N^+$. Since $\phi^+ : \mathbf{N}^* \to N^+$ is surjective, there exists $m \in \mathbf{N}^*$ such that $N = \phi^+(m)$. Let $p \in \mathbf{N}^*$ be such that $k_{p-1} < m \leq k_p$. Then, we have:

$$k_{p-1} + p < m + p < k_p + p$$

or equivalently:

$$k^*_{p-1} + 1 < m + p \leq k^*_p$$

From (2), it follows that:

$$\sigma(m+p) = \phi^+(m + p - p) = \phi^+(m) = N$$

Hence, we have found $n = m + p \in \mathbf{N}^*$ such that $\sigma(n) = N$.

We have proved that $\sigma : \mathbf{N}^* \to \mathbf{N}^*$ is surjective. Having proved that it is also injective, we conclude that it is a bijection.

5. Suppose $\sum_{k=1}^{+\infty} a_{\sigma(k)}$ converges. There exists $l \in \mathbf{R}$ such that for

all $\epsilon > 0$, there exists $N \geq 1$ such that:

$$n \geq N \;\Rightarrow\; \left|\sum_{k=1}^{n} a_{\sigma(k)} - l\right| < \epsilon$$

Taking $\epsilon = A/2$, we have $N \geq 1$, with:

$$n \geq N \;\Rightarrow\; \left|\sum_{k=1}^{n} a_{\sigma(k)} - l\right| < A/2 \tag{5}$$

and also:

$$n \geq N \;,\; p \geq 1 \Rightarrow \left|\sum_{k=1}^{n+p} a_{\sigma(k)} - l\right| < A/2 \tag{6}$$

From the inequality, where $n, p \geq 1$:

$$\left|\sum_{k=n+1}^{n+p} a_{\sigma(k)}\right| \leq \left|\sum_{k=1}^{n+p} a_{\sigma(k)} - l\right| + \left|\sum_{k=1}^{n} a_{\sigma(k)} - l\right|$$

Using (5) and (6), we have found $N \geq 1$ such that:

$$n \geq N\ ,\ p \geq 1\ \Rightarrow\ \left|\sum_{k=n+1}^{n+p} a_{\sigma(k)}\right| < A$$

6. Suppose $(a_n)_{n\geq 1}$ has the permutation property. From definition (90), the series $\sum_{k=1}^{+\infty} a_{\tau(k)}$ converges, for all bijections $\tau : \mathbf{N}^* \to \mathbf{N}^*$. In particular, the series $\sum_{k=1}^{+\infty} a_{\sigma(k)}$ converges, where σ is the bijection defined in part 4.. From 5., there exists $N \geq 1$ such that:

$$n \geq N\ ,\ q \geq 1\ \Rightarrow\ \left|\sum_{k=n+1}^{n+q} a_{\sigma(k)}\right| < A \tag{7}$$

However, from 3., the sequence $(k_p)_{p\geq 1}$ is such that:

$$\left|\sum_{k=k_{p-1}+1}^{k_p} a_{\phi^+(k)}\right| \geq \sum_{k=k_{p-1}+1}^{k_p} a_{\phi^+(k)} \geq A \tag{8}$$

for all $p \geq 1$. Furthermore, if $k_{p-1} + 1 \leq k \leq k_p$ then we have $k^*_{p-1} + 2 \leq k + p \leq k^*_p$, and going back to the definition of σ in equation (2), we see that $\sigma(k+p) = \phi^+(k+p-p) = \phi^+(k)$. Hence, from (8) we obtain:

$$\left| \sum_{k=k_{p-1}+1}^{k_p} a_{\sigma(k+p)} \right| \geq A$$

or equivalently:

$$\left| \sum_{k=k^*_{p-1}+2}^{k^*_p} a_{\sigma(k)} \right| \geq A \tag{9}$$

Since $k^*_p \uparrow +\infty$, we can choose p sufficiently large so as to have $k^*_{p-1} + 1 \geq N$. Taking $q = k^*_p - k^*_{p-1} - 1 \geq 1$ and applying (7), we obtain:

$$\left| \sum_{k=k^*_{p-1}+2}^{k^*_p} a_{\sigma(k)} \right| < A$$

which contradicts (9). We conclude that the series $\sum_{k=1}^{+\infty} a_{\sigma(k)}$ does not converge, and consequently that $(a_n)_{n\geq 1}$ cannot have the permutation property.

7. Let $(a_n)_{n\geq 1}$ be a complex sequence which has the permutation property. From exercise (1), both $(Re(a_n))_{n\geq 1}$ and $(Im(a_n))_{n\geq 1}$ are real valued sequences which have the permutation property. In particular, the series $\sum_{k=1}^{+\infty} Re(a_k)$ converges. If we had $\sum_{k=1}^{+\infty} |Re(a_k)| = +\infty$, then from 6. of the present exercise, we would conclude that $(Re(a_n))_{n\geq 1}$ cannot have the permutation property. It follows that:

$$\sum_{k=1}^{+\infty} |Re(a_k)| < +\infty$$

and similarly:

$$\sum_{k=1}^{+\infty} |Im(a_k)| < +\infty$$

From $|a_k| \leq |Re(a_k)| + |Im(a_k)|$ for all $k \geq 1$, we conclude that:

$$\sum_{k=1}^{+\infty} |a_k| < +\infty$$

which shows that the series $\sum_{k=1}^{+\infty} a_k$ is absolutely convergent. This proves theorem (56).

Exercise 2

Exercise 3.

1. Define $\mu : \mathcal{F} \to [0, +\infty]$ by $\mu(\emptyset) = 0$ and $\mu(A) = +\infty$ for all $A \in \mathcal{F}$, $A \neq \emptyset$. Then μ is a measure on $(\Omega, \mathcal{F})$. However, μ is not a map with values in $\mathbf{C}$. Hence it cannot be a complex measure.

2. Let $\mu \in M^1(\Omega, \mathcal{F})$. Let $E_n = \emptyset$ for all $n \geq 1$. Then $(E_n)_{n\geq 1}$ is a measurable partition of $\emptyset$. It follows that the series $\sum_{n=1}^{+\infty} \mu(E_n)$ converges to $\mu(\emptyset)$. Since $\mu(E_n) = \mu(\emptyset)$ for all $n \geq 1$, this is only possible if $\mu(\emptyset) = 0$.

3. Let μ be a finite measure on $(\Omega, \mathcal{F})$. Then $\mu(\Omega) < +\infty$. Hence for all $A \in \mathcal{F}$, $\mu(A) \leq \mu(\Omega) < +\infty$. So μ has values in $\mathbf{R}^+$ and therefore in $\mathbf{C}$. Let $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E. Then $E = \uplus_{n=1}^{+\infty} E_n$ and μ being a measure:

$$\mu(E) = \sum_{n=1}^{+\infty} \mu(E_n) \tag{10}$$

Since $\mu(E) < +\infty$, the series $\sum_{n=1}^{+\infty} \mu(E_n)$ actually converges to $\mu(E)$ in $\mathbf{C}$. We have proved that μ is a complex measure with values in $\mathbf{R}^+$. Conversely, suppose μ is a complex measure with values in $\mathbf{R}^+$. Then it is a map $\mu : \mathcal{F} \to [0, +\infty]$ which from 2. satisfies $\mu(\emptyset) = 0$. Furthermore, if $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ is a measurable partition of E, then the series $\sum_{n=1}^{+\infty} \mu(E_n)$ converges to $\mu(E)$ in $\mathbf{C}$. So equation (10) holds, and μ is therefore a measure on $(\Omega, \mathcal{F})$. Since μ has values in $\mathbf{R}^+$, $\mu(\Omega) < +\infty$ and μ is therefore a finite measure.

4. Let $\mu \in M^1(\Omega, \mathcal{F})$. Let $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E. Then $(E_n)_{n\geq 1}$ is a sequence of pairwise disjoint elements of $\mathcal{F}$ with $E = \uplus_{n=1}^{+\infty} E_n$. Given $\sigma : \mathbf{N}^* \to \mathbf{N}^*$ bijective, $(E_{\sigma(n)})_{n\geq 1}$ is also a sequence of pairwise disjoint elements of $\mathcal{F}$ with $E = \uplus_{n=1}^{+\infty} E_{\sigma(n)}$. In other words, $(E_{\sigma(n)})_{n\geq 1}$ is a measurable partition of E. Since μ is a complex measure, the series $\sum_{n=1}^{+\infty} \mu(E_{\sigma(n)})$ converges to $\mu(E)$. It follows that the series $\sum_{n=1}^{+\infty} \mu(E_{\sigma(n)})$ converges for all bijections $\sigma : \mathbf{N}^* \to \mathbf{N}^*$. So

$(\mu(E_n))_{n\geq 1}$ is a complex sequence which has the permutation property. Applying theorem (56), we conclude that:

$$\sum_{n=1}^{+\infty} |\mu(E_n)| < +\infty$$

5. Since $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, $\nu(E) = \int_E f d\mu$ is a well-defined complex number for all $E \in \mathcal{F}$. So $\nu : \mathcal{F} \to \mathbf{C}$ is a well-defined map with values in $\mathbf{C}$. Let $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E. Then $(E_n)_{n\geq 1}$ is a sequence of pairwise disjoint elements of $\mathcal{F}$ such that $E = \uplus_{n=1}^{+\infty} E_n$. For all $N \geq 1$, define:

$$g_N = \sum_{n=1}^{N} f 1_{E_n}$$

From the linearity of the integral, we have:

$$\int g_N d\mu = \sum_{n=1}^{N} \int f 1_{E_n} d\mu = \sum_{n=1}^{N} \nu(E_n) \tag{11}$$

Let $\omega \in \Omega$. If $\omega \notin E$ then $f1_E(\omega) = 0$. Furthermore, $\omega \notin E_n$ for all $n \geq 1$ and consequently $g_N(\omega) = 0$ for all $N \geq 1$. In particular, $g_N(\omega) \to f1_E(\omega)$ as $N \to +\infty$. If $\omega \in E$, then $f1_E(\omega) = f(\omega)$. Furthermore, there exists a unique $n_0 \geq 1$ such that $\omega \in E_{n_0}$. For all $N \geq n_0$, we have $g_N(\omega) = f(\omega)$. So $g_N(\omega) \to f1_E(\omega)$ as $N \to +\infty$. We have proved that for all $\omega \in \Omega$, $g_N(\omega) \to f1_E(\omega)$ as $N \to +\infty$. Since for all $N \geq 1$, we have $|g_N| \leq |f| \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, we can apply the dominated convergence theorem (23), to obtain:

$$\lim_{N \to +\infty} \int |g_N - f1_E| d\mu = 0$$

and in particular, using the integral modulus inequality (24):

$$\lim_{N \to +\infty} \int g_N d\mu = \int f1_E d\mu = \nu(E) \tag{12}$$

Comparing (11) with (12) we obtain:

$$\lim_{N\to+\infty}\sum_{n=1}^{N}\nu(E_n)=\nu(E)$$

This shows the series $\sum_{n=1}^{+\infty}\nu(E_n)$ converges to $\nu(E)$. This being true for all $E\in\mathcal{F}$ and measurable partition $(E_n)_{n\geq 1}$ of E, we have proved that ν is a complex measure on $(\Omega,\mathcal{F})$.

Exercise 3

Exercise 4.

1. Let $E \in \mathcal{F}$. Define $E_1 = E$ and $E_n = \emptyset$ for $n \geq 2$. From definition (91), $(E_n)_{n\geq 1}$ is a measurable partition of E. From definition (94), we have $\sum_{n=1}^{+\infty} |\mu(E_n)| \leq |\mu|(E)$. Using $\mu(\emptyset) = 0$ (see exercise (3)), we obtain $|\mu(E)| \leq |\mu|(E)$.

2. From 1. we have $|\mu(\emptyset)| \leq |\mu|(\emptyset)$ and therefore $0 \leq |\mu|(\emptyset)$. Let $(E_n)_{n\geq 1}$ be a measurable partition of $\emptyset$. Then $E_n = \emptyset$ for all $n \geq 1$. Hence, we have:

$$\sum_{n=1}^{+\infty} |\mu(E_n)| = 0 \tag{13}$$

It follows that 0 is an upper-bound of all sums involved in (13), where $(E_n)_{n\geq 1}$ is a measurable partition of $\emptyset$. From definition (94), $|\mu|(\emptyset)$ being the smallest of such upper-bound, we have $|\mu|(\emptyset) \leq 0$. We have proved that $|\mu|(\emptyset) = 0$.

Exercise 4

Exercise 5.

1. From exercise (4), $|\mu(E)| \leq |\mu|(E)$ for all $E \in \mathcal{F}$. In particular $0 \leq |\mu|(E)$. Hence, it is always possible to find $t \in \mathbf{R}$ such that $t < |\mu|(E)$. It follows that we can find a sequence $(t_n)_{n\geq 1}$ in $\mathbf{R}$, such that $t_n < |\mu|(E_n)$ for all $n \geq 1$.

2. Let $n \geq 1$. From definition (94), $|\mu|(E_n)$ is the smallest upper-bound of all sums $\sum_{p=1}^{+\infty} |\mu(E_n^p)|$ where $(E_n^p)_{p\geq 1}$ is a measurable partition of E_n. Since $t_n < |\mu|(E_n)$, t_n cannot be such upper-bound. We conclude that there exists a measurable partition $(E_n^p)_{p\geq 1}$ of E_n, such that:

$$t_n < \sum_{p=1}^{+\infty} |\mu(E_n^p)|$$

3. The family $(E_n^p)_{n,p\geq 1}$ is indexed by the countable set $\mathbf{N}^* \times \mathbf{N}^*$, and is a family of measurable sets (i.e. elements of $\mathcal{F}$). For all $n \geq 1$, $(E_n^p)_{p\geq 1}$ is a family of pairwise disjoint sets such that

$E_n = \uplus_{p\geq 1} E_n^p$. $(E_n)_{n\geq 1}$ is a family of pairwise disjoint sets, such that $E = \uplus_{n\geq 1} E_n$. It follows that $(E_n^p)_{n,p\geq 1}$ is a family of pairwise disjoint sets such that $E = \uplus_{n,p\geq 1} E_n^p$. This shows that $(E_n^p)_{n,p\geq 1}$ is a measurable partition of E.

4. Let $N \geq 1$. Using 2. we have:

$$\sum_{n=1}^{N} t_n < \sum_{n=1}^{N} \sum_{p=1}^{+\infty} |\mu(E_n^p)| \leq \sum_{n=1}^{+\infty} \sum_{p=1}^{+\infty} |\mu(E_n^p)| \leq |\mu|(E) \qquad (14)$$

where the last inequality follows from definition (94) and the fact that $(E_n^p)_{n,p\geq 1}$ is a measurable partition of E.

5. Suppose $|\mu|(E_k) = +\infty$ for some $k = 1, \ldots, N$. Then any choice of $t_k \in \mathbf{R}$ is such that $t_k < |\mu|(E_k)$. Since $\sum_{n=1}^{N} t_n < |\mu|(E)$ obtained in 4. is valid for any $t_1, \ldots, t_N$ in $\mathbf{R}$ such that for all n, $t_n < |\mu|(E_n)$, we see that $A < |\mu|(E)$ for any $A \in \mathbf{R}$ (choose $t_k = A - \sum_{n\neq k} t_n$). It follows that $|\mu|(E) = +\infty$, and

in particular:

$$\sum_{n=1}^{N} |\mu|(E_n) \leq |\mu|(E) \tag{15}$$

Suppose that $|\mu|(E_n) < +\infty$ for all n's. Then $\sum_{n=1}^{N} t_n < |\mu|(E)$ can be written as $\phi(t_1, \ldots, t_N) < |\mu|(E)$, where ϕ is the continuous map $\phi : \mathbf{R}^N \to \mathbf{R}$ defined by $\phi(t_1, \ldots, t_N) = t_1 + \ldots + t_N$. Given $k \geq 1$, the assumption $|\mu|(E_n) < \infty$ implies that we have $|\mu|(E_n) - 1/k < |\mu|(E_n)$, and consequently:

$$\phi(|\mu|(E_1) - 1/k, \ldots, |\mu|(E_N) - 1/k) < |\mu|(E) \tag{16}$$

Taking the limit as $k \to +\infty$ in (16), from the continuity of ϕ we obtain:

$$\phi(|\mu|(E_1), \ldots, |\mu|(E_N)) \leq |\mu|(E)$$

which shows that inequality (15) is true. We have proved that inequality (15) is true in all possible cases.

6. Let $p \geq 1$. $(E_n)_{n\geq 1}$ being a measurable partition of E, we have $E = \uplus_{n\geq 1} E_n$. It follows that $A_p = \uplus_{n\geq 1} A_p \cap E_n$. Since μ is

a complex measure, the series $\sum_{n=1}^{+\infty}\mu(A_p\cap E_n)$ converges to $\mu(A_p)$. Taking the limit as $N\to+\infty$ on both sides of:

$$\left|\sum_{n=1}^{N}\mu(A_p\cap E_n)\right|\leq\sum_{n=1}^{N}|\mu(A_p\cap E_n)|$$

we conclude that:

$$|\mu(A_p)|\leq\sum_{n=1}^{+\infty}|\mu(A_p\cap E_n)|$$

7. Let $n\geq 1$. $(A_p)_{p\geq 1}$ being a measurable partition of E, we have $E=\uplus_{p\geq 1}A_p$. It follows that $E_n=\uplus_{p\geq 1}A_p\cap E_n$. The family $(A_p\cap E_n)_{p\geq 1}$ is therefore a measurable partition of E_n. We conclude from definition (94) that;

$$\sum_{p=1}^{+\infty}|\mu(A_p\cap E_n)|\leq|\mu|(E_n)$$

8. Using 6. and 7. we have:

$$\sum_{p=1}^{+\infty} |\mu(A_p)| \leq \sum_{p=1}^{+\infty}\sum_{n=1}^{+\infty} |\mu(A_p \cap E_n)| \leq \sum_{n=1}^{+\infty} |\mu|(E_n)$$

where specifically, the second inequality was obtained by first inverting the order of summation, and then applying 7.

9. From exercise (4), $|\mu|(\emptyset) = 0$. Given $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ measurable partition of E, we showed in 5. that for all $N \geq 1$:

$$\sum_{n=1}^{N} |\mu|(E_n) \leq |\mu|(E) \tag{17}$$

Taking the limit as $N \to +\infty$ in (17), we obtain:

$$\sum_{n=1}^{+\infty} |\mu|(E_n) \leq |\mu|(E) \tag{18}$$

Also, if $(A_p)_{p\geq 1}$ is a measurable partition of E, then from 8.:

$$\sum_{p=1}^{+\infty} |\mu(A_p)| \leq \sum_{n=1}^{+\infty} |\mu|(E_n)$$

This shows that $\sum_{n=1}^{+\infty} |\mu|(E_n)$ is an upper-bound of all sums $\sum_{p=1}^{+\infty} |\mu(A_p)|$, where $(A_p)_{p\geq 1}$ is a measurable partition of E. $|\mu|(E)$ being the smallest of all such upper-bounds, we have:

$$|\mu|(E) \leq \sum_{n=1}^{+\infty} |\mu|(E_n) \tag{19}$$

From (18) and (19) we conclude that:

$$|\mu|(E) = \sum_{n=1}^{+\infty} |\mu|(E_n)$$

We have proved that $|\mu| : \mathcal{F} \to [0, +\infty]$ is a measure on $(\Omega, \mathcal{F})$.

Exercise 5

Exercise 6.

1. Since $F \in C^1([a,b];\mathbf{R})$, the derivative F' exists and is continuous on [a,b]. In particular, the map $F' : [a,b] \to \mathbf{R}$ is Borel measurable[5]. Furthermore, the interval $[a,b]$ being a compact topological space (theorem (34)), F' attains its maximum and its minimum (theorem (37)). In particular, F' is bounded on $[a,b]$. It follows that F' is an element of $L^1_{\mathbf{R}}([a,b],\mathcal{B}([a,b]),dx)$, and:

$$H(x) = \int_a^x F'(t)dt \stackrel{\triangle}{=} \int 1_{[a,x]}(t)F'(t)dt$$

is well-defined and $\mathbf{R}$-valued for all $x \in [a,b]$.

Let $x_0 \in [a,b]$. F' being continuous on $[a,b]$, given $\epsilon > 0$, there exists $\delta > 0$ such that:

$$x \in [a,b]\ ,\ |x - x_0| \leq \delta \ \Rightarrow\ |F'(x) - F'(x_0)| \leq \epsilon \qquad (20)$$

[5] See exercise (13) of Tutorial 4.

Let $h \in \mathbf{R} \setminus \{0\}$ be such that $x_0 + h \in [a, b]$. If $h > 0$, we have:

$$H(x_0 + h) - H(x_0) = \int 1_{]x_0, x_0+h]}(t)F'(t)dt$$

and if $h < 0$:

$$H(x_0 + h) - H(x_0) = -\int 1_{]x_0+h, x_0]}(t)F'(t)dt$$

where we have used the linearity of the integral, and the equality $1_B - 1_A = 1_{B\setminus A}$, valid whenever $A \subseteq B$. The Lebesgue measure on $[a, b]$ of the interval $]x_0, x_0 + h]$ being equal to h when $h > 0$, it is always possible to write $F'(x_0)$ as:

$$F'(x_0) = \frac{1}{h}\int 1_{]x_0, x_0+h]}(t)F'(x_0)dt$$

when $h > 0$, and similarly when $h < 0$:

$$F'(x_0) = -\frac{1}{h}\int 1_{]x_0+h, x_0]}F'(x_0)dt$$

It follows that in all cases, using theorem (24):

$$\left|\frac{H(x_0+h)-H(x_0)}{h}-F'(x_0)\right| \leq \frac{1}{|h|}\int 1_A(t)|F'(t)-F'(x_0)|dt$$

where $A =]x_0, x_0+h]$ if $h > 0$ and $A =]x_0+h, x_0]$ if $h < 0$. From (20), it appears that given $\epsilon > 0$, we have found $\delta > 0$ such that for all $h \neq 0$ with $x_0 + h \in [a, b]$:

$$|h| \leq \delta \ \Rightarrow \ \left|\frac{H(x_0+h)-H(x_0)}{h}-F'(x_0)\right| \leq \epsilon$$

This shows that for all $x_0 \in [a, b]$, H is differentiable at x_0 with $H'(x_0) = F'(x_0)$. We have proved that H is differentiable on $[a, b]$ with $H' = F'$. Since F' is continuous, we see that H' is continuous, and finally $H \in C^1([a, b]; \mathbf{R})$.

2. Define $G = F - H$. Then $G \in C^1([a, b]; \mathbf{R})$, and in particular G is continuous on $[a, b]$ and differentiable on $]a, b[$. Applying

taylor's theorem (39), there exists $c \in]a, b[$ such that:

$$G(b) - G(a) = G'(c)(b - a)$$

However from 1. $G'(c) = 0$ for all $c \in [a, b]$. We conclude that $G(b) = G(a)$, or equivalently:

$$F(b) - F(a) = H(b) - H(a) = \int_a^b F'(t)dt$$

3. Applying 2. to $F(\theta) = \sin\theta$ on $[-\pi/2, \pi/2]$, we obtain:

$$\frac{1}{2\pi}\int_{-\pi/2}^{+\pi/2} \cos\theta d\theta = \frac{1}{2\pi}(\sin(\pi/2) - \sin(-\pi/2)) = \frac{1}{\pi}$$

4. $u \in \mathbf{R}^n$ being given, let $\mu : \mathcal{B}(\mathbf{R}^n) \to [0, +\infty]$ be the map defined by $\mu(B) = dx(\{\tau_u \in B\})$ for all $B \in \mathcal{B}(\mathbf{R}^n)$. If $(B_n)_{n\geq 1}$ is a sequence of pairwise disjoint elements of $\mathcal{B}(\mathbf{R}^n)$, it follows that $(\tau_u^{-1}(B_n))_{n\geq 1}$ is also a sequence of pairwise disjoint elements of $\mathcal{B}(\mathbf{R}^n)$. Indeed, τ_u being a continuous map, it is also

Borel measurable. So each $\tau_u^{-1}(B_n)$ is an element of $\mathcal{B}(\mathbf{R}^n)$. Furthermore, for all $x \in \mathbf{R}^n$, $x \in \tau_u^{-1}(B_p) \cap \tau_u^{-1}(B_q)$ is equivalent to $\tau_u(x) \in B_p \cap B_q$, which implies that $p = q$. If we denote $B = \uplus_{n\geq 1} B_n$, then $\tau_u^{-1}(B) = \uplus_{n\geq 1}\tau_u^{-1}(B_n)$ and we see that:

$$\mu(B) = dx(\tau_u^{-1}(B)) = \sum_{n=1}^{+\infty} dx(\tau_u^{-1}(B_n)) = \sum_{n=1}^{+\infty} \mu(B_n)$$

Since furthermore it is clear that $\mu(\emptyset) = 0$, we have proved that μ is a measure on $\mathcal{B}(\mathbf{R}^n)$. Let $a_i \leq b_i$ for all $i \in \mathbf{N}_n$, and $B = [a_1, b_1] \times \ldots \times [a_n, b_n]$. Then:

$$\tau_u^{-1}(B) = [a_1 - u_1, b_1 - u_1] \times \ldots \times [a_n - u_n, b_n - u_n] \quad (21)$$

It follows from (21) and definition (63):

$$\mu([a_1, b_1] \times \ldots \times [a_n, b_n]) = dx(\tau_u^{-1}(B)) = \prod_{i=1}^{n}(b_i - a_i) \quad (22)$$

From definition (63), the Lebesgue measure on $\mathbf{R}^n$ is uniquely determined by property (22). We conclude that μ and the Lebesgue measure dx do in fact coincide, i.e. $\mu = dx$. We have proved that for all $u \in \mathbf{R}^n$ and $B \in \mathcal{B}(\mathbf{R}^n)$, $dx(\{\tau_u \in B\}) = dx(B)$ or in other words that the Lebesgue measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ is *invariant by translation*.

5. Let $u \in \mathbf{R}^n$ and $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$. We are aiming to prove that:

$$\int_{\mathbf{R}^n} f(x+u)dx = \int_{\mathbf{R}^n} f(x)dx \tag{23}$$

If $\tau_u : \mathbf{R}^n \to \mathbf{R}^n$ denotes the translation defined by $\tau_u(x) = x + u$, then τ_u is clearly continuous and therefore Borel measurable. It follows that the map $x \to f(x+u)$, being equal to $f \circ \tau_u$, is itself Borel measurable. Suppose equation (23) has been established for non-negative and measurable maps. Then, applying (23) to $|f|$, we obtain:

$$\int_{\mathbf{R}^n} |f(x+u)|dx = \int_{\mathbf{R}^n} |f(x)|dx < +\infty$$

which shows that $x \to f(x+u)$ is also integrable. Equation (23) is therefore meaningful for all $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$. Furthermore, writing $f = v_1 + iv_2$ and applying (23) to each positive and negative part of v_1 and v_2, we obtain:

$$\int_{\mathbf{R}^n} v_1^+(x+u)dx = \int_{\mathbf{R}^n} v_1^+(x)dx$$

with a similar equality for v_1^-, v_2^+ and v_2^-. From definition (48) of the Lebesgue integral, we have:

$$\int_{\mathbf{R}^n} f dx = \int_{\mathbf{R}^n} v_1^+ dx - \int_{\mathbf{R}^n} v_1^- dx + i\int_{\mathbf{R}^n} v_2^+ dx - i\int_{\mathbf{R}^n} v_2^- dx$$

with a similar equality involving $x \to f(x+u)$. We conclude that equation (23) is true for all $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$. We have shown that it is sufficient to prove (23) in the case when $f : (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n)) \to [0, +\infty]$ is a non-negative and measurable map. Suppose f is of the form $f = 1_B$ for some $B \in \mathcal{B}(\mathbf{R}^n)$.

Using the invariance of the Lebesgue measure proved in 4.:

$$\int_{\mathbf{R}^n} f(x+u)dx = dx(\{\tau_u \in B\}) = dx(B) = \int_{\mathbf{R}^n} f(x)dx$$

and (23) is shown to be true. If f is a simple function, then (23) is also true by linearity. Suppose f is a non-negative and measurable map. From theorem (18), there exists a sequence $(s_n)_{n\geq 1}$ of simple functions such that $s_n \uparrow f$. Given $n \geq 1$:

$$\int_{\mathbf{R}^n} s_n(x+u)dx = \int_{\mathbf{R}^n} s_n(x)dx \tag{24}$$

However, from the monotone convergence theorem (19):

$$\lim_{n\to+\infty} \int_{\mathbf{R}^n} s_n(x)dx = \int_{\mathbf{R}^n} f(x)dx$$

with a similar convergence involving $s_n(x+u)$ and $f(x+u)$. Taking the limit in (24) as $n \to +\infty$, we obtain (23).

6. Let $\alpha \in \mathbf{R}$ and define $f(\theta) = \cos^+(\theta - \alpha)1_{[-\pi,+\pi]}(\theta)$. Then:

$$\int_{-\pi}^{+\pi} \cos^+(\alpha - \theta)d\theta = \int_{-\pi}^{+\pi} \cos^+(\theta - \alpha)d\theta = \int_{\mathbf{R}} f(\theta)d\theta$$

Furthermore:

$$\int_{\mathbf{R}} f(\theta+\alpha)d\theta = \int_{\mathbf{R}} (\cos^+ \theta)1_{[-\pi,+\pi]}(\theta+\alpha)d\theta = \int_{-\pi-\alpha}^{+\pi-\alpha} \cos^+ \theta d\theta$$

Applying 5. to $f \in L^1_{\mathbf{R}}(\mathbf{R}, \mathcal{B}(\mathbf{R}), d\theta)$ and $u = \alpha$ we obtain:

$$\int_{\mathbf{R}} f(\theta)d\theta = \int_{\mathbf{R}} f(\theta + \alpha)d\theta$$

and we conclude that:

$$\int_{-\pi}^{+\pi} \cos^+(\alpha - \theta)d\theta = \int_{-\pi-\alpha}^{+\pi-\alpha} \cos^+ \theta d\theta$$

7. Let $\alpha \in \mathbf{R}$ and $k \in \mathbf{Z}$ be such that $k \leq \alpha/2\pi < k+1$. From $k \leq \alpha/2\pi$ we obtain $2k\pi \leq \alpha$ and consequently $-\pi - \alpha \leq -2k\pi - \pi$

together with $\pi - \alpha \leq -2k\pi + \pi$. From $\alpha/2\pi < k+1$ we obtain $\alpha < 2k\pi + 2\pi$ and consequently $-2k\pi - \pi < \pi - \alpha$. Finally:

$$-\pi - \alpha \leq -2k\pi - \pi < \pi - \alpha \leq -2k\pi + \pi$$

8. Define $f(\theta) = (\cos^+ \theta)1_{[-\pi-\alpha,-2k\pi-\pi]}(\theta)$. Applying 5. to the map $f \in L^1_{\mathbf{R}}(\mathbf{R}, \mathcal{B}(\mathbf{R}), d\theta)$ and $u = -2\pi$, we obtain:

$$\int_{-\pi-\alpha}^{-2k\pi-\pi} \cos^+ \theta d\theta = \int_{\mathbf{R}} f(\theta) d\theta = \int_{\mathbf{R}} f(\theta - 2\pi) d\theta = \int_{\pi-\alpha}^{-2k\pi+\pi} \cos^+ \theta d\theta$$

9. From 7. we have:

$$\int_{-\pi-\alpha}^{+\pi-\alpha} \cos^+ \theta d\theta = \int_{-\pi-\alpha}^{-2k\pi-\pi} \cos \theta d\theta + \int_{-2k\pi-\pi}^{+\pi-\alpha} \cos^+ \theta d\theta$$

However, from 8., we have:

$$\int_{-\pi-\alpha}^{-2k\pi-\pi} \cos^+ \theta d\theta = \int_{\pi-\alpha}^{-2k\pi+\pi} \cos^+ \theta d\theta$$

It follows that:

$$\int_{-\pi-\alpha}^{+\pi-\alpha} \cos^+ \theta d\theta = \int_{-2k\pi-\pi}^{-2k\pi+\pi} \cos^+ \theta d\theta \tag{25}$$

Define $f(\theta) = (\cos^+ \theta)1_{[-2k\pi-\pi,-2k\pi+\pi]}(\theta)$. Applying 5. to the map $f \in L^1_{\mathbf{R}}(\mathbf{R}, \mathcal{B}(\mathbf{R}), d\theta)$ and $u = -2k\pi$, we obtain:

$$\int_{-2k\pi-\pi}^{-2k\pi+\pi} \cos^+ \theta d\theta = \int_{\mathbf{R}} f(\theta) d\theta = \int_{\mathbf{R}} f(\theta - 2k\pi) d\theta = \int_{-\pi}^{+\pi} \cos^+ \theta d\theta$$

Using (25), we conclude that:

$$\int_{-\pi-\alpha}^{+\pi-\alpha} \cos^+ \theta d\theta = \int_{-\pi}^{+\pi} \cos^+ \theta d\theta$$

10. For all $\alpha \in \mathbf{R}$, using 6. and 9.:

$$\int_{-\pi}^{+\pi} \cos^+(\alpha - \theta) d\theta = \int_{-\pi}^{+\pi} \cos^+ \theta d\theta$$

However, given $\theta \in [-\pi, +\pi]$, we have $\cos\theta \geq 0$ if and only if $\theta \in [-\pi/2, +\pi/2]$. It follows that:

$$\int_{-\pi}^{+\pi} \cos^+ \theta d\theta = \int_{-\pi/2}^{+\pi/2} \cos\theta d\theta$$

Finally, using 3. we conclude that:

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} \cos^+(\alpha - \theta) d\theta = \frac{1}{2\pi}\int_{-\pi/2}^{+\pi/2} \cos\theta d\theta = \frac{1}{\pi}$$

Exercise 6

Exercise 7.

1. Let $\theta \in [-\pi, \pi]$. Since $|e^{-i\theta}| = 1$, we have:

$$\begin{aligned} \left| \sum_{k \in S(\theta)} z_k \right| &= \left| \sum_{k \in S(\theta)} z_k e^{-i\theta} \right| \\ &= \left| \sum_{k \in S(\theta)} |z_k| e^{i(\alpha_k - \theta)} \right| \\ &\geq Re \left(\sum_{k \in S(\theta)} |z_k| e^{i(\alpha_k - \theta)} \right) \\ &= \sum_{k \in S(\theta)} |z_k| \cos(\alpha_k - \theta) \end{aligned}$$

The fact that $\cos(\alpha_k - \theta) > 0$ for all $k \in S(\theta)$ was not used.

2. The map $\phi(\theta) = \sum_{k=1}^{N} |z_k| \cos^+(\alpha_k - \theta)$ being continuous and

defined on the compact interval $[-\pi, \pi]$, from theorem (37), it attains its maximum. In other words, there exists $\theta_0 \in [-\pi, \pi]$ such that:

$$\phi(\theta_0) = \sup_{\theta \in [-\pi,\pi]} \phi(\theta)$$

3. Using 10. of exercise (6), for all $k = 1, \ldots, N$:

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} \cos^+(\alpha_k - \theta)d\theta = \frac{1}{\pi}$$

   It follows that:

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} \phi(\theta)d\theta = \sum_{k=1}^{N} |z_k| \frac{1}{2\pi}\int_{-\pi}^{+\pi} \cos^+(\alpha_k - \theta)d\theta = \frac{1}{\pi}\sum_{k=1}^{N} |z_k|$$

4. Applying 1. to θ_0 as in 2., we have:

$$\left|\sum_{k \in S(\theta_0)} z_k\right| \geq \sum_{k \in S(\theta_0)} |z_k| \cos(\alpha_k - \theta_0)$$

Since $k \in S(\theta_0)$ is equivalent to $\cos(\alpha_k - \theta_0) > 0$, we have:

$$\sum_{k \in S(\theta_0)} |z_k| \cos(\alpha_k - \theta_0) = \sum_{k=1}^{N} |z_k| \cos^+(\alpha_k - \theta_0) = \phi(\theta_0)$$

where ϕ is defined as in 2. Furthermore, using 2. and 3.:

$$\phi(\theta_0) \geq \frac{1}{2\pi} \int_{-\pi}^{+\pi} \phi(\theta) d\theta = \frac{1}{\pi} \sum_{k=1}^{N} |z_k|$$

We conclude that:

$$\left| \sum_{k \in S(\theta_0)} z_k \right| \geq \frac{1}{\pi} \sum_{k=1}^{N} |z_k|$$

The purpose of this exercise is to provide us with a very useful inequality. We are all familiar with the fact that:

$$\left|\sum_{k=1}^{N} z_k\right| \leq \sum_{k=1}^{N} |z_k|$$

and we may informally say that the modulus of $\sum_{k=1}^{N} z_k$ *is controlled* by the sum $\sum_{k=1}^{N} |z_k|$. By showing that:

$$\sum_{k=1}^{N} |z_k| \leq \pi \left|\sum_{k\in S(\theta_0)} z_k\right|$$

this exercise allows us to *control* $\sum_{k=1}^{N} |z_k|$ in terms of something *formally very close* to the modulus of $\sum_{k=1}^{N} z_k$, i.e. the modulus of $\sum_{k\in S} z_k$, for some subset S of $\{1, \ldots, N\}$.

Exercise 7

Exercise 8.

1. Since $\mu(E) \in \mathbf{C}$, $t = \pi(1 + |\mu(E)|)$ is an element of $\mathbf{R}^+$. In particular, $t < +\infty$. From definition (94), $|\mu|(E)$ is the smallest upper-bound of all sums $\sum_{n=1}^{+\infty} |\mu(E_n)|$, as $(E_n)_{n\geq 1}$ ranges over all measurable partitions of E. Having assumed $|\mu|(E) = +\infty$, it follows that $t < |\mu|(E)$ and consequently t cannot be such upper-bound. We conclude that there exists a measurable partition $(E_n)_{n\geq 1}$ of E, such that:

$$t < \sum_{n=1}^{+\infty} |\mu(E_n)| \tag{26}$$

2. The series $\sum_{n=1}^{+\infty} |\mu(E_n)|$ being the supremum of all partial sums $\sum_{n=1}^{N} |\mu(E_n)|$ for $N \geq 1$, it is the smallest upper-bound of such partial sums. It follows from (26) that t cannot be such upper-

bound. We conclude that there exists $N \geq 1$ such that:

$$t < \sum_{n=1}^{N} |\mu(E_n))|$$

3. Applying 4. of exercise (7) to $z_1 = \mu(E_1), \ldots, z_N = \mu(E_N)$, there exists a subset S of $\{1, \ldots, N\}$ such that:

$$\sum_{n=1}^{N} |\mu(E_n)| \leq \pi \left| \sum_{n \in S} \mu(E_n) \right|$$

4. Let $A = \uplus_{n \in S} E_n$. μ being a complex measure, it is finitely additive and therefore $\mu(A) = \sum_{n \in S} \mu(E_n)$. Using 2. and 3. we obtain:

$$|\mu(A)| \geq \frac{1}{\pi} \sum_{n=1}^{N} |\mu(E_n)| > \frac{t}{\pi}$$

5. Let $B = E \setminus A$. Since $A \subseteq E$, we have $E = A \uplus B$. It follows that $\mu(E) = \mu(A) + \mu(B)$ and consequently

$$|\mu(A)| = |\mu(E) - \mu(B)| \leq |\mu(E)| + |\mu(B)|$$

We conclude that $|\mu(B)| \geq |\mu(A)| - |\mu(E)|$.

6. Since $A \subseteq E$ and $B = E \setminus A$, $E = A \uplus B$. From 4. we obtain:

$$|\mu(A)| > \frac{t}{\pi} = 1 + |\mu(E)| \geq 1$$

and from 4. and 5. we obtain:

$$|\mu(B)| \geq |\mu(A)| - |\mu(E)| > \frac{t}{\pi} - |\mu(E)| = 1$$

We conclude that $|\mu(A)| > 1$ and $|\mu(B)| > 1$.

7. From exercise (5), the total variation $|\mu|$ is a measure on $(\Omega, \mathcal{F})$. From $E = A \uplus B$ we obtain $|\mu|(E) = |\mu|(A) + |\mu|(B)$. Since $|\mu|(E) = +\infty$ we conclude that $|\mu|(A)$ and $|\mu|(B)$ cannot be both finite, i.e. $|\mu|(A) = +\infty$ or $|\mu|(B) = +\infty$. This exercise

shows that if $E \in \mathcal{F}$ is such that $|\mu|(E) = +\infty$, then E can be *partitioned* in two components A and B (i.e. $E = A \uplus B$) such that $|\mu(A)| > 1$ and $|\mu(B)| > 1$, and with $|\mu|(A) = +\infty$ or $|\mu|(B) = +\infty$.

Exercise 8

Exercise 9.

1. Since $|\mu|(\Omega) = +\infty$, applying exercise (8), there exists $A, B \in \mathcal{F}$ such that $\Omega = A \uplus B$, $|\mu(A)| > 1$, $|\mu(B) > 1$ and $|\mu|(A) = +\infty$ or $|\mu|(B) = +\infty$. If $|\mu|(B) = +\infty$, take $A_1 = A$ and $B_1 = B$. Otherwise, take $A_1 = B$ and $B_1 = A$. In any case, we have $A_1, B_1 \in \mathcal{F}$, $\Omega = A_1 \uplus B_1$, $|\mu(A_1)| > 1$ and $|\mu|(B_1) = +\infty$.

2. Given $n \geq 1$, let P_n denote the following statement: there exist $A_1, \ldots, A_n$ pairwise disjoint elements of $\mathcal{F}$ with $|\mu(A_k)| > 1$ for all $k \in \mathbf{N}_n$, and such that if $B_n = (A_1 \uplus \ldots \uplus A_n)^c$, then we have $|\mu|(B_n) = +\infty$. Note that from 1., the statement P_1 is true. Suppose the statement P_n is true for some $n \geq 1$. Applying exercise (8), there exist $A, B \in \mathcal{F}$ such that $B_n = A \uplus B$, $|\mu(A)| > 1$, $|\mu(B)| > 1$ and $|\mu|(A) = +\infty$ or $|\mu|(B) = +\infty$. Without loss of generality, we can assume that $|\mu|(B) = +\infty$. Define $A_{n+1} = A$. Then $|\mu(A_{n+1})| > 1$ and furthermore for all $k \in \mathbf{N}_n$, since $A_k \subseteq B_n^c$ and $A_{n+1} \subseteq B_n$, we have $A_k \cap A_{n+1} = \emptyset$. Having assumed P_n to be true, $A_1, \ldots, A_n$ are pairwise dis-

joint, and it follows that $A_1, \ldots, A_{n+1}$ are also pairwise disjoint elements of $\mathcal{F}$. Finally, if $B_{n+1} = (A_1 \uplus \ldots \uplus A_{n+1})^c$, then $B_{n+1}^c = B_n^c \uplus A_{n+1}$ and consequently:

$$B_{n+1}^c = (A^c \cap B^c) \uplus A = (A^c \cap B^c) \uplus (A \cap B^c) = B^c$$

since $A \cap B = \emptyset$. It follows that $|\mu|(B_{n+1}) = |\mu|(B) = +\infty$. This shows that having assumed the statement P_n to be true, the sequence $A_1, \ldots, A_n$ can be extended to $A_1, \ldots, A_{n+1}$ which satisfies the requirements of statement P_{n+1}. By induction, we can therefore construct a sequence $(A_n)_{n\geq 1}$ of pairwise disjoint elements of $\mathcal{F}$, such that $|\mu(A_n)| > 1$ for all $n \geq 1$.

3. Since $|\mu(A_n)| > 1$ for all $n \geq 1$, the series $\sum_{n=1}^{+\infty} \mu(A_n)$ cannot be a convergent series. In particular, it does not converge to $\mu(A)$ where $A = \uplus_{n\geq 1} A_n$. This contradicts definition (92) and the fact that μ is a complex measure.

4. The initial assumption of $|\mu|(\Omega) = +\infty$ in 1. has lead to the contradiction shown in 3.. We conclude that $|\mu|(\Omega) < +\infty$ for

all complex measure μ. We showed on exercise (5) that the total variation $|\mu|$ of a complex measure μ was a measure. This exercise shows that $|\mu|$ is in fact a finite measure, which proves theorem (57).

Exercise 9

Exercise 10. Let $\lambda, \mu \in M^1(\Omega, \mathcal{F})$ and $E \in \mathcal{F}$. Let $(E_n)_{n\geq 1}$ be a measurable partition of E. Then, the series $\sum_{n=1}^{+\infty} \lambda(E_n)$ and $\sum_{n=1}^{+\infty} \mu(E_n)$ converge to $\lambda(E)$ and $\mu(E)$ respectively. It follows that the series $\sum_{n=1}^{+\infty}(\lambda + \mu)(E_n)$ converges to $(\lambda + \mu)(E)$ and $\lambda + \mu$ is therefore a complex measure on $(\Omega, \mathcal{F})$. If $\alpha \in \mathbf{C}$, then the series $\sum_{n=1}^{+\infty}(\alpha\mu)(E_n)$ converges to $(\alpha\mu)(E)$ and $\alpha\mu$ is therefore a complex measure on $(\Omega, \mathcal{F})$. This shows that $M^1(\Omega, \mathcal{F})$ is a sub-vector space over $\mathbf{C}$, of the set $\mathbf{C}^{\mathcal{F}}$ of all maps $\mu : \mathcal{F} \to \mathbf{C}$.

Exercise 10

Exercise 11.

1. Given $f \in L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$, the condition $\|f\|_p = 0$ is equivalent to $\int |f|^p d\mu = 0$. In particular, it does not guarantee that $f = 0$, but only that $f = 0$ μ-almost surely. Hence, property (i) of definition (95) is not satisfied in general, and $\|\cdot\|_p$ may fail to be a norm on $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$.

2. Let $\langle\cdot,\cdot\rangle$ be an inner-product on a $\mathbf{K}$-vector space $\mathcal{H}$, and let $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$. The fact that given $x \in \mathcal{H}$ $\|x\| = 0$ is equivalent to $x = 0$, is a consequence of property (v) of definition (81). So (i) of definition (95) is satisfied. Given $\alpha \in \mathbf{K}$, using (i) and (iii) of definition (81), we have:

$$\langle \alpha x, \alpha x\rangle = \alpha\bar{\alpha}\langle x, x\rangle = |\alpha|^2\langle x, x\rangle$$

and consequently $\|\alpha x\| = |\alpha|\|x\|$. So (ii) of definition (95) is also satisfied. Finally, the triangle inequality:

$$\|x + y\| \leq \|x\| + \|y\|$$

has been proved in exercise (17) of Tutorial 10. So (iii) of definition (95) is also satisfied. We have proved that $\|\cdot\|$ is indeed a norm on $\mathcal{H}$.

3. Suppose $|\mu|(\Omega) = 0$. Then for all $E \in \mathcal{F}$, we have:

$$|\mu(E)| \leq |\mu|(E) \leq |\mu|(\Omega) = 0$$

and consequently $\mu = 0$. Conversely, if $\mu = 0$ it follows immediately from definition (94) that $|\mu| = 0$ and in particular $\|\mu\| = |\mu|(\Omega) = 0$. So property (i) of definition (95) is satisfied. Let $\alpha \in \mathbf{C}$. Given $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ measurable partition of E, using definition (94) we have:

$$\sum_{n=1}^{+\infty} |\alpha\mu(E_n)| = |\alpha| \sum_{n=1}^{+\infty} |\mu(E_n)| \leq |\alpha||\mu|(E)$$

It follows that $|\alpha||\mu|(E)$ is an upper-bound of all $\sum_{n=1}^{+\infty} |\alpha\mu(E_n)|$ as $(E_n)_{n\geq 1}$ ranges over all measurable partitions of E. From definition (94), $|\alpha\mu|(E)$ being the smallest of such upper-bounds, we obtain $|\alpha\mu|(E) \leq |\alpha||\mu|(E)$. In the case when $\alpha \neq 0$, replacing α by α^{-1} and μ by $\alpha\mu$, we have:

$$|\alpha||\mu|(E) = |\alpha||\alpha^{-1}(\alpha\mu)|(E) \leq |\alpha||\alpha|^{-1}|\alpha\mu|(E)$$

and consequently $|\alpha||\mu|(E) \leq |\alpha\mu|(E)$. This being also true for $\alpha = 0$, we have proved that $|\alpha\mu|(E) = |\alpha||\mu|(E)$ for all complex measure μ, $E \in \mathcal{F}$ and $\alpha \in \mathbf{C}$. Taking $E = \Omega$ we obtain:

$$\|\alpha\mu\| = |\alpha\mu|(\Omega) = |\alpha||\mu|(\Omega) = |\alpha|\|\mu\|$$

and property (ii) of definition (95) is therefore satisfied. Let μ and λ be two complex measures and $E \in \mathcal{F}$. Let $(E_n)_{n\geq 1}$ be a measurable partition of E. We have:

$$\sum_{n=1}^{+\infty} |(\lambda + \mu)(E_n)| \leq \sum_{n=1}^{+\infty} |\lambda(E_n)| + \sum_{n=1}^{+\infty} |\mu(E_n)| \leq |\lambda|(E) + |\mu|(E)$$

and $|\lambda|(E)+|\mu|(E)$ is an upper-bound of all $\sum_{n=1}^{+\infty} |(\lambda+\mu)(E_n)|$, as $(E_n)_{n\geq 1}$ ranges over all measurable partitions of E. From definition (94), $|\lambda+\mu|(E)$ being the smallest of such upper-bounds, we obtain:

$$|\lambda+\mu|(E) \leq |\lambda|(E) + |\mu|(E)$$

In particular for $E=\Omega$, we have $\|\lambda+\mu\| \leq \|\lambda\| + \|\mu\|$. This shows that property (iii) of definition (95) is satisfied. We have proved that $\|\mu\| = |\mu|(\Omega)$ defines a norm on $M^1(\Omega, \mathcal{F})$.

Exercise 11

Exercise 12. Let $\mu \in M^1(\Omega, \mathcal{F})$ and $\mu^+ = (|\mu| + \mu)/2$. From theorem (57), the total variation $|\mu|$ is a finite measure on $(\Omega, \mathcal{F})$, or in other words, a complex measure with values in $\mathbf{R}^+$. Since μ is a signed measure, it is a complex measure with values in $\mathbf{R}$. It follows that μ^+ is a complex measure with values in $\mathbf{R}$. Furthermore, the fact that μ is a signed measure allows us to write $-\mu(E) \leq |\mu(E)|$ for all $E \in \mathcal{F}$. Since $|\mu(E)| \leq |\mu|(E)$ can be seen as an easy consequence of definition (94), we conclude that $-\mu(E) \leq |\mu|(E)$, or equivalently $\mu^+(E) \geq 0$ for all $E \in \mathcal{F}$. So μ^+ is a complex measure with values in $\mathbf{R}^+$, or in other words, it is a finite measure on $(\Omega, \mathcal{F})$. Since $\mu(E) \leq |\mu(E)|$ for all $E \in \mathcal{F}$, we obtain similarly that $\mu^- = (|\mu| - \mu)/2$ is a finite measure on $(\Omega, \mathcal{F})$. The fact that $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$ is clear.

Exercise 12

Exercise 13.

1. Let (e_1, e_2) be the canonical basis of $\mathbf{R}^2$. For all $(x, y) \in \mathbf{R}^2$ and $(x', y') \in \mathbf{R}^2$, we have:

$$\begin{aligned} |l(x,y) - l(x',y')| &= |(x-x')l(e_1) + (y-y')l(e_2)| \\ &\leq \alpha(|x-x'| + |y-y'|) \end{aligned}$$

where $\alpha = \max(|l(e_1)|, |l(e_2)|)$. Since the metric d defined by:

$$d[(x,y),(x',y')] = |x-x'| + |y-y'|$$

induces the product topology on $\mathbf{R}^2$, we conclude that l is a continuous mapping.

2. Let $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E. μ being a complex measure on $(\Omega, \mathcal{F})$, the series $\sum_{n=1}^{+\infty} \mu(E_n)$ converges to $\mu(E)$ in $\mathbf{C} = \mathbf{R}^2$. Since l is a continuous mapping, the series $\sum_{n=1}^{+\infty} l \circ \mu(E_n)$ converges to $l \circ \mu(E)$ in $\mathbf{R}$. This being true for all $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ measurable partition of E, $l \circ \mu$ is a

complex measure with values in $\mathbf{R}$. In other words, $l \circ \mu$ is a signed measure on $(\Omega, \mathcal{F})$.

3. Let $\mu \in M^1(\Omega, \mathcal{F})$. It is always possible to write:

$$\mu = Re(\mu) + iIm(\mu)$$

Since $Re, Im : \mathbf{R}^2 \to \mathbf{R}$ are two linear mappings, it follows from 2. that $Re(\mu)$ and $Im(\mu)$ are two signed measures on $(\Omega, \mathcal{F})$. From exercise (12), $Re(\mu)$ and $Im(\mu)$ can be decomposed as $Re(\mu) = Re(\mu)^+ - Re(\mu)^-$ and $Im(\mu) = Im(\mu)^+ - Im(\mu)^-$. Taking $\mu_1 = Re(\mu)^+$, $\mu_2 = Re(\mu)^-$, $\mu_3 = Im(\mu)^+$ and finally $\mu_4 = Im(\mu)^-$, we obtain:

$$\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$$

where μ_1, μ_2, μ_3 and μ_4 are finite measures on $(\Omega, \mathcal{F})$.

Exercise 13