16. Differentiation

Definition 115 *Let $(\Omega, \mathcal{T})$ be a topological space. A map $f : \Omega \to \bar{\mathbf{R}}$ is said to be* **lower-semi-continuous** *(l.s.c), if and only if:*

$$\forall \lambda \in \mathbf{R}\ ,\ \{\lambda < f\} \text{ is open}$$

We say that f is **upper-semi-continuous** *(u.s.c), if and only if:*

$$\forall \lambda \in \mathbf{R}\ ,\ \{f < \lambda\} \text{ is open}$$

EXERCISE 1. Let $f : \Omega \to \bar{\mathbf{R}}$ be a map, where Ω is a topological space.

1. Show that f is l.s.c if and only if $\{\lambda < f\}$ is open for all $\lambda \in \bar{\mathbf{R}}$.

2. Show that f is u.s.c if and only if $\{f < \lambda\}$ is open for all $\lambda \in \bar{\mathbf{R}}$.

3. Show that every open set U in $\bar{\mathbf{R}}$ can be written:

$$U = V^+ \cup V^- \cup \bigcup_{i \in I}]\alpha_i, \beta_i[$$

for some index set I, $\alpha_i, \beta_i \in \mathbf{R}$, $V^+ = \emptyset$ or $V^+ =]\alpha, +\infty]$, $(\alpha \in \mathbf{R})$ and $V^- = \emptyset$ or $V^- = [-\infty, \beta[$, $(\beta \in \mathbf{R})$.

4. Show that f is continuous if and only if it is both l.s.c and u.s.c.

5. Let $u : \Omega \to \mathbf{R}$ and $v : \Omega \to \bar{\mathbf{R}}$. Let $\lambda \in \mathbf{R}$. Show that:

$$\{\lambda < u + v\} = \bigcup_{\substack{(\lambda_1, \lambda_2) \in \mathbf{R}^2 \\ \lambda_1 + \lambda_2 = \lambda}} \{\lambda_1 < u\} \cap \{\lambda_2 < v\}$$

6. Show that if both u and v are l.s.c, then $u + v$ is also l.s.c.

7. Show that if both u and v are u.s.c, then $u + v$ is also u.s.c.

8. Show that if f is l.s.c, then αf is l.s.c, for all $\alpha \in \mathbf{R}^+$.

9. Show that if f is u.s.c, then αf is u.s.c, for all $\alpha \in \mathbf{R}^+$.

10. Show that if f is l.s.c, then $-f$ is u.s.c.

11. Show that if f is u.s.c, then $-f$ is l.s.c.

12. Show that if V is open in Ω, then $f = 1_V$ is l.s.c.

13. Show that if F is closed in Ω, then $f = 1_F$ is u.s.c.

Exercise 2. Let $(f_i)_{i\in I}$ be an a arbitrary family of maps $f_i : \Omega \to \bar{\mathbf{R}}$, defined on a topological space Ω.

1. Show that if all f_i's are l.s.c, then $f = \sup_{i\in I} f_i$ is l.s.c.

2. Show that if all f_i's are u.s.c, then $f = \inf_{i\in I} f_i$ is u.s.c.

Exercise 3. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let f be an element of $L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$, such that $f \geq 0$.

1. Let $(s_n)_{n\geq 1}$ be a sequence of simple functions on $(\Omega, \mathcal{B}(\Omega))$ such that $s_n \uparrow f$. Define $t_1 = s_1$ and $t_n = s_n - s_{n-1}$ for all $n \geq 2$. Show that t_n is a simple function on $(\Omega, \mathcal{B}(\Omega))$, for all $n \geq 1$.

2. Show that f can be written as:

$$f = \sum_{n=1}^{+\infty} \alpha_n 1_{A_n}$$

where $\alpha_n \in \mathbf{R}^+ \setminus \{0\}$ and $A_n \in \mathcal{B}(\Omega)$, for all $n \geq 1$.

3. Show that $\mu(A_n) < +\infty$, for all $n \geq 1$.

4. Show that there exist K_n compact and V_n open in Ω such that:

$$K_n \subseteq A_n \subseteq V_n \quad , \quad \mu(V_n \setminus K_n) \leq \frac{\epsilon}{\alpha_n 2^{n+1}}$$

for all $\epsilon > 0$ and $n \geq 1$.

5. Show the existence of $N \geq 1$ such that:

$$\sum_{n=N+1}^{+\infty} \alpha_n \mu(A_n) \leq \frac{\epsilon}{2}$$

6. Define $u = \sum_{n=1}^{N} \alpha_n 1_{K_n}$. Show that u is u.s.c.

7. Define $v = \sum_{n=1}^{+\infty} \alpha_n 1_{V_n}$. Show that v is l.s.c.

8. Show that we have $0 \leq u \leq f \leq v$.

9. Show that we have:

$$v = u + \sum_{n=N+1}^{+\infty} \alpha_n 1_{K_n} + \sum_{n=1}^{+\infty} \alpha_n 1_{V_n \setminus K_n}$$

10. Show that $\int v d\mu \leq \int u d\mu + \epsilon < +\infty$.

11. Show that $u \in L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$.

12. Explain why v may fail to be in $L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$.

13. Show that v is μ-a.s. equal to an element of $L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$.

14. Show that $\int (v-u)d\mu \leq \epsilon$.

15. Prove the following:

Theorem 94 (Vitali-Caratheodory) *Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$ and f be an element of $L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$. Then, for all $\epsilon > 0$, there exist measurable maps $u, v : \Omega \to \bar{\mathbf{R}}$, which are μ-a.s. equal to elements of $L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$, such that $u \leq f \leq v$, u is u.s.c, v is l.s.c, and furthermore:*

$$\int (v-u)d\mu \leq \epsilon$$

Definition 116 *Let $(\Omega, \mathcal{T})$ be a topological space. We say that $(\Omega, \mathcal{T})$ is* **connected**, *if and only if the only subsets of Ω which are both open and closed are Ω and $\emptyset$.*

EXERCISE 4. Let $(\Omega, \mathcal{T})$ be a topological space.

1. Show that $(\Omega, \mathcal{T})$ is connected if and only if whenever $\Omega = A \uplus B$ where A, B are disjoint open sets, we have $A = \emptyset$ or $B = \emptyset$.

2. Show that $(\Omega, \mathcal{T})$ is connected if and only if whenever $\Omega = A \uplus B$ where A, B are disjoint closed sets, we have $A = \emptyset$ or $B = \emptyset$.

Definition 117 *Let $(\Omega, \mathcal{T})$ be a topological space, and $A \subseteq \Omega$. We say that A is a* **connected subset** *of Ω, if and only if the induced topological space $(A, \mathcal{T}_{|A})$ is connected.*

EXERCISE 5. Let A be open and closed in $\mathbf{R}$, with $A \neq \emptyset$ and $A^c \neq \emptyset$.

1. Let $x \in A^c$. Show that $A \cap [x, +\infty[$ or $A \cap]-\infty, x]$ is non-empty.

2. Suppose $B = A \cap [x, +\infty[\neq \emptyset$. Show that B is closed and that we have $B = A \cap]x, +\infty[$. Conclude that B is also open.

3. Let $b = \inf B$. Show that $b \in B$ (and in particular $b \in \mathbf{R}$).

4. Show the existence of $\epsilon > 0$ such that $]b - \epsilon, b + \epsilon[\subseteq B$.

5. Conclude with the following:

Theorem 95 *The topological space* $(\mathbf{R}, \mathcal{T}_{\mathbf{R}})$ *is connected.*

EXERCISE 6. Let $(\Omega, \mathcal{T})$ be a topological space and $A \subseteq \Omega$ be a connected subset of Ω. Let B be a subset of Ω such that $A \subseteq B \subseteq \bar{A}$. We assume that $B = V_1 \uplus V_2$ where V_1, V_2 are disjoint open sets in B.

1. Show there is U_1, U_2 open in Ω, with $V_1 = B \cap U_1$, $V_2 = B \cap U_2$.

2. Show that $A \cap U_1 = \emptyset$ or $A \cap U_2 = \emptyset$.

3. Suppose that $A \cap U_1 = \emptyset$. Show that $\bar{A} \subseteq U_1^c$.

4. Show then that $V_1 = B \cap U_1 = \emptyset$.

5. Conclude that B and $\bar{A}$ are both connected subsets of Ω.

Exercise 7. Prove the following:

Theorem 96 *Let $(\Omega, \mathcal{T})$, $(\Omega', \mathcal{T}')$ be two topological spaces, and f be a continuous map, $f : \Omega \to \Omega'$. If $(\Omega, \mathcal{T})$ is connected, then $f(\Omega)$ is a connected subset of Ω'.*

Definition 118 *Let $A \subseteq \bar{\mathbf{R}}$. We say that A is an* **interval***, if and only if for all $x, y \in A$ with $x \leq y$, we have $[x, y] \subseteq A$, where:*

$$[x, y] \stackrel{\triangle}{=} \{z \in \bar{\mathbf{R}} \ : \ x \leq z \leq y\}$$

Exercise 8. Let $A \subseteq \bar{\mathbf{R}}$.

1. If A is an interval, and $\alpha = \inf A$, $\beta = \sup A$, show that:

$$]\alpha, \beta[\subseteq A \subseteq [\alpha, \beta]$$

2. Show that A is an interval if and only if, it is of the form $[\alpha, \beta]$, $[\alpha, \beta[$, $]\alpha, \beta]$ or $]\alpha, \beta[$, for some $\alpha, \beta \in \bar{\mathbf{R}}$.

3. Show that an interval of the form $]-\infty, \alpha[$, where $\alpha \in \mathbf{R}$, is homeomorphic to $]-1, \alpha'[$, for some $\alpha' \in \mathbf{R}$.

4. Show that an interval of the form $]\alpha, +\infty[$, where $\alpha \in \mathbf{R}$, is homeomorphic to $]\alpha', 1[$, for some $\alpha' \in \mathbf{R}$.

5. Show that an interval of the form $]\alpha, \beta[$, where $\alpha, \beta \in \mathbf{R}$ and $\alpha < \beta$, is homeomorphic to $]-1, 1[$.

6. Show that $]-1, 1[$ is homeomorphic to $\mathbf{R}$.

7. Show an non-empty open interval in $\mathbf{R}$, is homeomorphic to $\mathbf{R}$.

8. Show that an open interval in $\mathbf{R}$, is a connected subset of $\mathbf{R}$.

9. Show that an interval in $\mathbf{R}$, is a connected subset of $\mathbf{R}$.

EXERCISE 9. Let $A \subseteq \mathbf{R}$ be a non-empty connected subset of $\mathbf{R}$, and $\alpha = \inf A$, $\beta = \sup A$. We assume there exists $x_0 \in A^c \cap]\alpha, \beta[$.

1. Show that $A \cap]x_0, +\infty[$ or $A \cap] - \infty, x_0[$ is empty.

2. Show that $A \cap]x_0, +\infty[= \emptyset$ leads to a contradiction.

3. Show that $]\alpha, \beta[\subseteq A \subseteq [\alpha, \beta]$.

4. Show the following:

Theorem 97 *For all $A \subseteq \mathbf{R}$, A is a connected subset of $\mathbf{R}$, if and only if A is an interval.*

Exercise 10. Prove the following:

Theorem 98 *Let $f : \Omega \to \mathbf{R}$ be a continuous map, where $(\Omega, \mathcal{T})$ is a connected topological space. Let $a, b \in \Omega$ such that $f(a) \leq f(b)$. Then, for all $z \in [f(a), f(b)]$, there exists $x \in \Omega$ such that $z = f(x)$.*

Exercise 11. Let $a, b \in \mathbf{R}$, $a < b$, and $f : [a, b] \to \mathbf{R}$ be a map such that $f'(x)$ exists for all $x \in [a, b]$.

1. Show that $f' : ([a, b], \mathcal{B}([a, b])) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is measurable.

2. Show that $f' \in L^1_{\mathbf{R}}([a, b], \mathcal{B}([a, b]), dx)$ is equivalent to:

$$\int_a^b |f'(t)| dt < +\infty$$

3. We assume from now on that $f' \in L^1_{\mathbf{R}}([a, b], \mathcal{B}([a, b]), dx)$. Given $\epsilon > 0$, show the existence of $g : [a, b] \to \bar{\mathbf{R}}$, almost surely equal

to an element of $L^1_{\mathbf{R}}([a,b],\mathcal{B}([a,b]),dx)$, such that $f' \leq g$ and g is l.s.c, with:

$$\int_a^b g(t)dt \leq \int_a^b f'(t)dt + \epsilon$$

4. By considering $g+\alpha$ for some $\alpha > 0$, show that without loss of generality, we can assume that $f' < g$ with the above inequality still holding.

5. We define the complex measure $\nu = \int g dx \in M^1([a,b],\mathcal{B}([a,b]))$. Show that:

$$\forall \epsilon' > 0 \ , \ \exists \delta > 0 \ , \ \forall E \in \mathcal{B}([a,b]) \ , \ dx(E) \leq \delta \ \Rightarrow \ |\nu(E)| < \epsilon'$$

6. For all $\eta > 0$ and $x \in [a,b]$, we define:

$$F_\eta(x) \stackrel{\triangle}{=} \int_a^x g(t)dt - f(x) + f(a) + \eta(x-a)$$

Show that $F_\eta : [a,b] \to \mathbf{R}$ is a continuous map.

7. η being fixed, let $x = \sup F_\eta^{-1}(\{0\})$. Show that $x \in [a, b]$ and $F_\eta(x) = 0$.

8. We assume that $x \in [a, b[$. Show the existence of $\delta > 0$ such that for all $t \in]x, x + \delta[\cap[a, b]$, we have:

$$f'(x) < g(t) \quad \text{and} \quad \frac{f(t) - f(x)}{t - x} < f'(x) + \eta$$

9. Show that for all $t \in]x, x+\delta[\cap[a, b]$, we have $F_\eta(t) > F_\eta(x) = 0$.

10. Show that there exists t_0 such that $x < t_0 < b$ and $F_\eta(t_0) > 0$.

11. Show that $F_\eta(b) < 0$ leads to a contradiction.

12. Conclude that $F_\eta(b) \geq 0$, even if $x = b$.

13. Show that $f(b) - f(a) \leq \int_a^b f'(t)dt$, and conclude:

Theorem 99 (Fundamental Calculus) *Let $a, b \in \mathbf{R}$, $a < b$, and $f : [a, b] \to \mathbf{R}$ be a map which is differentiable at every point of $[a, b]$, and such that:*

$$\int_a^b |f'(t)|dt < +\infty$$

Then, we have:

$$f(b) - f(a) = \int_a^b f'(t)dt$$

Exercise 12. Let $\alpha > 0$, and $k_\alpha : \mathbf{R}^n \to \mathbf{R}^n$ defined by $k_\alpha(x) = \alpha x$.

1. Show that $k_\alpha : (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n)) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ is measurable.
2. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$, we have:
$$dx(\{k_\alpha \in B\}) = \frac{1}{\alpha^n} dx(B)$$
3. Show that for all $\epsilon > 0$ and $x \in \mathbf{R}^n$:
$$dx(B(x, \epsilon)) = \epsilon^n dx(B(0, 1))$$

Definition 119 *Let μ be a complex measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, $n \geq 1$, with total variation $|\mu|$. We call* **maximal function** *of μ, the map $M\mu : \mathbf{R}^n \to [0, +\infty]$, defined by:*

$$\forall x \in \mathbf{R}^n \ , \ (M\mu)(x) \stackrel{\triangle}{=} \sup_{\epsilon > 0} \frac{|\mu|(B(x,\epsilon))}{dx(B(x,\epsilon))}$$

where $B(x,\epsilon)$ is the open ball in $\mathbf{R}^n$, of center x and radius ϵ, with respect to the usual metric of $\mathbf{R}^n$.

Exercise 13. Let μ be a complex measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$.

1. Let $\lambda \in \mathbf{R}$. Show that if $\lambda < 0$, then $\{\lambda < M\mu\} = \mathbf{R}^n$.
2. Show that if $\lambda = 0$, then $\{\lambda < M\mu\} = \mathbf{R}^n$ if $\mu \neq 0$, and $\{\lambda < M\mu\}$ is the empty set if $\mu = 0$.
3. Suppose $\lambda > 0$. Let $x \in \{\lambda < M\mu\}$. Show the existence of $\epsilon > 0$ such that $|\mu|(B(x,\epsilon)) = t dx(B(x,\epsilon))$, for some $t > \lambda$.

4. Show the existence of $\delta > 0$ such that $(\epsilon+\delta)^n < \epsilon^n t/\lambda$.

5. Show that if $y \in B(x,\delta)$, then $B(x,\epsilon) \subseteq B(y,\epsilon+\delta)$.

6. Show that if $y \in B(x,\delta)$, then:

$$|\mu|(B(y,\epsilon+\delta)) \geq \frac{\epsilon^n t}{(\epsilon+\delta)^n} dx(B(y,\epsilon+\delta)) > \lambda dx(B(y,\epsilon+\delta))$$

7. Conclude that $B(x,\delta) \subseteq \{\lambda < M\mu\}$, and that the maximal function $M\mu : \mathbf{R}^n \to [0,+\infty]$ is l.s.c, and therefore measurable.

EXERCISE 14. Let $B_i = B(x_i,\epsilon_i)$, $i = 1,\dots,N$, $N \geq 1$, be a finite collection of open balls in $\mathbf{R}^n$. Assume without loss of generality that $\epsilon_N \leq \dots \leq \epsilon_1$. We define a sequence (J_k) of sets by $J_0 = \{1,\dots,N\}$ and for all $k \geq 1$:

$$J_k \triangleq \begin{cases} J_{k-1} \cap \{j : \ j > i_k \ , \ B_j \cap B_{i_k} = \emptyset\} & \text{if } J_{k-1} \neq \emptyset \\ \emptyset & \text{if } J_{k-1} = \emptyset \end{cases}$$

where we have put $i_k = \min J_{k-1}$, whenever $J_{k-1} \neq \emptyset$.

1. Show that if $J_{k-1} \neq \emptyset$ then $J_k \subset J_{k-1}$ (strict inclusion), $k \geq 1$.
2. Let $p = \min\{k \geq 1 : \ J_k = \emptyset\}$. Show that p is well-defined.
3. Let $S = \{i_1, \ldots, i_p\}$. Explain why S is well defined.
4. Suppose that $1 \leq k < k' \leq p$. Show that $i_{k'} \in J_k$.
5. Show that $(B_i)_{i \in S}$ is a family of pairwise disjoint open balls.
6. Let $i \in \{1, \ldots, N\} \setminus S$, and define k_0 to be the minimum of the set $\{k \in \mathbf{N}_p : \ i \notin J_k\}$. Explain why k_0 is well-defined.
7. Show that $i \in J_{k_0 - 1}$ and $i_{k_0} \leq i$.
8. Show that $B_i \cap B_{i_{k_0}} \neq \emptyset$.
9. Show that $B_i \subseteq B(x_{i_{k_0}}, 3\epsilon_{i_{k_0}})$.

10. Conclude that there exists a subset S of $\{1,\ldots,N\}$ such that $(B_i)_{i\in S}$ is a family of pairwise disjoint balls, and:

$$\bigcup_{i=1}^{N} B(x_i,\epsilon_i) \subseteq \bigcup_{i\in S} B(x_i,3\epsilon_i)$$

11. Show that:

$$dx\left(\bigcup_{i=1}^{N} B(x_i,\epsilon_i)\right) \leq 3^n \sum_{i\in S} dx(B(x_i,\epsilon_i))$$

Exercise 15. Let μ be a complex measure on $\mathbf{R}^n$. Let $\lambda > 0$ and K be a non-empty compact subset of $\{\lambda < M\mu\}$.

1. Show that K can be covered by a finite collection $B_i = B(x_i,\epsilon_i)$, $i = 1,\ldots,N$ of open balls, such that:

$$\forall i = 1,\ldots,N\ ,\ \lambda dx(B_i) < |\mu|(B_i)$$

2. Show the existence of $S \subseteq \{1, \ldots, N\}$ such that:

$$dx(K) \leq 3^n \lambda^{-1} |\mu| \left(\bigcup_{i \in S} B(x_i, \epsilon_i) \right)$$

3. Show that $dx(K) \leq 3^n \lambda^{-1} \|\mu\|$

4. Conclude with the following:

Theorem 100 *Let μ be a complex measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, $n \geq 1$, with maximal function $M\mu$. Then, for all $\lambda \in \mathbf{R}^+ \setminus \{0\}$, we have:*

$$dx(\{\lambda < M\mu\}) \leq 3^n \lambda^{-1} \|\mu\|$$

Definition 120 *Let $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, and μ be the complex measure $\mu = \int f dx$ on $\mathbf{R}^n$, $n \geq 1$. We call* **maximal function** *of f, denoted Mf, the maximal function $M\mu$ of μ.*

EXERCISE 16. Let $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, $n \geq 1$.

1. Show that for all $x \in \mathbf{R}^n$:

$$(Mf)(x) = \sup_{\epsilon > 0} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |f| dx$$

2. Show that for all $\lambda > 0$, $dx(\{\lambda < Mf\}) \leq 3^n \lambda^{-1} \|f\|_1$.

Definition 121 *Let $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, $n \geq 1$. We say that $x \in \mathbf{R}^n$ is a* **Lebesgue point** *of f, if and only if we have:*

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |f(y) - f(x)| dy = 0$$

EXERCISE 17. Let $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, $n \geq 1$.

1. Show that if f is continuous at $x \in \mathbf{R}^n$, then x is a Lebesgue point of f.

2. Show that if $x \in \mathbf{R}^n$ is a Lebesgue point of f, then:

$$f(x) = \lim_{\epsilon \downarrow\downarrow 0} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} f(y)dy$$

Exercise 18. Let $n \geq 1$ and $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$. For all $\epsilon > 0$ and $x \in \mathbf{R}^n$, we define:

$$(T_\epsilon f)(x) \stackrel{\triangle}{=} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |f(y) - f(x)|dy$$

and we put, for all $x \in \mathbf{R}^n$:

$$(Tf)(x) \stackrel{\triangle}{=} \limsup_{\epsilon \downarrow\downarrow 0} (T_\epsilon f)(x) \stackrel{\triangle}{=} \inf_{\epsilon > 0} \sup_{u \in]0,\epsilon[} (T_u f)(x)$$

1. Given $\eta > 0$, show the existence of $g \in C^c_{\mathbf{C}}(\mathbf{R}^n)$ such that:

$$\|f - g\|_1 \leq \eta$$

2. Let $h = f - g$. Show that for all $\epsilon > 0$ and $x \in \mathbf{R}^n$:

$$(T_\epsilon h)(x) \leq \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |h| dx + |h(x)|$$

3. Show that $Th \leq Mh + |h|$.

4. Show that for all $\epsilon > 0$, we have $T_\epsilon f \leq T_\epsilon g + T_\epsilon h$.

5. Show that $Tf \leq Tg + Th$.

6. Using the continuity of g, show that $Tg = 0$.

7. Show that $Tf \leq Mh + |h|$.

8. Show that for all $\alpha > 0$, $\{2\alpha < Tf\} \subseteq \{\alpha < Mh\} \cup \{\alpha < |h|\}$.

9. Show that $dx(\{\alpha < |h|\}) \leq \alpha^{-1} \|h\|_1$.

10. Conclude that for all $\alpha > 0$ and $\eta > 0$, there is $N_{\alpha,\eta} \in \mathcal{B}(\mathbf{R}^n)$ such that $\{2\alpha < Tf\} \subseteq N_{\alpha,\eta}$ and $dx(N_{\alpha,\eta}) \leq \eta$.

11. Show that for all $\alpha > 0$, there exists $N_\alpha \in \mathcal{B}(\mathbf{R}^n)$ such that $\{2\alpha < Tf\} \subseteq N_\alpha$ and $dx(N_\alpha) = 0$.

12. Show there is $N \in \mathcal{B}(\mathbf{R}^n)$, $dx(N) = 0$, such that $\{Tf > 0\} \subseteq N$.

13. Conclude that $Tf = 0$, $dx-$a.s.

14. Conclude with the following:

Theorem 101 *Let $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, $n \geq 1$. Then, dx-almost surely, any $x \in \mathbf{R}^n$ is a Lebesgue points of f, i.e.*

$$dx\text{-}a.s.\ ,\ \lim_{\epsilon \downarrow\downarrow 0} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |f(y) - f(x)| dy = 0$$

Exercise 19. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\Omega' \in \mathcal{F}$. We define $\mathcal{F}' = \mathcal{F}_{|\Omega'}$ and $\mu' = \mu_{|\mathcal{F}'}$. For all maps $f : \Omega' \to [0, +\infty]$ (or

C), we define $\tilde{f} : \Omega \to [0, +\infty]$ (or **C**), by:

$$\tilde{f}(\omega) \stackrel{\triangle}{=} \begin{cases} f(\omega) & \text{if} \quad \omega \in \Omega' \\ 0 & \text{if} \quad \omega \notin \Omega' \end{cases}$$

1. Show that $\mathcal{F}' \subseteq \mathcal{F}$ and conclude that μ' is therefore a well-defined measure on $(\Omega', \mathcal{F}')$.

2. Let $A \in \mathcal{F}'$ and $1'_A$ be the characteristic function of A defined on Ω'. Let 1_A be the characteristic function of A defined on Ω. Show that $\tilde{1}'_A = 1_A$.

3. Let $f : (\Omega', \mathcal{F}') \to [0, +\infty]$ be a non-negative and measurable map. Show that $\tilde{f} : (\Omega, \mathcal{F}) \to [0, +\infty]$ is also non-negative and measurable, and that we have:

$$\int_{\Omega'} f d\mu' = \int_{\Omega} \tilde{f} d\mu$$

4. Let $f \in L^1_{\mathbf{C}}(\Omega', \mathcal{F}', \mu')$. Show that $\tilde{f} \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, and:

$$\int_{\Omega'} f d\mu' = \int_{\Omega} \tilde{f} d\mu$$

Definition 122 *$b : \mathbf{R}^+ \to \mathbf{C}$ is* **absolutely continuous**, *if and only if b is right-continuous of finite variation, and b is absolutely continuous with respect to $a(t) = t$.*

Exercise 20. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map.

1. Show that b is absolutely continuous, if and only if there is $f \in L^{1,\text{loc}}_{\mathbf{C}}(t)$ such that $b(t) = \int_0^t f(s)ds$, for all $t \in \mathbf{R}^+$.
2. Show that b absolutely continuous $\Rightarrow$ b continuous with $b(0) = 0$.

Exercise 21. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be an absolutely continuous map. Let $f \in L^{1,\text{loc}}_{\mathbf{C}}(t)$ be such that $b = f.t$. For all $n \geq 1$, we define

$f_n : \mathbf{R} \to \mathbf{C}$ by:

$$f_n(t) \stackrel{\triangle}{=} \left\{ \begin{array}{lll} f(t)1_{[0,n]}(t) & \text{if} & t \in \mathbf{R}^+ \\ 0 & \text{if} & t < 0 \end{array} \right.$$

1. Let $n \geq 1$. Show $f_n \in L^1_{\mathbf{C}}(\mathbf{R}, \mathcal{B}(\mathbf{R}), dx)$ and for all $t \in [0, n]$:

$$b(t) = \int_0^t f_n dx$$

2. Show the existence of $N_n \in \mathcal{B}(\mathbf{R})$ such that $dx(N_n) = 0$, and for all $t \in N_n^c$, t is a Lebesgue point of f_n.

3. Show that for all $t \in \mathbf{R}$, and $\epsilon > 0$:

$$\frac{1}{\epsilon} \int_t^{t+\epsilon} |f_n(s) - f_n(t)| ds \leq \frac{2}{dx(B(t,\epsilon))} \int_{B(t,\epsilon)} |f_n(s) - f_n(t)| ds$$

4. Show that for all $t \in N_n^c$, we have:

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} f_n(s) ds = f_n(t)$$

5. Show similarly that for all $t \in N_n^c$, we have:

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{1}{\epsilon} \int_{t-\epsilon}^{t} f_n(s)ds = f_n(t)$$

6. Show that for all $t \in N_n^c \cap [0, n[$, $b'(t)$ exists and $b'(t) = f(t)$.[1]

7. Show the existence of $N \in \mathcal{B}(\mathbf{R}^+)$, such that $dx(N) = 0$, and:

$$\forall t \in N^c \ , \ b'(t) \text{ exists with } b'(t) = f(t)$$

8. Conclude with the following:

[1] $b'(0)$ being a r.h.s derivative only.

Theorem 102 *A map $b : \mathbf{R}^+ \to \mathbf{C}$ is absolutely continuous, if and only if there exists $f \in L^{1,loc}_{\mathbf{C}}(t)$ such that:*

$$\forall t \in \mathbf{R}^+ \ , \ b(t) = \int_0^t f(s)ds$$

in which case, b is almost surely differentiable with $b' = f$ dx-a.s.

Solutions to Exercises

Exercise 1.

1. Let $f : \Omega \to \bar{\mathbf{R}}$ be a map, where Ω is a topological space. Suppose that $\{\lambda < f\}$ is open for all $\lambda \in \bar{\mathbf{R}}$. Then in particular, $\{\lambda < f\}$ is open for all $\lambda \in \mathbf{R}$. So f is l.s.c. Conversely, suppose f is l.s.c. Then $\{\lambda < f\}$ is open for all $\lambda \in \mathbf{R}$, and since:

$$\{-\infty < f\} = \bigcup_{\lambda \in \mathbf{R}} \{\lambda < f\}$$

   it follows that $\{-\infty < f\}$ is also open. Furthermore, $\{+\infty < f\}$ is the empty set, and in particular, $\{+\infty < f\}$ is open. We conclude that $\{\lambda < f\}$ is open for all $\lambda \in \bar{\mathbf{R}}$. We have proved that f is l.s.c if and only if $\{\lambda < f\}$ is open for all $\lambda \in \bar{\mathbf{R}}$.

2. Similarly to 1. we have:

$$\{f < +\infty\} = \bigcup_{\lambda \in \mathbf{R}} \{f < \lambda\}$$

and $\{f < -\infty\} = \emptyset$ which is open. We conclude that f is u.s.c if and only if $\{f < \lambda\}$ is open for all $\lambda \in \bar{\mathbf{R}}$.

3. Let U be open in $\bar{\mathbf{R}}$. If $+\infty \in U$, let $V^+ =]\alpha, +\infty]$ where $\alpha \in \mathbf{R}$ is such that $]\alpha, +\infty] \subseteq U$. Otherwise, let $V^+ = \emptyset$. If $-\infty \in U$, let $V^- = [-\infty, \beta[$, where $\beta \in \mathbf{R}$ is such that $[-\infty, \beta[\subseteq U$. Otherwise, let $V^- = \emptyset$. Then, we have:

$$U = V^+ \cup V^- \cup (U \cap \mathbf{R})$$

and $U \cap \mathbf{R}$ is an open subset of $\mathbf{R}$ (possibly empty). For all $x \in U \cap \mathbf{R}$, let $\alpha_x, \beta_x \in \mathbf{R}$ be such that $x \in]\alpha_x, \beta_x[\subseteq U \cap \mathbf{R}$. Then, we have:

$$U \cap \mathbf{R} = \bigcup_{x \in U \cap \mathbf{R}}]\alpha_x, \beta_x[$$

where it is understood that if $U \cap \mathbf{R} = \emptyset$, the corresponding union is the empty set. Taking $I = U \cap \mathbf{R}$, we conclude that:

$$U = V^+ \cup V^- \cup \bigcup_{i \in I}]\alpha_i, \beta_i[$$

4. Suppose that f is continuous. For all $\lambda \in \mathbf{R}$, the interval $]\lambda, +\infty]$ is an open subset of $\bar{\mathbf{R}}$. It follows that $\{\lambda < f\} = f^{-1}(]\lambda, +\infty])$ is open. This being true for all $\lambda \in \mathbf{R}$, f is l.s.c. Similarly, the interval $[-\infty, \lambda[$ is an open subset of $\bar{\mathbf{R}}$. It follows that $\{f < \lambda\} = f^{-1}([-\infty, \lambda[)$ is open. This being true for all $\lambda \in \mathbf{R}$, f is u.s.c. Hence, if f is continuous, it is both l.s.c and u.s.c. Conversely, suppose f is both l.s.c. and u.s.c. Let U be an open subset of $\bar{\mathbf{R}}$. Using the decomposition obtained in 3. we have:

$$\begin{aligned}
f^{-1}(U) &= f^{-1}\left(V^+ \cup V^- \cup \bigcup_{i\in I}]\alpha_i, \beta_i[\right) \\
&= f^{-1}(V^+) \cup f^{-1}(V^-) \cup \bigcup_{i\in I} f^{-1}(]\alpha_i, \beta_i[) \\
&= f^{-1}(V^+) \cup f^{-1}(V^-) \cup \bigcup_{i\in I} \{\alpha_i < f\} \cap \{f < \beta_i\}
\end{aligned}$$

Since $f^{-1}(V^+)$ is either $\{\alpha < f\}$ or $\emptyset$, and $f^{-1}(V^-)$ is either $\{f < \beta\}$ or $\emptyset$, it follows that $f^{-1}(U)$ is a union of open sets in

Ω, and is therefore open. Having proved that $f^{-1}(U)$ is open for all U open in $\bar{\mathbf{R}}$, we conclude that f is continuous. So f is continuous, if and only if it is both l.s.c and u.s.c.

5. Let $u : \Omega \to \mathbf{R}$ and $v : \Omega \to \bar{\mathbf{R}}$. Let $\lambda \in \mathbf{R}$. Note that having restricted the range of u to be a subset of $\mathbf{R}$, the map $u + v$ is well defined, as there can be no occurrence of $(+\infty) + (-\infty)$. We claim that:

$$\{\lambda < u + v\} = \bigcup_{\substack{(\lambda_1, \lambda_2) \in \mathbf{R}^2 \\ \lambda_1 + \lambda_2 = \lambda}} \{\lambda_1 < u\} \cap \{\lambda_2 < v\}$$

It is clear that if $\omega \in \Omega$ is such that $\lambda_1 < u(\omega)$ and $\lambda_2 < v(\omega)$ for some $\lambda_1, \lambda_2 \in \mathbf{R}$ with $\lambda_1 + \lambda_2 = \lambda$, then $\lambda < u(\omega) + v(\omega)$. This shows the inclusion $\supseteq$. To show the reverse inclusion, suppose that $\omega \in \Omega$ is such that $\lambda < u(\omega) + v(\omega)$. Then, we have $\lambda - u(\omega) < v(\omega)$, and there exists $\lambda_2 \in \mathbf{R}$ such that:

$$\lambda - u(\omega) < \lambda_2 < v(\omega)$$

Define $\lambda_1 = \lambda - \lambda_2$. Then $\lambda_2 < v(\omega)$ and $\lambda_1 < u(\omega)$ where λ_1, λ_2 are elements of $\mathbf{R}$ such that $\lambda_1 + \lambda_2 = \lambda$. This shows the inclusion $\subseteq$.

6. Suppose that both u and v are l.s.c. Then for all $\lambda_1, \lambda_2 \in \mathbf{R}$, $\{\lambda_1 < u\}$ and $\{\lambda_2 < v\}$ are open subsets of Ω. It follows from 5. that $\{\lambda < u + v\}$ is also an open subset of Ω, for all $\lambda \in \mathbf{R}$. So $u + v$ is l.s.c.

7. Suppose that both u and v are u.s.c. Similarly to 5. we have:

$$\{u + v < \lambda\} = \bigcup_{\substack{(\lambda_1, \lambda_2) \in \mathbf{R}^2 \\ \lambda_1 + \lambda_2 = \lambda}} \{u < \lambda_1\} \cap \{v < \lambda_2\}$$

and consequently $\{u + v < \lambda\}$ is an open subset of Ω, for all $\lambda \in \mathbf{R}$. So $u + v$ is u.s.c. Anticipating on questions 10. and 11., an alternative proof goes as follows: if u and v are u.s.c, then $-u$ and $-v$ are l.s.c. so $-u - v$ is l.s.c. and finally $u + v$ is u.s.c.

8. Suppose f is l.s.c and let $\alpha \in \mathbf{R}^+$. If $\alpha = 0$, then $\alpha f = 0$ and consequently αf is continuous and in particular l.s.c. We assume that $\alpha > 0$. Then for all $\omega \in \Omega$, $\lambda < \alpha f(\omega)$ is equivalent to $\lambda/\alpha < f(\omega)$ (this is certainly true when $f(\omega) \in \mathbf{R}$, and one can easily check that it is still true when $f(\omega) \in \{-\infty, +\infty\}$). It follows that $\{\lambda < \alpha f\} = \{\lambda/\alpha < f\}$ and consequently $\{\lambda < \alpha f\}$ is an open subset of Ω. This being true for all $\lambda \in \mathbf{R}$, we conclude that αf is l.s.c.

9. Suppose that f is u.s.c and $\alpha \in \mathbf{R}^+$. If $\alpha = 0$ then αf is u.s.c. We assume that $\alpha > 0$. Then $\{\alpha f < \lambda\} = \{f < \lambda/\alpha\}$ and consequently $\{\alpha f < \lambda\}$ is open for all $\lambda \in \mathbf{R}$. So αf is u.s.c.

10. Suppose that f is l.s.c. Then $\{-f < \lambda\} = \{-\lambda < f\}$ for all $\lambda \in \mathbf{R}$, and consequently $\{-f < \lambda\}$ is an open subset of Ω. So $-f$ is u.s.c.

11. Suppose that f is u.s.c. Then $\{\lambda < -f\} = \{f < -\lambda\}$ for all $\lambda \in \mathbf{R}$, and consequently $\{\lambda < -f\}$ is an open subset of Ω. So

$-f$ is l.s.c.

12. Let V be an open subset of Ω and $f = 1_V$. Let $\lambda \in \mathbf{R}$. If $\lambda < 0$ we have $\{\lambda < f\} = \Omega$. If $0 \leq \lambda < 1$ we have $\{\lambda < f\} = V$. If $1 \leq \lambda$ we have $\{\lambda < f\} = \emptyset$. In any case, $\{\lambda < f\}$ is an open subset of Ω. So f is l.s.c. The characteristic function of an open subset of Ω is lower-semi-continuous

13. Let F be a closed subset of Ω. Let $\lambda \in \mathbf{R}$. Then $\{f < \lambda\}$ is either $\emptyset$, F^c or Ω, depending respectively on whether $\lambda \leq 0$, $0 < \lambda \leq 1$ and $1 < \lambda$. In any case, $\{f < \lambda\}$ is an open subset of Ω. So f is u.s.c. The characteristic function of a closed subset of Ω is upper-semi-continuous.

Exercise 1

Exercise 2.

1. Let $(f_i)_{i\in I}$ be a family of maps $f_i : \Omega \to \bar{\mathbf{R}}$, where Ω is a topological space. Let $f = \sup_{i\in I} f_i$. We assume that all f_i's are l.s.c. For all $\lambda \in \mathbf{R}$, we claim that:

$$\{\lambda < f\} = \bigcup_{i\in I}\{\lambda < f_i\} \tag{1}$$

Indeed, suppose that $\omega \in \Omega$ is such that $\lambda < f(\omega)$. Since $f(\omega)$ is the lowest upper-bound of all $f_i(\omega)$'s, λ cannot be such an upper-bound. Hence, there exists $i \in I$ such that $\lambda < f_i(\omega)$. This shows the inclusion $\subseteq$. To show the reverse inclusion, suppose $\omega \in \Omega$ is such that $\lambda < f_i(\omega)$ for some $i \in I$. Since $f_i(\omega) \le f(\omega)$, in particular we have $\lambda < f(\omega)$. This shows the inclusion $\supseteq$. Having proved equation (1) and since all f_i's are l.s.c, $\{\lambda < f\}$ is an open subset of Ω for all $\lambda \in \mathbf{R}$. It follows that f is l.s.c. The supremum of l.s.c functions is l.s.c.

2. Suppose that all f_i's are u.s.c and $f = \inf_{i\in I} f_i$. Given $\lambda \in \mathbf{R}$:

$$\{f < \lambda\} = \bigcup_{i\in I} \{f_i < \lambda\}$$

and consequently $\{f < \lambda\}$ is an open subset of Ω. It follows that f is u.s.c. The infimum of u.s.c functions is u.s.c.

Exercise 2

Exercise 3.

1. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Let $f \in L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$, $f \geq 0$, where μ is a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. From theorem (18), there exists a sequence $(s_n)_{n\geq 1}$ of simple functions on $(\Omega, \mathcal{B}(\Omega))$ such that $s_n \uparrow f$ (i.e. $s_n \leq s_{n+1}$ for all $n \geq 1$ and $s_n \to f$ pointwise). We define $t_1 = s_1$ and $t_n = s_n - s_{n-1}$ for all $n \geq 2$. In order to show that t_n is a simple function for all $n \geq 1$, we need to show that if s, t are simple functions on $(\Omega, \mathcal{B}(\Omega))$ with $s \leq t$, then $t - s$ is also a simple function on $(\Omega, \mathcal{B}(\Omega))$. Since s and t are measurable with values in $\mathbf{R}^+$, and $s \leq t$, the map $t - s$ is also measurable with values in $\mathbf{R}^+$. From:
$$t - s = \sum_{\alpha \in (t-s)(\Omega)} \alpha 1_{\{t-s=\alpha\}}$$
we conclude that $t - s$ is a simple function on $(\Omega, \mathcal{B}(\Omega))$.

2. Since each t_n is a simple function on $(\Omega, \mathcal{B}(\Omega))$, for all $n \geq 1$

there exists an integer $p_n \geq 1$ and some $\alpha_n^1, \ldots, \alpha_n^{p_n} \in \mathbf{R}^+$ and $A_n^1, \ldots, A_n^{p_n} \in \mathcal{B}(\Omega)$ such that:

$$t_n = \sum_{k=1}^{p_n} \alpha_n^k 1_{A_n^k}$$

Note that it is always possible to assume $\alpha_n^k \neq 0$, by setting $A_n^k = \emptyset$ if necessary. Since $s_N = \sum_{n=1}^N t_n$ for all $N \geq 1$, from $s_N \to f$ we obtain:

$$f = \sum_{n=1}^{+\infty} t_n = \sum_{n=1}^{+\infty} \sum_{k=1}^{p_n} \alpha_n^k 1_{A_n^k}$$

This last sum having a countable number of (non-negative) terms, it can be re-expressed as:

$$f = \sum_{n=1}^{+\infty} \alpha_n 1_{A_n}$$

where $\alpha_n \in \mathbf{R}^+ \setminus \{0\}$ and $A_n \in \mathcal{B}(\Omega)$ for all $n \geq 1$.

3. Since $f \in L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$ and $f \geq 0$, from 2. we have:

$$\begin{aligned}\sum_{n=1}^{+\infty} \alpha_n \mu(A_n) &= \sum_{n=1}^{+\infty} \alpha_n \int 1_{A_n} d\mu \\ &= \int \left(\sum_{n=1}^{+\infty} \alpha_n 1_{A_n} \right) d\mu \\ &= \int f d\mu < +\infty\end{aligned}$$

where the second equality is obtained from the linearity of the integral and an immediate application of the monotone convergence theorem (19). Since for all $n \geq 1$ we have $\alpha_n > 0$, we conclude that $\mu(A_n) < +\infty$.

4. Let $\epsilon > 0$ and $n \geq 1$. Define $\epsilon' = \epsilon/(\alpha_n 2^{n+2})$. Since $(\Omega, \mathcal{T})$ is metrizable and σ-compact, while μ is a locally finite measure on

$(\Omega, \mathcal{B}(\Omega))$, from theorem (73) μ is a regular measure. Hence:

$$\begin{aligned} \mu(A_n) &= \sup\{\mu(K) : K \subseteq A_n \ , \ K \text{ compact}\} \\ &= \inf\{\mu(V) : A_n \subseteq V \ , \ V \text{ open}\} \end{aligned}$$

Since $\mu(A_n) < +\infty$, we have $\mu(A_n) < \mu(A_n) + \epsilon'$, and $\mu(A_n)$ being the greatest lower-bound of all $\mu(V)$'s as V runs through the set of all open subsets of Ω with $A_n \subseteq V$, $\mu(A_n) + \epsilon'$ cannot be such a lower-bound. There exists V_n open subset of Ω such that $A_n \subseteq V_n$, and:

$$\mu(V_n) < \mu(A_n) + \epsilon'$$

Similarly, from the fact that $\mu(A_n) - \epsilon' < \mu(A_n)$, there exists K_n compact subset of Ω such that $K_n \subseteq A_n$, and:

$$\mu(A_n) - \epsilon' < \mu(K_n)$$

From $K_n \subseteq A_n$ note in particular that $\mu(K_n) < +\infty$, and consequently we have $K_n \subseteq A_n \subseteq V_n$ with:

$$\mu(V_n \setminus K_n) = \mu(V_n) - \mu(K_n) < 2\epsilon' = \frac{\epsilon}{\alpha_n 2^{n+1}}$$

5. Having proved in 3. that $\sum_{n\geq 1} \alpha_n \mu(A_n) < +\infty$, given $\epsilon > 0$ there exists $N \geq 1$ such that:

$$\left|\sum_{n=1}^{+\infty} \alpha_n \mu(A_n) - \sum_{n=1}^{N} \alpha_n \mu(A_n)\right| \leq \frac{\epsilon}{2}$$

or equivalently:

$$\sum_{n=N+1}^{+\infty} \alpha_n \mu(A_n) \leq \frac{\epsilon}{2}$$

6. Let $u = \sum_{n=1}^{N} \alpha_n 1_{K_n}$. Since $(\Omega, \mathcal{T})$ is metrizable, in particular it is a Hausdorff topological space. Since K_n is a compact subset of Ω, from theorem (35) K_n is a closed subset of Ω. It follows from 13. of exercise (1) that 1_{K_n} is upper-semi-continuous. Using 7. and 9. of exercise (1), we conclude that u is also u.s.c.

7. Let $v = \sum_{n=1}^{+\infty} \alpha_n 1_{V_n}$. Since V_n is an open subset of Ω, from 12. of exercise (1) the map 1_{V_n} is lower-semi-continuous. It follows from 6. and 8. of this same exercise that every partial sum $\sum_{n=1}^{k} \alpha_n 1_{V_n}$ is itself l.s.c. Since v is the supremum of these partial sums, we conclude from exercise (2) that v is l.s.c.

8. Since $K_n \subseteq A_n \subseteq V_n$ and $\alpha_n \in \mathbf{R}^+$ for all $n \geq 1$:

$$
\begin{aligned}
0 &\leq \sum_{n=1}^{N} \alpha_n 1_{K_n} = u \\
&\leq \sum_{n=1}^{N} \alpha_n 1_{A_n} \\
&\leq \sum_{n=1}^{+\infty} \alpha_n 1_{A_n} = f \\
&\leq \sum_{n=1}^{+\infty} \alpha_n 1_{V_n} = v
\end{aligned}
$$

We conclude that $0 \leq u \leq f \leq v$.

9. Since $K_n \subseteq V_n$ for all $n \geq 1$, we have:

$$\begin{aligned} v = \sum_{n=1}^{+\infty} \alpha_n 1_{V_n} &= \sum_{n=1}^{+\infty} \alpha_n (1_{K_n} + 1_{V_n \setminus K_n}) \\ &= \sum_{n=1}^{+\infty} \alpha_n 1_{K_n} + \sum_{n=1}^{+\infty} \alpha_n 1_{V_n \setminus K_n} \\ &= u + \sum_{n=N+1}^{+\infty} \alpha_n 1_{K_n} + \sum_{n=1}^{+\infty} \alpha_n 1_{V_n \setminus K_n} \end{aligned}$$

10. Since $K_n \subseteq A_n$ for all $n \geq 1$, using 5. we have:

$$\sum_{n=N+1}^{+\infty} \alpha_n \mu(K_n) \leq \sum_{n=N+1}^{+\infty} \alpha_n \mu(A_n) \leq \frac{\epsilon}{2}$$

Hence, using 9. and 4. we obtain:

$$\begin{aligned}
\int v d\mu &= \int \left(u + \sum_{n=N+1}^{+\infty} \alpha_n 1_{K_n} + \sum_{n=1}^{+\infty} \alpha_n 1_{V_n \setminus K_n} \right) d\mu \\
&= \int u d\mu + \sum_{n=N+1}^{+\infty} \alpha_n \int 1_{K_n} d\mu + \sum_{n=1}^{+\infty} \alpha_n \int 1_{V_n \setminus K_n} d\mu \\
&= \int u d\mu + \sum_{n=N+1}^{+\infty} \alpha_n \mu(K_n) + \sum_{n=1}^{+\infty} \alpha_n \mu(V_n \setminus K_n) \\
&\leq \int u d\mu + \frac{\epsilon}{2} + \sum_{n=1}^{+\infty} \alpha_n \cdot \frac{\epsilon}{\alpha_n 2^{n+1}} \\
&= \int u d\mu + \epsilon
\end{aligned}$$

where the second equality stems from the linearity of the integral and an application of the monotone convergence theorem (19).

Note that since $\mu(K_n) < +\infty$ for all $n \geq 1$, in particular:

$$\int u d\mu = \sum_{n=1}^{N} \alpha_n \mu(K_n) < +\infty$$

Hence, we conclude that:

$$\int v d\mu \leq \int u d\mu + \epsilon < +\infty$$

11. The map u is $\mathbf{R}$-valued, Borel measurable with:

$$\int |u| d\mu = \int u d\mu < +\infty$$

So $u \in L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$.

12. The map v is Borel measurable with:

$$\int |v| d\mu = \int v d\mu < +\infty$$

However, it has values in $[0, +\infty]$, i.e. $v(\omega) = +\infty$ is possible for some $\omega \in \Omega$. The condition $\int v d\mu < +\infty$ does imply that $v(\omega) < +\infty$ for μ-almost every $\omega \in \Omega$. As we shall see in the next question, v is therefore μ-almost surely equal to an element of $L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$. But strictly speaking, it may not be itself an element of this space, because its range $v(\Omega)$ may fail to be a subset of $\mathbf{R}$.

13. Since $\int v d\mu < +\infty$, we have $v < +\infty$ μ-a.s since:

$$(+\infty) \cdot \mu(\{v = +\infty\}) = \int_{\{v=+\infty\}} v d\mu \leq \int v d\mu < +\infty$$

Hence, if $N = \{v = +\infty\}$, we have $N \in \mathcal{B}(\Omega)$ and $\mu(N) = 0$. Let $v^* = v 1_{N^c}$. Then v^* has values in $\mathbf{R}$, is Borel measurable and:

$$\int |v^*| d\mu = \int v 1_{N^c} d\mu = \int v d\mu < +\infty$$

So $v^* \in L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$. Since $v^* = v$ μ-a.s. we conclude that v is μ-almost surely equal to an element of $L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$.

14. Note that from 8. we have $0 \leq u \leq v$ and consequently $v - u$ is non-negative and measurable, and the integral $\int (v-u)d\mu$ makes sense. In fact, even if $u \leq v$ did not hold, since $u \in L^1$ and v is almost surely equal to an element of L^1, it would be possible to give meaning to $\int (v-u)d\mu$ in the obvious way. Now from 10. we have:

$$\begin{aligned} \int u d\mu + \int (v-u) d\mu &= \int v d\mu \\ &\leq \int u d\mu + \epsilon \end{aligned}$$

and since $\int u d\mu < +\infty$ we conclude that $\int (v-u)d\mu \leq \epsilon$.

15. Having considered a metrizable and σ-compact topological space $(\Omega, \mathcal{T})$ and a locally finite measure μ on $(\Omega, \mathcal{B}(\Omega))$, given $\epsilon > 0$ and $f \in L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$ with $f \geq 0$, we have found two measurable maps $u, v : \Omega \to [0, +\infty]$ (where in fact u has values in $\mathbf{R}^+$), which are μ-almost surely equal to elements of $L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$

(in fact u is itself an element of L^1) and such that $u \leq f \leq v$, u is u.s.c, v is l.s.c. and:

$$\int (v-u)d\mu \leq \epsilon$$

Now let $f \in L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$ which we no longer assume to be non-negative. Let f^+ and f^- be respectively the positive and negative parts of f. Then $f = f^+ - f^-$ and given $\epsilon > 0$, it is possible to apply the result of this exercise to f^+ and f^- separately, with $\epsilon/2$ instead of ϵ. Hence, there exist four measurable maps u^+, v^+, u^- and v^- where u^+, u^- have values in $\mathbf{R}^+$ and v^+, v^- have values in $[0, +\infty]$, which are μ-almost surely equal elements of L^1, and satisfy the conditions $u^+ \leq f^+ \leq v^+$, $u^- \leq f^- \leq v^-$, u^+, u^- are u.s.c, v^+, v^- are l.s.c, and:

$$\int (v^+ - u^+)d\mu \leq \frac{\epsilon}{2}$$

together with:

$$\int (v^- - u^-)d\mu \leq \frac{\epsilon}{2}$$

We define $u = u^+ - v^-$ and $v = v^+ - u^-$. Since u^+, u^- have values in $\mathbf{R}$, given $\omega \in \Omega$, the differences $u^+(\omega) - v^-(\omega)$ and $v^+(\omega) - u^-(\omega)$ are always well-defined elements of $\bar{\mathbf{R}}$. It follows that $u, v : \Omega \to \bar{\mathbf{R}}$ are well-defined measurable maps. Furthermore, it is clear that both u and v are μ-almost surely equal to an element of L^1. From $u^+ \leq f^+ \leq v^+$, $u^- \leq f^- \leq v^-$ and $f = f^+ - f^-$ we obtain $u \leq f \leq v$. Furthermore, since u^+ is $\mathbf{R}$-valued and u.s.c while v^- is l.s.c, from exercise (1) $u = u^+ - v^-$ is u.s.c, and similarly $v = v^+ - u^-$ is l.s.c. Finally, since $u \leq f \leq v$ and f is $\mathbf{R}$-valued, given $\omega \in \Omega$ the difference $v(\omega) - u(\omega)$ is always a well-defined element of $[0, +\infty]$. So $v - u$ is a well-defined non-negative and measurable map, and the integral $\int (v - u)d\mu$ is meaningful. We have:

$$\int (v - u)d\mu \quad = \quad \int (v^+ - u^- - u^+ + v^-)d\mu$$

$$
\begin{aligned}
&= \int (v^+ - u^+ + v^- - u^-)d\mu \\
&= \int (v^+ - u^+)d\mu + \int (v^- - u^-)d\mu \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}
$$

This completes the proof of theorem (94).

Exercise 3

Exercise 4.

1. Let $(\Omega, \mathcal{T})$ be a topological space. Suppose it is connected and $\Omega = A \uplus B$ where A, B are disjoint open sets. Then $A^c = B$ so A is closed and consequently A is both open and closed. Hence, Ω being connected, we have $A = \emptyset$ or $A = \Omega$, i.e. $A = \emptyset$ or $B = \emptyset$. Conversely, suppose $\Omega = A \uplus B$ with A, B disjoint open sets implies that $A = \emptyset$ or $B = \emptyset$. Then if A is both open and closed in Ω, with have $\Omega = A \uplus A^c$ where A, A^c are disjoint open sets. So $A = \emptyset$ or $A^c = \emptyset$, i.e. $A = \emptyset$ or $A = \Omega$. This shows that Ω is connected. We have proved that Ω is connected if and only if whenever $\Omega = A \uplus B$ with A, B disjoint open sets, we have $A = \emptyset$ or $B = \emptyset$.

2. If $\Omega = A \uplus B$ with A, B disjoint open sets, then $\Omega = A^c \uplus B^c$ with A^c, B^c disjoint closed sets, and conversely if $\Omega = A \uplus B$ with A, B disjoint closed sets, then $\Omega = A^c \uplus B^c$ with A^c, B^c

disjoint open sets. Hence, the statements:

$$(i)\ \Omega = A \uplus B\ ,\ A, B \text{ disjoint and open } \Rightarrow\ A = \emptyset \text{ or } B = \emptyset$$
$$(ii)\ \Omega = A \uplus B\ ,\ A, B \text{ disjoint and closed } \Rightarrow\ A = \emptyset \text{ or } B = \emptyset$$

are equivalent. We conclude from 1. that Ω is connected, if and only if whenever $\Omega = A \uplus B$ with A, B disjoint closed sets, we have $A = \emptyset$ or $B = \emptyset$.

Exercise 4

Exercise 5.

1. Let A be an open and closed subset of $\mathbf{R}$, with $A \neq \emptyset$ and $A^c \neq \emptyset$. Let $x \in A^c$. We have:

$$A = (A \cap]-\infty, x]) \cup (A \cap [x, +\infty[)$$

and since $A \neq \emptyset$, we have $A \cap]-\infty, x] \neq \emptyset$ or $A \cap [x, +\infty[\neq \emptyset$.

2. Let $B = A \cap [x, +\infty[$ and suppose $B \neq \emptyset$. Both A and $[x, +\infty[$ are closed subsets of $\mathbf{R}$. So B is a closed subset of $\mathbf{R}$. However, since $x \in A^c$, we have:

$$\begin{aligned} B &= A \cap [x, +\infty[\\ &= (A \cap \{x\}) \cup (A \cap]x, +\infty[) \\ &= A \cap]x, +\infty[\end{aligned}$$

and since both A and $]x, +\infty[$ are open subsets of $\mathbf{R}$, B is also an open subset of $\mathbf{R}$. Note that the assumption $B \neq \emptyset$ has not been used so far.

3. Let $b = \inf B$. We have proved in exercise (9) (part 5) of Tutorial 8 that if B is a non-empty closed subset of $\bar{\mathbf{R}}$, then $\inf B \in B$. Unfortunately, this result does not apply to non-empty closed subsets of $\mathbf{R}$ (indeed $\mathbf{R}$, is a non-empty closed subset of $\mathbf{R}$ and $\inf \mathbf{R} = -\infty \notin \mathbf{R}$). So we cannot apply exercise (9) of Tutorial 8, at least not without a little bit of care. However, the following can be done: since $B \neq \emptyset$, there exists $y \in B = A \cap [x, +\infty[$. Then it is clear that $B^* = A \cap [x, y]$ is a non-empty closed subset of $\bar{\mathbf{R}}$, and consequently since $b = \inf B^*$, applying exercise (9) of Tutorial 8, we have $b \in B^*$. So $b \in B \subseteq \mathbf{R}$. For those who wish to have a more detailed argument, the following can be said: the fact that $B^* \neq \emptyset$ is a consequence of $y \in B^*$. If we define $b^* = \inf B^*$, the fact that $b^* = b$ can be shown as follows: since $B^* \subseteq B$, any lower-bound of B is also a lower-bound of B^*, and consequently b is a lower-bound of B^* which shows that $b \leq b^*$. To show the reverse inequality, consider $u \in B$. Then if $u \leq y$ we have $u \in B^*$ and therefore $b^* \leq u$. But if $y < u$, then $b^* \leq y < u$ and we see

that $b^* \leq u$ is true in all cases. So b^* is a lower-bound of B which shows that $b^* \leq b$. We have proved that $b = b^*$. To show that B^* is a closed subset of $\bar{\mathbf{R}}$, we first argue that it is a closed subset of $\mathbf{R}$ since A is closed and $[x, y]$ is closed. However, the topology of $\mathbf{R}$ is induced by the topology of $\bar{\mathbf{R}}$. It is a simple exercise to show that any closed subset of $\mathbf{R}$ can be written as $F \cap \mathbf{R}$ where F is a closed subset of $\bar{\mathbf{R}}$. Hence, there is a closed subset F of $\bar{\mathbf{R}}$ such that $B^* = F \cap \mathbf{R}$. But then:

$$
\begin{aligned}
B^* &= A \cap [x, y] \\
&= A \cap [x, y] \cap [x, y] \\
&= B^* \cap [x, y] \\
&= (F \cap \mathbf{R}) \cap [x, y] \\
&= F \cap [x, y]
\end{aligned}
$$

and since $[x, y]$ is also closed in $\bar{\mathbf{R}}$, we conclude that B^* is indeed closed in $\bar{\mathbf{R}}$. This concludes our proof that $b \in B$. All this may seem like a lot of work, made necessary by our desperate attempt to apply exercise (9) of Tutorial 8. For those who believe that a direct proof is more convenient, here is the following: Since $B = A \cap [x, +\infty[$, it is clear that x is a lower bound of B and consequently $x \leq b$. To show that $b \in B$, we only need to show that $b \in A$. Since $B \neq \emptyset$, there exist $y \in B \subseteq \mathbf{R}$ and from $b \leq y$ we obtain in particular $b < +\infty$. Hence, there exists a sequence $(t_n)_{n \geq 1}$ in $\mathbf{R}$ such that $t_n \downarrow\downarrow b$ (i.e. $t_n \to b$ with $b < t_{n+1} \leq t_n$ for all $n \geq 1$). Since $b < t_n$, it is impossible that t_n be a lower-bound of B. Hence, for all $n \geq 1$ there exists some $x_n \in B \subseteq A$ such that $b \leq x_n < t_n$. From $t_n \to b$ we see that $x_n \to b$ and since $x_n \in A$ while A is a closed subset of $\mathbf{R}$, we conclude that $b \in A$. This completes our second proof of $b \in B$.

4. Having proved in 2. that B is an open subset of $\mathbf{R}$, since $b \in B$ there exists $\epsilon > 0$ such that $]b - \epsilon, b + \epsilon[\subseteq B$.

5. To show that $(\mathbf{R}, \mathcal{T}_{\mathbf{R}})$ is connected, we need to show that if A is an open and closed subset of $\mathbf{R}$, then $A = \emptyset$ or $A = \mathbf{R}$. Suppose this is not the case and $A \neq \emptyset$ together with $A^c \neq \emptyset$. We have

shown in 2. that $A\cap[x,+\infty[\neq\emptyset$ or $A\cap]-\infty,x]\neq\emptyset$. If we assume that $B=A\cap[x,+\infty[$ and $B\neq\emptyset$, then $b=\inf B\in\mathbf{R}$ and we have proved in 4. that there exists $\epsilon>0$ such that $]b-\epsilon,b+\epsilon[\subseteq B$. This is a contradiction. Indeed , since $b-\epsilon/2<b$, the fact that $b-\epsilon/2\in B$ contradicts the fact that b is a lower-bound of B. So the only possible case is that $C\neq\emptyset$ where $C=A\cap]-\infty,x]$. However, if $c=\sup C$, then a similar proof to that of 3. will show that $c\in C$ (in particular $c\in\mathbf{R}$) and C being open in $\mathbf{R}$, there exists $\epsilon>0$ with $]c-\epsilon,c+\epsilon[\subseteq C$, leading to a contradiction. Hence, we see that all possible cases lead to a contradiction. We conclude that the initial assumption is absurd, i.e. that $A=\emptyset$ or $A=\mathbf{R}$. So $(\mathbf{R},\mathcal{T}_{\mathbf{R}})$ is a connected topological space, which completes the proof of theorem (95).

Exercise 5

Exercise 6.

1. Let $(\Omega, \mathcal{T})$ be a topological space and $A \subseteq \Omega$ be a connected subset of Ω. Let B be a subset of Ω such that $A \subseteq B \subseteq \bar{A}$, where $\bar{A}$ is the closure of A in Ω. Let V_1, V_2 be disjoint open subsets of B such that $B = V_1 \uplus V_2$. From definition (23) of the induced topology $\mathcal{T}_{|B}$, there exist U_1, U_2 open subsets of Ω such that $V_1 = B \cap U_1$ and $V_2 = B \cap U_2$.

2. Since $A \subseteq B$, using 1. we have:

$$
\begin{aligned}
A &= A \cap B \\
&= A \cap (V_1 \uplus V_2) \\
&= A \cap [(B \cap U_1) \uplus (B \cap U_2)] \\
&= (A \cap B \cap U_1) \uplus (A \cap B \cap U_2) \\
&= (A \cap U_1) \uplus (A \cap U_2)
\end{aligned}
$$

   Now since U_1, U_2 are open subsets of Ω, $A \cap U_1$ and $A \cap U_2$ are open subsets of A. Furthermore, since V_1 and V_2 are disjoint, we have $V_1 \cap V_2 = B \cap U_1 \cap U_2 = \emptyset$. and in particular since $A \subseteq B$, $A \cap U_1 \cap U_2 = \emptyset$. So $A \cap U_1$ and $A \cap U_2$ are disjoint open subsets of A with $A = (A \cap U_1) \uplus (A \cap U_2)$. Having assumed that A is a connected subset of Ω, the topological space $(A, \mathcal{T}_{|A})$ is connected and consequently using exercise (4), it follows that $A \cap U_1 = \emptyset$ or $A \cap U_2 = \emptyset$.

3. Suppose that $A \cap U_1 = \emptyset$. Let $x \in \bar{A}$. Then for all U open subsets of Ω with $x \in U$, we have $A \cap U \neq \emptyset$. Hence, since U_1 is an open subset of Ω and $A \cap U_1 = \emptyset$, it is necessary that $x \notin U_1$. So $x \in U_1^c$ and we have proved that $\bar{A} \subseteq U_1^c$.

4. Having assumed that $B \subseteq \bar{A}$, it follows from 3. that $B \subseteq U_1^c$, i.e. $V_1 = B \cap U_1 = \emptyset$.

5. From 3. and 4. we have seen that if $A \cap U_1 = \emptyset$, then $V_1 = \emptyset$. Similarly, if $A \cap U_2 = \emptyset$, then $V_2 = \emptyset$. However, we have shown in 2. that $A \cap U_1 = \emptyset$ or $A \cap U_2 = \emptyset$. So $V_1 = \emptyset$ or $V_2 = \emptyset$. Having considered $B \subseteq \Omega$ such that $A \subseteq B \subseteq \bar{A}$, and V_1, V_2

disjoint open subsets of B such that $B = V_1 \uplus V_2$, we have proved that $V_1 = \emptyset$ or $V_2 = \emptyset$. From exercise (4), this shows that the topological space $(B, \mathcal{T}_{|B})$ is connected, or equivalently that B is a connected subset of Ω. Hence, if A is a connected subset of Ω and $A \subseteq B \subseteq \bar{A}$, then B is also a connected subset of Ω. In particular, $\bar{A}$ is a connected subset of Ω.

Exercise 6

Exercise 7. Let $(\Omega, \mathcal{T})$ and $(\Omega', \mathcal{T}')$ be two topological spaces, and f be a continuous map $f : \Omega \to \Omega'$. We assume that $(\Omega, \mathcal{T})$ is connected. We claim that $f(\Omega)$ is a connected subset of Ω', or equivalently that the topological space $(f(\Omega), \mathcal{T}'_{|f(\Omega)})$ is connected. In order to prove this, we shall use exercise (4) and consider A, B two disjoint open subsets of $f(\Omega)$ such that $f(\Omega) = A \uplus B$. There exist U', V' open subsets of Ω' such that $A = f(\Omega) \cap U'$ and $B = f(\Omega) \cap V'$. Since f is continuous, $f^{-1}(U')$ and $f^{-1}(V')$ are open subsets of Ω. Furthermore, it is clear that:

$$f^{-1}(U') = f^{-1}(f(\Omega) \cap U') = f^{-1}(A)$$

and similarly $f^{-1}(V') = f^{-1}(B)$. So $f^{-1}(A)$ and $f^{-1}(B)$ are open subsets of Ω. Since A and B are disjoint, $f^{-1}(A)$ and $f^{-1}(B)$ are also disjoint. Since $f(\Omega) = A \uplus B$, for all $x \in \Omega$ we have $f(x) \in A$ or $f(x) \in B$. So $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. It follows that $f^{-1}(A)$ and $f^{-1}(B)$ are two disjoint open subsets of Ω, such that $\Omega = f^{-1}(A) \uplus f^{-1}(B)$. Since Ω is connected, from exercise (4) it follows that $f^{-1}(A) = \emptyset$ or $f^{-1}(B) = \emptyset$. Suppose that $f^{-1}(A) = \emptyset$.

We claim that $A = \emptyset$. Otherwise there exists $y \in A \subseteq f(\Omega)$. Let $x \in \Omega$ be such that $y = f(x)$. Then $f(x) \in A$ and consequently $x \in f^{-1}(A)$ which contradicts $f^{-1}(A) = \emptyset$. So $f^{-1}(A) = \emptyset$ implies that $A = \emptyset$, and similarly $f^{-1}(B) = \emptyset$ implies that $B = \emptyset$. It follows that $A = \emptyset$ or $B = \emptyset$. Having assumed that $f(\Omega) = A \uplus B$ where A, B are disjoint open subsets of $f(\Omega)$, we have proved that $A = \emptyset$ or $B = \emptyset$. From exercise (4), this shows that the topological space $(f(\Omega), \mathcal{T}'_{|f(\Omega)})$ is connected, or equivalently that $f(\Omega)$ is a connected subset of Ω'. This completes the proof of theorem (96).

Exercise 7

Exercise 8.

1. Let $A \subseteq \bar{\mathbf{R}}$ and suppose that A is an interval. Let $\alpha = \inf A$ and $\beta = \sup A$. We claim that:

$$]\alpha, \beta[\subseteq A \subseteq [\alpha, \beta]$$

If $A = \emptyset$, then $\alpha = +\infty$ and $\beta = -\infty$, so there is nothing to prove. So we assume that $A \neq \emptyset$. Then there is $x \in A$, and we have $\alpha \leq x$ as well as $x \leq \beta$. In particular, $\alpha \leq \beta$. Let $z \in A$. Since α is a lower-bound of A, $\alpha \leq z$. Since β is an upper-bound of A, $z \leq \beta$. So $z \in [\alpha, \beta]$ and we have proved that $A \subseteq [\alpha, \beta]$. Suppose $z \in]\alpha, \beta[$. From $\alpha < z$ we see that z cannot be a lower-bound of A (α is the greatest of such lower-bounds). There exists $x \in A$ such that $\alpha \leq x < z$. From $z < \beta$ we see that z cannot be an upper-bound of A. There exists $y \in A$ such that $z < y \leq \beta$. From $x < z < y$ we obtain in particular $z \in [x, y]$. Since $x, y \in A$ and A is assumed to be an interval, it follows from definition (118) that $z \in A$. We have proved that $]\alpha, \beta[\subseteq A$.

2. Let $A \subseteq \bar{\mathbf{R}}$. Suppose that A is of the form $[\alpha,\beta]$, $[\alpha,\beta[$, $]\alpha,\beta]$ or $]\alpha,\beta[$ for some $\alpha,\beta \in \bar{\mathbf{R}}$. Suppose there exist $x, y \in A$ with $x \leq y$. Then for all $z \in [x, y]$ we have $x \leq z \leq y$. If $\alpha \leq x$ then $\alpha \leq z$. If $\alpha < x$ then $\alpha < z$. If $y \leq \beta$ then $z \leq \beta$. If $y < \beta$ then $z < \beta$. In any case, we see that $z \in A$. This shows that $[x, y] \subseteq A$ for all $x, y \in A$, $x \leq y$, and consequently from definition (118), A is an interval. Note that A can be the empty set without anything being flawed in the argument just given. Conversely, suppose that A is an interval. From 1. we have:

$$]\alpha,\beta[\subseteq A \subseteq [\alpha,\beta]$$

where $\alpha = \inf A$ and $\beta = \sup A$. We shall distinguish four cases: suppose $\alpha \in A$ and $\beta \in A$. Then:

$$[\alpha,\beta] =]\alpha,\beta[\cup \{\alpha\} \cup \{\beta\} \subseteq A \subseteq [\alpha,\beta]$$

and consequently $A = [\alpha,\beta]$. Suppose $\alpha \in A$ and $\beta \notin A$. Then:

$$[\alpha,\beta[=]\alpha,\beta[\cup \{\alpha\} \subseteq A \subseteq [\alpha,\beta] \setminus \{\beta\} = [\alpha,\beta[$$

and consequently $A = [\alpha, \beta[$. Suppose $\alpha \notin A$ and $\beta \in A$. Then:

$$]\alpha, \beta] =]\alpha, \beta[\cup \{\beta\} \subseteq A \subseteq [\alpha, \beta] \setminus \{\alpha\} =]\alpha, \beta]$$

and consequently $A =]\alpha, \beta]$. Finally suppose $\alpha \notin A$ and $\beta \notin A$:

$$]\alpha, \beta[\subseteq A \subseteq [\alpha, \beta] \setminus \{\alpha, \beta\} =]\alpha, \beta[$$

and consequently $A =]\alpha, \beta[$. Hence, we have proved that A is of the form $[\alpha, \beta]$, $[\alpha, \beta[$, $]\alpha, \beta]$ or $]\alpha, \beta[$. Note that if $A = \emptyset$, there is nothing flawed in the argument just given.

3. Let $A =]-\infty, \alpha[$ where $\alpha \in \mathbf{R}$. Consider $\phi : \mathbf{R} \to]-1, 1[$ defined by $\phi(x) = x/(1 + |x|)$. Then ϕ is a bijection with $\phi^{-1}(y) = y/(1 - |y|)$. Let $\psi = \phi_{|A}$ be the restriction of ϕ to A. Then ψ is injective, and it is therefore a bijection from A to $\psi(A)$. We claim that $\psi(A) =] - 1, \phi(\alpha)[$. Since $|\phi(x)| < 1$ for all $x \in \mathbf{R}$, it is clear that $\psi(A) \subseteq] - 1, 1[$. Since $\phi(x) = 1 - 1/(1 + x)$ for $x > 0$ and $\phi(x) = 1 + 1/(1 - x)$ for $x < 0$, it is clear that ϕ is increasing. So $\psi(A) \subseteq] - 1, \phi(\alpha)[$. To show the reverse

inclusion, consider $y \in]-1, \phi(\alpha)[$. Since ϕ^{-1} is also increasing, from $y < \phi(\alpha)$ we obtain $\phi^{-1}(y) < \alpha$. Hence, $\phi^{-1}(y) \in A$ and $y = \psi(\phi^{-1}(y)) \in \psi(A)$. We have proved that $\psi(A) =]-1, \phi(\alpha)[$ and ψ is consequently a bijection from A to $]-1, \phi(\alpha)[$. Since ϕ is continuous, $\psi = \phi_{|A}$ is also continuous. Since ϕ^{-1} is continuous, $\psi^{-1} = (\phi^{-1})_{|\psi(A)}$ is also continuous. We conclude that $\psi : A \to]-1, \phi(\alpha)[$ is a homeomorphism. We have proved that for all $\alpha \in \mathbf{R}$, $]-\infty, \alpha[$ is homeomorphic to $]-1, \alpha'[$ for some $\alpha' \in \mathbf{R}$.

4. Let $A =]\alpha, +\infty[$ where $\alpha \in \mathbf{R}$. Then if $\phi : \mathbf{R} \to]-, 1, 1[$ is defined as in 3. and $\psi = \phi_{|A}$, then $\psi(A) =]\phi(\alpha), 1[$ and ψ is a homeomorphism from A to $]\phi(\alpha), 1[$. Hence, for all $\alpha \in \mathbf{R}$, $]\alpha, +\infty[$ is homeomorphic to $]\alpha', 1[$ for some $\alpha' \in \mathbf{R}$.

5. Let $A =]\alpha, \beta[$, $\alpha, \beta \in \mathbf{R}$, $\alpha < \beta$. Define $\phi :]-1, 1[\to]\alpha, \beta[$ by:

$$\phi(x) = \alpha + \frac{\beta - \alpha}{2}(x + 1)$$

Then it is easy to show that ϕ is a continuous bijection, and that

ϕ^{-1} is continuous. So $\phi :]-1,1[\to]\alpha,\beta[$ is a homeomorphism.

6. $\phi(x) = x/(1+|x|)$ is a homeomorphism between $\mathbf{R}$ and $]-1,1[$.

7. Let A be a non-empty open interval in $\mathbf{R}$, i.e. a non-empty interval of $\bar{\mathbf{R}}$ which is an open subset of $\mathbf{R}$. Being an interval, from 2. it is of the form $[\alpha,\beta]$, $[\alpha,\beta[$, $]\alpha,\beta]$ or $]\alpha,\beta[$ for some $\alpha,\beta \in \bar{\mathbf{R}}$. Suppose A is of the form $[\alpha,\beta]$. Being non-empty with have $\alpha \leq \beta$. So $\alpha \in [\alpha,\beta] \subseteq \mathbf{R}$. Being an open subset of $\mathbf{R}$, there exists $\epsilon > 0$ such that $]\alpha-\epsilon,\alpha+\epsilon[\subset [\alpha,\beta]$. This is a contradiction since $\alpha \in \mathbf{R}$. So A cannot be of the form $[\alpha,\beta]$ and we prove similarly that it cannot be of the form $[\alpha,\beta[$ and $]\alpha,\beta]$ either. So A is of the form $]\alpha,\beta[$ for some $\alpha,\beta \in \bar{\mathbf{R}}$, $\alpha < \beta$. Suppose $\alpha = -\infty$ and $\beta = +\infty$. Then $A = \mathbf{R}$ which is clearly homeomorphic to $\mathbf{R}$. Suppose $\alpha = -\infty$ and $\beta \in \mathbf{R}$. Then from 3. A is homeomorphic to $]-1,\alpha'[$ for some $\alpha' \in \mathbf{R}$, which is itself homeomorphic to $]-1,1[$, as we have proved in 5. Having proved in 6. that $]-1,1[$ is homeomorphic to $\mathbf{R}$, we conclude that A is homeomorphic to $\mathbf{R}$. Suppose $\alpha \in \mathbf{R}$ and $\beta = +\infty$.

Then from 4. 5. and 6. we see that A is homeomorphic to $\mathbf{R}$. Suppose $\alpha \in \mathbf{R}$ and $\beta \in \mathbf{R}$. Then from 5. and 6. we see that A is homeomorphic to $\mathbf{R}$. Hence, in all possible cases, we see that A is homeomorphic to $\mathbf{R}$. We have proved that any non-empty open interval in $\mathbf{R}$ is homeomorphic to $\mathbf{R}$.

8. Let A be an open interval of $\mathbf{R}$. If $A = \emptyset$, then the induced topology on A is reduced to $\{\emptyset\}$, and $(\emptyset, \{\emptyset\})$ is a connected topological space. So A is a connected subset of $\mathbf{R}$. If $A \neq \emptyset$, then from 7. A is homeomorphic to $\mathbf{R}$. In particular, there exists $f : \mathbf{R} \to A$ which is continuous and surjective. From theorem (95), $\mathbf{R}$ is connected. Since f is continuous, from theorem (96) $f(\mathbf{R})$ is a connected subset of A. Since f is surjective, $f(\mathbf{R}) = A$ and consequently A is connected. We have proved that any open interval of $\mathbf{R}$ is a connected subset of $\mathbf{R}$.

9. Let A be an interval of $\mathbf{R}$, i.e. an interval of $\bar{\mathbf{R}}$ with $A \subseteq \mathbf{R}$. If $A = \emptyset$ then A is connected. So we assume that $A \neq \emptyset$. From 1.

there exist $\alpha, \beta \in \bar{\mathbf{R}}$ such that:

$$]\alpha, \beta[\subseteq A \subseteq [\alpha, \beta]$$

and since $A \neq \emptyset$ we have $\alpha \leq \beta$. Since $]\alpha, \beta[$ is an open interval in $\mathbf{R}$, from 8. it is a connected subset of $\mathbf{R}$. Suppose $\alpha = -\infty$ and $\beta = +\infty$. Then $A = \mathbf{R}$ and:

$$]\alpha, \beta[\subseteq A \subseteq]\alpha, \beta[= \overline{]\alpha, \beta[}$$

Suppose $\alpha = -\infty$ and $\beta \in \mathbf{R}$. Since $A \subseteq \mathbf{R}$ we have:

$$]\alpha, \beta[\subseteq A \subseteq]\alpha, \beta] = \overline{]\alpha, \beta[}$$

Suppose $\alpha \in \mathbf{R}$ and $\beta = +\infty$. Then:

$$]\alpha, \beta[\subseteq A \subseteq [\alpha, \beta[= \overline{]\alpha, \beta[}$$

And finally suppose that $\alpha, \beta \in \mathbf{R}$. Then:

$$]\alpha, \beta[\subseteq A \subseteq [\alpha, \beta] = \overline{]\alpha, \beta[}$$

It follows that $]\alpha, \beta[\subseteq A \subseteq \overline{]\alpha, \beta[}$ in all possible cases, where $\overline{]\alpha, \beta[}$ denotes the closure of $]\alpha, \beta[$ in $\mathbf{R}$. Having proved that $]\alpha, \beta[$ is a connected subset of $\mathbf{R}$, from exercise (6) we conclude that A is a connected subset of $\mathbf{R}$. We have proved that any interval in $\mathbf{R}$ is a connected subset of $\mathbf{R}$.

Exercise 8

Exercise 9.

1. Let $A \subseteq \mathbf{R}$ be a non-empty connected subset of $\mathbf{R}$. Let $\alpha = \inf A$ and $\beta = \sup A$. We assume that there exists $x_0 \in A^c \cap]\alpha, \beta[$. In particular, we have $x_0 \in A^c$ and consequently, since $A \subseteq \mathbf{R}$:
$$A = (A \cap]-\infty, x_0[) \uplus (A \cap]x_0, +\infty[) \tag{2}$$
However, $]-\infty, x_0[$ and $]x_0, +\infty[$ being open subsets of $\mathbf{R}$, the sets $A \cap]-\infty, x_0[$ and $A \cap]x_0, +\infty[$ are open in A, and they are clearly disjoint. Since A is connected, it follows from exercise (4) that $A \cap]-\infty, x_0[= \emptyset$ or $A \cap]x_0, +\infty[= \emptyset$.

2. Suppose $A \cap]x_0, +\infty[= \emptyset$. From (2) we have $A = A \cap]-\infty, x_0[$, and consequently x_0 is an upper-bound of A. Since β is the smallest of such upper-bounds, we obtain $\beta \leq x_0$ contradicting $x_0 \in]\alpha, \beta[$.

3. Similarly, if $A \cap]-\infty, x_0[= \emptyset$, then x_0 is a lower-bound of A and consequently $x_0 \leq \alpha$ contradicting $x_0 \in]\alpha, \beta[$. We have seen

in 1. that $A\cap]-\infty, x_0[=\emptyset$ or $A\cap]x_0, +\infty[=\emptyset$. However, both of these cases lead to a contradiction. We conclude that our initial assumption was absurd, i.e. that there exists no x_0 in $A^c\cap]\alpha, \beta[$. In other words, $A^c\cap]\alpha, \beta[=\emptyset$ or equivalently $]\alpha, \beta[\subseteq A$. The fact that $A \subseteq [\alpha, \beta]$ follows immediately from the fact that α and β are respectively a lower-bound and an upper-bound of A. We have proved that $]\alpha, \beta[\subseteq A \subseteq [\alpha, \beta]$.

4. Let $A \subseteq \mathbf{R}$. Suppose that A is a connected subset of $\mathbf{R}$. If $A = \emptyset$ then in particular A is an interval, as can be seen from definition (118). If $A \neq \emptyset$, then A is a non-empty connected subset of $\mathbf{R}$, and we have just proved that $]\alpha, \beta[\subseteq A \subseteq [\alpha, \beta]$ where $\alpha = \inf A$ and $\beta = \sup A$. In a similar fashion to 2. of exercise (8) (depending on whether α, β lie in A or not), we conclude that A is of the form $[\alpha, \beta]$, $[\alpha, \beta[$, $]\alpha, \beta]$ or $]\alpha, \beta[$. From this same exercise, this is equivalent to A being an interval. So any connected subset of $\mathbf{R}$ is an interval. Conversely, suppose that A is an interval of $\mathbf{R}$. Then from exercise (8), A is a

connected subset of $\mathbf{R}$. We have proved that for all $A \subseteq \mathbf{R}$, A is connected, if and only if A is an interval. This completes the proof of theorem (97).

Exercise 9

Exercise 10. Let $f : \Omega \to \mathbf{R}$ be a continuous map, where $(\Omega, \mathcal{T})$ is a connected topological space. Let $a, b \in \Omega$ with $f(a) \leq f(b)$. From theorem (96), $f(\Omega)$ is a connected subset of $\mathbf{R}$. From theorem (97), $f(\Omega)$ is therefore an interval of $\mathbf{R}$. Since $f(a), f(b)$ are elements of $f(\Omega)$ and $f(a) \leq f(b)$, it follows from definition (118) that for all $z \in [f(a), f(b)]$ we have $z \in f(\Omega)$. So there exists $x \in \Omega$ such that $z = f(x)$. This completes the proof of theorem (98).

Exercise 10

Exercise 11.

1. Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a, b] \to \mathbf{R}$ be a map such that $f'(x)$ exists for all $x \in [a, b]$. Note in particular that f is continuous and therefore measurable. For all $n \geq 1$, let $\phi_n : [a, b] \to [a, b]$:

$$\forall x \in [a,b] \ , \ \phi_n(x) = \left\{ \begin{array}{lcr} x + \frac{(b-x)}{n} & , & \text{if } x \in [a, b[\\ b - \frac{(b-a)}{n} & , & \text{if } x = b \end{array} \right.$$

Then ϕ_n is well-defined on $[a, b]$ and has indeed values in $[a, b]$. The particular definition of ϕ_n is however not very important. What we need to note is that ϕ_n is Borel measurable, satisfies $\phi_n(x) \to x$ while $\phi_n(x) \neq x$ for all $x \in [a, b]$. Given $n \geq 1$, we now define $g_n : [a, b] \to \mathbf{R}$ as:

$$\forall x \in [a,b] \ , \ g_n(x) = \frac{f \circ \phi_n(x) - f(x)}{\phi_n(x) - x}$$

Then $g_n : ([a, b], \mathcal{B}([a, b])) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is well-defined and measurable, and furthermore $g_n(x) \to f'(x)$ for all $x \in [a, b]$. It follows that f' is the pointwise limit of the sequence $(g_n)_{n\geq 1}$, and we conclude from theorem (17) that f' is itself Borel measurable.

2. Since f' is measurable and $\mathbf{R}$-valued, the condition:

$$\int_a^b |f'(t)|dt < +\infty$$

is equivalent to $f' \in L^1_{\mathbf{R}}([a,b],\mathcal{B}([a,b]),dx)$.

3. We assume that $f' \in L^1_{\mathbf{R}}([a,b],\mathcal{B}([a,b]),dx)$. Let $\epsilon > 0$. The topological space $[a,b]$ is metrizable and compact, and in particular σ-compact. The Lebesgue measure dx on $[a,b]$ is finite, and in particular locally finite. Since $f' \in L^1_{\mathbf{R}}([a,b],\mathcal{B}([a,b]),dx)$, we can apply Vitali-Caratheodory theorem (94): there exists measurable maps $u,v:[a,b]\to\bar{\mathbf{R}}$ which are almost surely equal to elements of L^1, such that $u \leq f' \leq v$, u is u.s.c, v is l.s.c and furthermore:

$$\int_a^b (v(t)-u(t))dt \leq \epsilon$$

In particular, denoting $g = v$, we have found $g : [a,b] \to \bar{\mathbf{R}}$ almost surely equal to an element of L^1, such that $f' \leq g$ and g is l.s.c. Note that the integral $\int_a^b g(t)dt$ is meaningful, and:

$$\begin{aligned}
\int_a^b g(t)dt &= \int_a^b (f'(t) + g(t) - f'(t))dt \\
&= \int_a^b f'(t)dt + \int_a^b (g(t) - f'(t))dt \\
&\leq \int_a^b f'(t)dt + \int_a^b (v(t) - u(t))dt \\
&\leq \int_a^b f'(t)dt + \epsilon
\end{aligned}$$

4. Let $\alpha > 0$. Since $f' \leq g$ we have $f' < g + \alpha$. Indeed, suppose $f'(x) = g(x) + \alpha$, $x \in [a,b]$. Then $f'(x) = g(x) = g(x) + \alpha$ and consequently $g(x) \in \{-\infty, +\infty\}$ contradicting the fact that f' is $\mathbf{R}$-valued. Having proved that $f' < g + \alpha$, note that $g + \alpha$ is

also a lower-semi-continuous map, which furthermore is almost surely equal to an element of L^1, since the Lebesgue measure on $[a, b]$ is finite. Furthermore, we have:

$$\begin{aligned}\int_a^b (g+\alpha)(t)dt &= \int_a^b g(t)dt + \alpha(b-a) \\ &\leq \int_a^b f'(t)dt + \epsilon + \alpha(b-a)\end{aligned}$$

Hence, taking $\alpha > 0$ small enough, it is possible to achieve:

$$\int_a^b (g+\alpha)(t)dt \leq \int_a^b f'(t)dt + 2\epsilon$$

Replacing g by $g+\alpha$, we have found $g : [a, b] \to \bar{\mathbf{R}}$ almost surely equal to an element of L^1, which is l.s.c. and satisfies $f' < g$ together with:

$$\int_a^b g(t)dt \leq \int_a^b f'(t)dt + 2\epsilon$$

Since $\epsilon > 0$ was arbitrary, it is possible to find g such that:

$$\int_a^b g(t)dt \leq \int_a^b f'(t)dt + \epsilon$$

In other words, without loss of generality, we have been able to find a map g as in 3., with the additional condition $f' < g$.

5. Let ν be the complex measure defined by $\nu = \int g dx$. Note that strictly speaking, g is not an element of L^1 (it may have values in $\{-\infty, +\infty\}$). If h is an element of $L^1_{\mathbf{R}}([a,b], \mathcal{B}([a,b]), dx)$ such that $g = h$ dx-almost surely, then for all $E \in \mathcal{B}([a,b])$, $\nu(E)$ is defined as:

$$\nu(E) = \int_E h(x)dx$$

Note that ν is in fact a signed measure (i.e. a complex measure with values in $\mathbf{R}$). Since $dx(E) = 0$ implies $\nu(E) = 0$, the measure ν is absolutely continuous with respect to the Lebesgue measure on $[a,b]$. From theorem (58), we have:

$$\forall \epsilon' > 0 \ , \ \exists \delta > 0 \ , \ \forall E \in \mathcal{B}([a,b]) \ , \ dx(E) \leq \delta \ \Rightarrow \ |\nu(E)| \leq \epsilon'$$

6. Let $\eta > 0$ and $x \in [a,b]$. We define:

$$F_\eta(x) = \int_a^x g(t)dt - f(x) + f(a) + \eta(x-a)$$

Then $F_\eta : [a,b] \to \mathbf{R}$ is well-defined, and we claim that it is continuous. It is sufficient to show that $x \to \int_a^x g(t)dt$ is continuous. Let $\epsilon' > 0$ be given, and consider $\delta > 0$ such that the statement of 5. is satisfied. Let $u, u' \in [a,b]$ such that $|u' - u| \leq \delta$. Without loss of generality, we may assume that $u \leq u'$. Then $dx(]u,u']) \leq \delta$ and consequently from 5., $|\nu(]u,u'])| \leq \epsilon'$. So:

$$\left| \int_a^{u'} g(t)dt - \int_a^u g(t)dt \right| \ = \ \left| \int_{[a,u']} g(t)dt - \int_{[a,u]} g(t)dt \right|$$

$$= \left| \int_{]u,u']} g(t)dt \right| = |\nu(]u,u'])| \leq \epsilon'$$

This shows that $x \to \int_a^x g(t)dt$ is indeed continuous on $[a,b]$ (in fact uniformly continuous), and $F_\eta : [a,b] \to \mathbf{R}$ is indeed a continuous map.

7. Given $\eta > 0$, let $x = \sup F_\eta^{-1}(\{0\})$. It is clear that $F_\eta(a) = 0$ and consequently $a \in F_\eta^{-1}(\{0\})$. So $a \leq x$. Since $F_\eta^{-1}(\{0\}) \subseteq [a,b]$, in particular b is an upper-bound of $F_\eta^{-1}(\{0\})$. So $x \leq b$. We have proved that $x \in [a,b]$. In particular, $x \in \mathbf{R}$ and for all $n \geq 1$ we have $x - 1/n < x$. Since x is the lowest upper-bound of $F_\eta^{-1}(\{0\})$, $x - 1/n$ cannot be such an upper-bound. There exists $x_n \in F_\eta^{-1}(\{0\})$ such that $x - 1/n < x_n \leq x$. We have thus constructed a sequence $(x_n)_{n\geq 1}$ in $F_\eta^{-1}(\{0\})$ such that $x_n \to x$ as $n \to +\infty$. Since $F_\eta(x_n) = 0$ for all $n \geq 1$, from the continuity of F_η we obtain $F_\eta(x) = 0$.

8. Suppose $x \in [a,b[$. Having proved in 4. that $f' < g$, in particular

$f'(x) < g(x)$. Since g is l.s.c, the set $\{f'(x) < g\}$ is an open subset of $[a, b]$, which contains x. Hence, there exists $\delta_1 > 0$ such that:

$$]x - \delta_1, x + \delta_1[\cap[a, b] \subseteq \{f'(x) < g\}$$

In particular we have:

$$t \in]x, x + \delta_1[\cap[a, b] \;\Rightarrow\; f'(x) < g(t)$$

Furthermore, by definition of the derivative $f'(x)$, since $\eta > 0$, there exists $\delta_2 > 0$ such that:

$$t \in]x - \delta_2, x + \delta_2[\cap[a, b], t \neq x \;\Rightarrow\; \left|\frac{f(t) - f(x)}{t - x} - f'(x)\right| < \eta$$

In particular, we have:

$$t \in]x, x + \delta_2[\cap[a, b] \;\Rightarrow\; \frac{f(t) - f(x)}{t - x} < f'(x) + \eta$$

Taking $\delta = \min(\delta_1, \delta_2)$, for all $t \in]x, x+\delta[\cap[a,b]$ we have:

$$f'(x) < g(t) \text{ and } \frac{f(t)-f(x)}{t-x} < f'(x) + \eta$$

Note that this conclusion is not very interesting if $x = b$, which is why we have assumed $x \in [a, b[$.

9. Let $t \in]x, x+\delta[\cap[a,b]$. Using 8. we have:

$$\begin{aligned} F_\eta(t) &= \int_a^t g(u)du - f(t) + f(a) + \eta(t-a) \\ &= F_\eta(x) + \int_x^t g(u)du + f(x) - f(t) + \eta(t-x) \\ &> F_\eta(x) + \int_x^t g(u)du - f'(x)(t-x) \\ &\geq F_\eta(x) + \int_x^t f'(x)du - f'(x)(t-x) \\ &= F_\eta(x) = 0 \end{aligned}$$

10. From 9. we have found $\delta > 0$ such that $F_\eta(t) > 0$ for all t in the set $]x, x+\delta[\cap[a,b]$. Having assumed in 8. that $x \in [a,b[$, in particular $x < b$. So it is possible to find $t_0 \in]x,b[$ such that $t_0 \in]x, x+\delta[\cap[a,b]$. In particular $F_\eta(t_0) > 0$. We have proved the existence of $t_0 \in]x,b[$ such that $F_\eta(t_0) > 0$.

11. Suppose $F_\eta(b) < 0$. From 10. we have $t_0 \in]x,b[$ such that $F_\eta(t_0) > 0$. From 6. the map $F_\eta : [a,b] \to \mathbf{R}$ is continuous. Let $h = (F_\eta)_{|[t_0,b]}$ be the restriction of F_η to the interval $[t_0,b]$. Then h is also continuous. From theorem (97), $[t_0,b]$ is a connected topological space. Since $0 \in [F_\eta(b), F_\eta(t_0)]$, from theorem (98) there exists $u \in [t_0,b]$ such that $F_\eta(u) = 0$. Since $x = \sup F_\eta^{-1}(\{0\})$, in particular $u \leq x$. Hence, we obtain the contradiction $x < t_0 \leq u \leq x$.

12. From 11. we see that $F_\eta(b) \geq 0$ must be true when $x \in [a,b[$. Having proved in 7. that $F_\eta(x) = 0$, if $x = b$, $F_\eta(b) = 0$ and in particular $F_\eta(b) \geq 0$ is still true. So $F_\eta(b) \geq 0$ in all cases.

13. From $F_\eta(b) \geq 0$ we obtain:

$$\int_a^b g(t)dt - f(b) + f(a) + \eta(b-a) \geq 0$$

This being true for all $\eta > 0$, we have:

$$f(b) - f(a) \leq \int_a^b g(t)dt$$

Hence, using 3. we obtain:

$$f(b) - f(a) \leq \int_a^b f'(t)dt + \epsilon$$

and this being true for all $\epsilon > 0$, we have proved that:

$$f(b) - f(a) \leq \int_a^b f'(t)dt \tag{3}$$

Having considered $a, b \in \mathbf{R}$, $a < b$ and $f : [a, b] \to \mathbf{R}$ a map such that $f'(x)$ exists for all $x \in [a,b]$ and:

$$\int_a^b |f'(t)|dt < +\infty$$

we have been able to prove inequality (3). Applying this result to $-f$ instead of f, we obtain:

$$\int_a^b f'(t)dt \leq f(b) - f(a)$$

and finally we conclude that:

$$f(b) - f(a) = \int_a^b f'(t)dt$$

This completes the proof of theorem (99).

Exercise 11

Exercise 12.

1. Let $\alpha > 0$ and $k_\alpha : \mathbf{R}^n \to \mathbf{R}^n$ defined by $k_\alpha(x) = \alpha x$. Then k_α is continuous, and in particular Borel measurable.

2. Let $\mu : \mathcal{B}(\mathbf{R}^n) \to [0, +\infty]$ be defined by:

$$\forall B \in \mathcal{B}(\mathbf{R}^n)\ ,\ \mu(B) = \alpha^n dx(\{k_\alpha \in B\})$$

where dx is the Lebesgue measure on $\mathbf{R}^n$. Note that μ is well-defined since $\{k_\alpha \in B\}$ is a Borel set for all $B \in \mathcal{B}(\mathbf{R}^n)$, k_α being measurable. It is clear that $\mu(\emptyset) = 0$ and furthermore, if $(B_p)_{p\geq 1}$ is sequence of pairwise disjoint elements of $\mathcal{B}(\mathbf{R}^n)$ and $B = \uplus_{p\geq 1} B_p$, we have:

$$\mu(B) \quad = \quad \alpha^n dx\left(k_\alpha^{-1}\left(\biguplus_{p\geq 1} B_p\right)\right)$$

$$
\begin{aligned}
&= \alpha^n dx\left(\biguplus_{p\geq 1} k_\alpha^{-1}(B_p)\right)\\
&= \alpha^n\left(\sum_{p=1}^{+\infty} dx(k_\alpha^{-1}(B_p))\right)\\
&= \sum_{p=1}^{+\infty}\alpha^n dx(\{k_\alpha\in B_p\})\\
&= \sum_{p=1}^{+\infty}\mu(B_p)
\end{aligned}
$$

So μ is a measure on $\mathbf{R}^n$. Let $a_i, b_i \in \mathbf{R}$, $a_i \leq b_i$ for $i \in \mathbf{N}_n$. For all $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ the inequality $a_i \leq \alpha x_i \leq b_i$ is equivalent to $a_i/\alpha \leq x_i \leq b_i/\alpha$. Hence:

$$
\mu([a_1, b_1]\times\ldots\times[a_n, b_n]) \quad = \quad \alpha^n dx\left(\left\{\alpha x\in\prod_{i=1}^n [a_i, b_i]\right\}\right)
$$

$$\begin{aligned} &= \alpha^n dx \left(\prod_{i=1}^n \left[\frac{a_i}{\alpha}, \frac{b_i}{\alpha} \right] \right) \\ &= \alpha^n \prod_{i=1}^n \left(\frac{b_i}{\alpha} - \frac{a_i}{\alpha} \right) \\ &= \prod_{i=1}^n (b_i - a_i) \end{aligned}$$

From the uniqueness property of definition (63) we conclude that $\mu = dx$. Hence, we have proved that for all $B \in \mathcal{B}(\mathbf{R}^n)$:

$$dx(\{k_\alpha \in B\}) = \frac{1}{\alpha^n}\mu(B) = \frac{1}{\alpha^n}dx(B)$$

3. Let $\epsilon > 0$ and $x \in \mathbf{R}^n$. Let $B(x, \epsilon)$ be the open ball:

$$B(x, \epsilon) = \{y \in \mathbf{R}^n : \|x - y\| < \epsilon\}$$

where $\| \cdot \|$ denotes the usual Euclidean norm on $\mathbf{R}^n$. Given $u \in \mathbf{R}^n$ we consider $\tau_u : \mathbf{R}^n \to \mathbf{R}^n$ the translation mapping of vector u defined by $\tau_u(x) = u + x$. Then τ_u is clearly continuous, hence Borel measurable. Furthermore, for all $a, b \in \mathbf{R}^n$ such that $a_i \leq b_i$ for all $i \in \mathbf{N}_n$, we have:

$$\begin{aligned} dx\left(\left\{\tau_u \in \prod_{i=1}^n [a_i, b_i]\right\}\right) &= dx\left(\prod_{i=1}^n [a_i - u_i, b_i - u_i]\right) \\ &= \prod_{i=1}^n (b_i - a_i) \end{aligned}$$

and in a similar fashion to 2. we conclude from the uniqueness property of definition (63) that for all $B \in \mathcal{B}(\mathbf{R}^n)$:

$$dx(\{\tau_u \in B\}) = dx(B)$$

This equality expresses the idea that the Lebesgue measure is *invariant by translation*. We shall see more on the subject in Tutorial 17. In the meantime, using 2. we obtain:

$$dx(B(x, \epsilon)) = dx(\{\tau_{-x} \in B(0, \epsilon)\})$$

$$
\begin{aligned}
&= dx(B(0,\epsilon)) \\
&= dx(\{k_{1/\epsilon} \in B(0,1)\}) \\
&= \epsilon^n dx(B(0,1))
\end{aligned}
$$

So we have proved that $dx(B(x,\epsilon)) = \epsilon^n dx(B(0,1))$.

Exercise 12

Exercise 13.

1. Let μ be a complex measure on $\mathbf{R}^n$. Let $\lambda \in \mathbf{R}$ and suppose that $\lambda < 0$. Let $x \in \mathbf{R}^n$ and $\epsilon > 0$. Since $B(x,\epsilon)$ is an open subset of $\mathbf{R}^n$, in particular it is a Borel subset of $\mathbf{R}^n$. So $|\mu|(B(x,\epsilon))$ and $dx(B(x,\epsilon))$ are well-defined quantities of $[0,+\infty]$. In fact, from theorem (57), the total variation $|\mu|$ is a finite measure on $\mathbf{R}^n$, so $|\mu|(B(x,\epsilon))$ is an element of $\mathbf{R}^+$ (this is not relevant to the present question, but the fact that $|\mu|$ is a finite measure should not be forgotten). From the inclusions:

$$[-1/2\sqrt{n}, 1/2\sqrt{n}]^n \subseteq B(0,1) \subseteq [-1,1]^n$$

we obtain the crude estimates:

$$\left(\frac{1}{\sqrt{n}}\right)^n \leq dx(B(0,1)) \leq 2^n$$

and it follows from 3. of exercise (12) that $dx(B(x,\epsilon))$ is an element of $]0,+\infty[$. Hence, we see that $|\mu|(B(x,\epsilon))/dx(B(x,\epsilon))$

is a well-defined element of $\mathbf{R}^+$. Since $(M\mu)(x)$ is an upper-bound of all such ratios for $\epsilon > 0$, we have:

$$\lambda < 0 \leq \frac{|\mu|(B(x,\epsilon))}{dx(B(x,\epsilon))} \leq (M\mu)(x)$$

So $x \in \{\lambda < M\mu\}$. This being true for all $x \in \mathbf{R}^n$, we conclude that $\{\lambda < M\mu\} = \mathbf{R}^n$.

2. Suppose $\lambda = 0$ and $\mu \neq 0$. There exists $E \in \mathcal{B}(\mathbf{R}^n)$ such that $\mu(E) \neq 0$. Since $|\mu(E)| \leq |\mu|(E)$, in particular $|\mu|(E) > 0$. Let $x \in \mathbf{R}^n$. Since $B(x,p) \uparrow \mathbf{R}^n$ as $p \to +\infty$, from theorem (7):

$$0 < |\mu|(E) = \lim_{p \to +\infty} |\mu|(E \cap B(x,p))$$

In particular, there exists $p \geq 1$ such that $|\mu|(E \cap B(x,p)) > 0$ and consequently $|\mu|(B(x,p)) > 0$. Hence, we have:

$$0 < \frac{|\mu|(B(x,p))}{dx(B(x,p))} \leq (M\mu)(x)$$

and we have proved that $x \in \{\lambda < M\mu\} = \{0 < M\mu\}$. This being true for all $x \in \mathbf{R}^n$, we have $\{\lambda < M\mu\} = \mathbf{R}^n$. Suppose now that $\lambda = 0$ with $\mu = 0$. Then $|\mu| = 0$ and it is clear that $(M\mu)(x) = 0$ for all $x \in \mathbf{R}^n$. So $\{\lambda < M\mu\} = \emptyset$.

3. Suppose $\lambda > 0$. Let $x \in \{\lambda < M\mu\}$. Then $\lambda < (M\mu)(x)$. Since $(M\mu)(x)$ is the smallest upper-bound of all ratios:

$$|\mu|(B(x,\epsilon))/dx(B(x,\epsilon))$$

as $\epsilon > 0$, λ cannot be such an upper-bound. There exists $\epsilon > 0$ such that $\lambda < |\mu|(B(x,\epsilon))/dx(B(x,\epsilon))$. Defining:

$$t = |\mu|(B(x,\epsilon))/dx(B(x,\epsilon))$$

we have $t > \lambda$ and $|\mu|(B(x,\epsilon)) = t dx(B(x,\epsilon))$.

4. Since $1 < t/\lambda$ we have $\epsilon^n < \epsilon^n t/\lambda$. Furthermore, it is clear that $\lim_{\delta\downarrow 0}(\epsilon+\delta)^n = \epsilon^n$. Hence, we have $(\epsilon+\delta)^n < \epsilon^n t/\lambda$, for $\delta > 0$ small enough.

5. Suppose $y \in B(x,\delta)$ and let $z \in B(x,\epsilon)$. Then:

$$\|z-y\| \leq \|z-x\| + \|x-y\| < \epsilon + \delta$$

So $z \in B(y,\epsilon+\delta)$ and we have proved that $B(x,\epsilon) \subseteq B(y,\epsilon+\delta)$.

6. Let $y \in B(x,\delta)$. Since $B(x,\epsilon) \subseteq B(y,\epsilon+\delta)$, we have:

$$\begin{aligned} |\mu|(B(y,\epsilon+\delta)) &\geq |\mu|(B(x,\epsilon)) \\ &= tdx(B(x,\epsilon)) \\ &= \epsilon^n tdx(B(0,1)) \\ &= \frac{\epsilon^n t}{(\epsilon+\delta)^n} dx(B(y,\epsilon+\delta)) \\ &> \lambda dx(B(y,\epsilon+\delta)) \end{aligned}$$

where the second and third equalities stem from exercise (12).

7. For all $y \in B(x,\delta)$, from 6. we have:

$$\lambda < \frac{|\mu|(B(y,\epsilon+\delta))}{dx(B(y,\epsilon+\delta))} \leq (M\mu)(y)$$

So in particular $y \in \{\lambda < M\mu\}$ and we have proved that $B(x, \delta) \subseteq \{\lambda < M\mu\}$. Having considered $x \in \{\lambda < M\mu\}$ we have found $\delta > 0$ such that $B(x, \delta) \subseteq \{\lambda < M\mu\}$. This shows that $\{\lambda < M\mu\}$ is an open subset of $\mathbf{R}^n$, for all $\lambda \in \mathbf{R}$ with $\lambda > 0$. In fact, it follows from 1. and 2. that $\{\lambda < M\mu\}$ is also open if $\lambda \leq 0$. We conclude that $\{\lambda < M\mu\}$ is open for all $\lambda \in \mathbf{R}$, i.e. that the maximal function $M\mu$ is lower-semi-continuous. In particular, $\{\lambda < M\mu\}$ is a Borel subset of $\mathbf{R}^n$ for all $\lambda \in \mathbf{R}$ and from theorem (15), $M\mu$ is measurable.

Exercise 13

Exercise 14.

1. Let $B_i = B(x_i, \epsilon_i)$, $i = 1, \ldots, N$, be a finite collection of open balls in $\mathbf{R}^n$ where we have assumed that $\epsilon_N \le \ldots \le \epsilon_1$. We define $J_0 = \{1, \ldots, N\}$ and for all $k \ge 1$:

$$J_k \stackrel{\triangle}{=} \begin{cases} J_{k-1} \cap \{j : \ j > i_k \ , \ B_j \cap B_{i_k} = \emptyset\} & \text{if } J_{k-1} \neq \emptyset \\ \emptyset & \text{if } J_{k-1} = \emptyset \end{cases}$$

   where $i_k = \min J_{k-1}$ if $J_{k-1} \neq \emptyset$. Suppose $k \ge 1$ and $J_{k-1} \neq \emptyset$. The fact that $J_k \subseteq J_{k-1}$ is clear. However, the inclusion is strict. Indeed, since $i_k = \min J_{k-1}$, in particular $i_k \in J_{k-1}$. However, it is clear that $i_k \notin J_k$. We have proved that $J_k \subset J_{k-1}$.

2. Since $(J_k)_{k \ge 0}$ is a strictly decreasing sequence (in the inclusion sense) and J_0 is a finite set, there exists $k \ge 1$ such that $J_k = \emptyset$. It follows that $p = \min\{k \ge 1 : J_k = \emptyset\}$, as the smallest element of a non-empty subset of $\mathbf{N}$, is well-defined.

3. Let $S = \{i_1, \ldots, i_p\}$ where $i_k = \min J_{k-1}$ for all $k \ge 1$ with $J_{k-1} \neq \emptyset$. In order to show that S is well-defined, we need to

ensure that i_k is meaningful for $k \in \mathbf{N}_p$, i.e. that $J_{k-1} \neq \emptyset$. But if $k \in \mathbf{N}_p$ and $J_{k-1} = \emptyset$, since p is the smallest element of $\{k \geq 1 : J_k = \emptyset\}$ we obtain $p \leq k-1$ and $k \leq p$ which is a contradiction. So S is well-defined.

4. Suppose $1 \leq k < k' \leq p$. We have $i_{k'} \in J_{k'-1} \subseteq J_k$. So $i_{k'} \in J_k$.

5. The family $(B_i)_{i \in S}$ is a family of open balls. Suppose $i, j \in S$ with $i < j$. There exist $1 \leq k < k' \leq p$ such that $i = i_k$ and $j = i_{k'}$. From 4. we have $j \in J_k$. This implies in particular that $B_j \cap B_{i_k} = \emptyset$. So $B_j \cap B_i = \emptyset$, and $(B_i)_{i \in S}$ is a family of pairwise disjoint open balls.

6. Let $i \in \{1, \ldots, N\} \setminus S$ and $k_0 = \min\{k \in \mathbf{N}_p : i \notin J_k\}$. In order to show that k_0 is well-defined, we need to check that $\{k \in \mathbf{N}_p : i \notin J_k\}$ is not empty. This is clear from the fact that $J_p = \emptyset$. So k_0 is well-defined. Note that this conclusion holds for any $i \in \{1, \ldots, N\}$.

7. k_0 being the smallest element of $\{k \in \mathbf{N}_p : i \notin J_k\}$, $k_0 - 1$ does not lie in this set. So either $k_0 - 1 = 0$ or $i \in J_{k_0-1}$. Since $J_0 = \{1, \ldots, N\}$, in any case we have $i \in J_{k_0-1}$. In particular $J_{k_0-1} \neq \emptyset$. So i_{k_0} is defined as the smallest element of J_{k_0-1}. From $i \in J_{k_0-1}$ we obtain $i_{k_0} \leq i$.

8. Since $J_{k_0-1} \neq \emptyset$, we have:

$$J_{k_0} = J_{k_0-1} \cap \{j : j > i_{k_0}, B_j \cap B_{i_{k_0}} = \emptyset\}$$

k_0 being the smallest element of $\{k \in \mathbf{N}_p : i \notin J_k\}$, in particular it is an element of this set and consequently we know that $i \notin J_{k_0}$. However, we have proved in 7. that $i \in J_{k_0-1}$. Furthermore, we know that $i_{k_0} \leq i$ and since by assumption $i \in \{1, \ldots, N\} \setminus S$, in particular i is not an element of S. So $i \neq i_{k_0}$ and therefore $i_{k_0} < i$. Since $i \notin J_{k_0}$ we conclude that $B_i \cap B_{i_{k_0}} \neq \emptyset$.

9. From 8. we have $B_i \cap B_{i_{k_0}} = B(x_i, \epsilon_i) \cap B(x_{i_{k_0}}, \epsilon_{i_{k_0}}) \neq \emptyset$. Let x be an arbitrary element of $B_i \cap B_{i_{k_0}}$. Then for all $y \in B_i$, since

$i_{k_0} < i$ and $\epsilon_N \leq \ldots \leq \epsilon_1$, we have:

$$\begin{aligned} \|y - x_{i_{k_0}}\| &\leq \|y - x_i\| + \|x_i - x\| + \|x - x_{i_{k_0}}\| \\ &< \epsilon_i + \epsilon_i + \epsilon_{i_{k_0}} \\ &\leq 3\epsilon_{i_{k_0}} \end{aligned}$$

So $y \in B(x_{i_{k_0}}, 3\epsilon_{i_{k_0}})$ and we have proved $B_i \subseteq B(x_{i_{k_0}}, 3\epsilon_{i_{k_0}})$.

10. For all $i \in \{1, \ldots, N\} \setminus S$, we found $k_0 \in \mathbf{N}_p$ such that $B_i \subseteq B(x_{i_{k_0}}, 3\epsilon_{i_{k_0}})$. In other words, if we denote $j(i) = i_{k_0}$, there exists some $j(i) \in S$ such that we have $B_i \subseteq B(x_{j(i)}, 3\epsilon_{j(i)})$. Hence:

$$\begin{aligned} \bigcup_{i=1}^{N} B(x_i, \epsilon_i) &= \bigcup_{i \in S} B(x_i, \epsilon_i) \cup \left(\bigcup_{i \notin S} B(x_i, \epsilon_i) \right) \\ &\subseteq \bigcup_{i \in S} B(x_i, \epsilon_i) \cup \left(\bigcup_{i \notin S} B(x_{j(i)}, 3\epsilon_{j(i)}) \right) \end{aligned}$$

$$\begin{aligned} &\subseteq \bigcup_{i\in S} B(x_i,\epsilon_i) \cup \left(\bigcup_{i\in S} B(x_i,3\epsilon_i)\right) \\ &= \bigcup_{i\in S} B(x_i,3\epsilon_i) \end{aligned}$$

So $S = \{i_1,\ldots,i_p\}$ is a subset of $\{1,\ldots,N\}$ such that $(B_i)_{i\in S}$ is a family of pairwise disjoint open balls, and:

$$\bigcup_{i=1}^{N} B(x_i,\epsilon_i) \subseteq \bigcup_{i\in S} B(x_i,3\epsilon_i)$$

11. Using 10. and exercise (12), we have:

$$\begin{aligned} dx\left(\bigcup_{i=1}^{N} B(x_i,\epsilon_i)\right) &\leq dx\left(\bigcup_{i\in S} B(x_i,3\epsilon_i)\right) \\ &\leq \sum_{i\in S} dx(B(x_i,3\epsilon_i)) \end{aligned}$$

$$\begin{aligned} &= \sum_{i \in S} 3^n \epsilon_i^n dx(B(0,1)) \\ &= 3^n \sum_{i \in S} dx(B(x_i, \epsilon_i)) \end{aligned}$$

where the second inequality stems from the fact that a measure is always sub-additive, as can be seen from exercise (13) of Tutorial 5.

Exercise 14

Exercise 15.

1. Let μ be a complex measure on $\mathbf{R}^n$. Let $\lambda > 0$ and K be a non-empty compact subset of $\{\lambda < M\mu\}$. Let $x \in K$. Then $x \in \{\lambda < M\mu\}$, i.e. $\lambda < (M\mu)(x)$. Since $(M\mu)(x)$ is the smallest upper-bound of all ratios:

$$|\mu|(B(x,\epsilon))/dx(B(x,\epsilon))$$

as $\epsilon > 0$, it is impossible for λ to be such an upper-bound. There exists $\epsilon_x > 0$ such that:

$$\lambda < \frac{|\mu|(B(x,\epsilon_x))}{dx(B(x,\epsilon_x))} \tag{4}$$

Now it is clear that $K \subseteq \cup_{x\in K} B(x,\epsilon_x)$. Since K is compact, there exist $N \geq 1$ and $x_1,\ldots,x_N \in K$ such that:

$$K \subseteq B(x_1,\epsilon_{x_1}) \cup \ldots \cup B(x_N,\epsilon_{x_N})$$

Defining $\epsilon_i = \epsilon_{x_i}$ and $B_i = B(x_i,\epsilon_i)$, the collection $(B_i)_{i\in\mathbf{N}_N}$ is therefore a covering of K. From (4), for all $i = 1,\ldots,N$ we have $\lambda dx(B_i) < |\mu|(B_i)$.

2. By re-indexing the B_i's if necessary, without loss of generality we can assume that $\epsilon_N \leq \ldots \leq \epsilon_1$. From exercise (14), there exists a subset S of $\{1, \ldots, N\}$ such that the B_i's for $i \in S$ are pairwise disjoint, and furthermore:

$$dx\left(\bigcup_{i=1}^{N} B(x_i, \epsilon_i)\right) \leq 3^n \sum_{i \in S} dx(B(x_i, \epsilon_i))$$

Hence, since $K \subseteq \cup_{i=1}^{N} B_i$, using 1. we obtain:

$$\begin{aligned} dx(K) &\leq dx\left(\bigcup_{i=1}^{N} B(x_i, \epsilon_i)\right) \\ &\leq 3^n \sum_{i \in S} dx(B(x_i, \epsilon_i)) \\ &< 3^n \sum_{i \in S} \frac{1}{\lambda} |\mu|(B(x_i, \epsilon_i)) \end{aligned}$$

$$= \quad \frac{3^n}{\lambda}|\mu|\left(\bigcup_{i\in S} B(x_i,\epsilon_i)\right)$$

where the last equality stems from the fact that all the B_i's, $i \in S$, are pairwise disjoint. We have effectively obtained a strict inequality, when only a large inequality was required.

3. Let $\|\mu\| = |\mu|(\mathbf{R}^n) < +\infty$ be the total mass of $|\mu|$. From 2.:

$$dx(K) \leq 3^n\lambda^{-1}|\mu|\left(\bigcup_{i\in S} B(x_i,\epsilon_i)\right) \leq 3^n\lambda^{-1}\|\mu\|$$

4. Having considered a complex measure μ on $\mathbf{R}^n$, with maximal function $M\mu$, given $\lambda \in \mathbf{R}^+ \setminus \{0\}$, for all K non-empty compact subset of $\{\lambda < M\mu\}$, we have proved that:

$$dx(K) \leq 3^n\lambda^{-1}\|\mu\|$$

Note that this inequality is still valid if $K = \emptyset$. The Lebesgue measure on $\mathbf{R}^n$ being locally finite, from theorem (74) it is innerregular. In particular, we have:

$$dx(\{\lambda < M\mu\}) = \sup\{dx(K) : K \subseteq \{\lambda < M\mu\}, K \text{ compact}\}$$

In other words, $dx(\{\lambda < M\mu\})$ is the smallest upper-bound of all $dx(K)$'s, as K runs through the set of all compact subsets of $\{\lambda < M\mu\}$. Having proved that $3^n\lambda^{-1}\|\mu\|$ is one of those upper-bounds, we conclude that:

$$dx(\{\lambda < M\mu\}) \leq 3^n\lambda^{-1}\|\mu\|$$

This completes the proof of theorem (100).

Exercise 15

Exercise 16.

1. Let $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, $n \geq 1$. From theorem (63), $\mu = \int f dx$ is a well-defined complex measure on $\mathbf{R}^n$, and its total variation $|\mu|$ is given by $|\mu| = \int |f| dx$. From definition (120), the maximal function Mf of f is exactly the maximal function $M\mu$ of μ. Hence, for all $x \in \mathbf{R}^n$:

$$\begin{aligned} (Mf)(x) &= (M\mu)(x) \\ &= \sup_{\epsilon>0} \frac{|\mu|(B(x,\epsilon))}{dx(B(x,\epsilon))} \\ &= \sup_{\epsilon>0} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |f| dx \end{aligned}$$

2. If $\mu = \int f dx$ then $|\mu| = \int |f| dx$ and consequently:

$$\|\mu\| = |\mu|(\mathbf{R}^n) = \int_{\mathbf{R}^n} |f| dx = \|f\|_1$$

Applying theorem (100) to μ, for all $\lambda > 0$ we obtain:

$$\begin{aligned} dx(\{\lambda < Mf\}) &= dx(\{\lambda < M\mu\}) \\ &\leq 3^n \lambda^{-1} \|\mu\| \\ &= 3^n \lambda^{-1} \|f\|_1 \end{aligned}$$

Exercise 16

Exercise 17.

1. Let $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, $n \geq 1$. Let $x \in \mathbf{R}^n$. We assume that f is continuous at x. Let $\eta > 0$. There is $\delta > 0$ such that:

$$\forall y \in \mathbf{R}^n \ , \ \|x - y\| \leq \delta \ \Rightarrow \ |f(x) - f(y)| \leq \eta$$

Suppose $\epsilon > 0$ is such that $0 < \epsilon < \delta$. Then:

$$\frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |f(y) - f(x)| dy \leq \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} \eta dy = \eta$$

We conclude that:

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |f(y) - f(x)| dy = 0$$

and x is therefore a Lebesgue point of f.

2. Let $x \in \mathbf{R}^n$. We assume that x is a Lebesgue point of f. For

all $\epsilon > 0$, denoting $B_\epsilon = B(x,\epsilon)$ we have:

$$\begin{aligned}\left|\frac{1}{dx(B_\epsilon)}\int_{B_\epsilon} f(y)dy - f(x)\right| &= \left|\frac{1}{dx(B_\epsilon)}\int_{B_\epsilon}(f(y)-f(x))dy\right| \\ &\leq \frac{1}{dx(B_\epsilon)}\int_{B_\epsilon}|f(y)-f(x)|dy\end{aligned}$$

Hence, from:

$$\lim_{\epsilon\downarrow\downarrow 0}\frac{1}{dx(B(x,\epsilon))}\int_{B(x,\epsilon)}|f(y)-f(x)|dy = 0$$

we conclude that:

$$f(x) = \lim_{\epsilon\downarrow\downarrow 0}\frac{1}{dx(B(x,\epsilon))}\int_{B(x,\epsilon)} f(y)dy$$

Exercise 17

Exercise 18.

1. Given $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, for all $\epsilon > 0$ and $x \in \mathbf{R}^n$, let:
$$(T_\epsilon f)(x) = \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |f(y) - f(x)| dy$$
and:
$$(Tf)(x) = \inf_{\epsilon > 0} \sup_{u \in]0,\epsilon[} (T_u f)(x)$$
From theorem (79), the space $C^c_{\mathbf{C}}(\mathbf{R}^n)$ of continuous $\mathbf{C}$-valued functions defined on $\mathbf{R}^n$ with compact support, is dense in L^1. Given $\eta > 0$, there exists $g \in C^c_{\mathbf{C}}(\mathbf{R}^n)$ such that $\|f - g\|_1 \leq \eta$.

2. Let $h = f - g$. For all $\epsilon > 0$ and $x \in \mathbf{R}^n$ we have:
$$\begin{aligned}(T_\epsilon h)(x) &= \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |h(y) - h(x)| dy \\ &\leq \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} (|h(y)| + |h(x)|) dy\end{aligned}$$

$$\begin{aligned} &= \frac{1}{dx(B(x,\epsilon))}\int_{B(x,\epsilon)} |h(y)|dy + |h(x)| \\ &= \frac{1}{dx(B(x,\epsilon))}\int_{B(x,\epsilon)} |h|dx + |h(x)| \end{aligned}$$

3. Let $x \in \mathbf{R}^n$. From exercise (16) we have:

$$(Mh)(x) = \sup_{\epsilon>0} \frac{1}{dx(B(x,\epsilon))}\int_{B(x,\epsilon)} |h|dx$$

In particular, for all $\epsilon > 0$, from 2. we obtain:

$$(T_\epsilon h)(x) \leq (Mh)(x) + |h(x)|$$

Hence, if $\epsilon > 0$ is given, $(Mh)(x) + |h(x)|$ is an upper-bound of all $(T_u h)(x)$ as $u \in]0,\epsilon[$. It follows that:

$$\sup_{u\in]0,\epsilon[} (T_u h)(x) \leq (Mh)(x) + |h(x)|$$

and we have:

$$\begin{aligned}(Th)(x) &= \inf_{\epsilon'>0}\sup_{u\in]0,\epsilon'[}(T_u h)(x)\\ &\leq \sup_{u\in]0,\epsilon[}(T_u h)(x)\\ &\leq (Mh)(x)+|h(x)|\end{aligned}$$

This being true for all $x\in\mathbf{R}^n$, $Th\leq Mh+|h|$.

4. Let $x\in\mathbf{R}^n$ and $\epsilon>0$. Let $B_\epsilon=B(x,\epsilon)$. Then:

$$\begin{aligned}(T_\epsilon f)(x) &= \frac{1}{dx(B_\epsilon)}\int_{B_\epsilon}|f(y)-f(x)|dy\\ &= \frac{1}{dx(B_\epsilon)}\int_{B_\epsilon}|g(y)-g(x)+h(y)-h(x)|dy\\ &\leq \frac{1}{dx(B_\epsilon)}\left(\int_{B_\epsilon}|g(y)-g(x)|dy+\int_{B_\epsilon}|h(y)-h(x)|dy\right)\\ &= (T_\epsilon g)(x)+(T_\epsilon h)(x)\end{aligned}$$

This being true for all $x \in \mathbf{R}^n$, $T_\epsilon f \leq T_\epsilon g + T_\epsilon h$.

5. Let $x \in \mathbf{R}^n$. Let $\epsilon_1, \epsilon_2 > 0$ be given and $\epsilon = \min(\epsilon_1, \epsilon_2)$. For all $u \in]0, \epsilon[$, using 4. we have:

$$\begin{array}{rcl} (T_u f)(x) & \leq & (T_u g)(x) + (T_u h)(x) \\ & \leq & \sup\limits_{u \in]0,\epsilon_1[} (T_u g)(x) + \sup\limits_{u \in]0,\epsilon_2[} (T_u h)(x) \end{array}$$

Hence, the right-hand-side of this inequality is an upper-bound of all $(T_u f)(x)$'s as $u \in]0, \epsilon[$. It follows that:

$$\begin{array}{rcl} (Tf)(x) & = & \inf\limits_{\epsilon' > 0} \sup\limits_{u \in]0,\epsilon'[} (T_u f)(x) \\ & \leq & \sup\limits_{u \in]0,\epsilon[} (T_u f)(x) \\ & \leq & \sup\limits_{u \in]0,\epsilon_1[} (T_u g)(x) + \sup\limits_{u \in]0,\epsilon_2[} (T_u h)(x) \end{array}$$

Suppose $\sup_{u \in]0,\epsilon_1[} (T_u g)(x) < +\infty$. Then this quantity can be safely subtracted from both sides of the previous inequality, to obtain:

$$(Tf)(x) - \sup_{u\in]0,\epsilon_1[} (T_u g)(x) \leq \sup_{u\in]0,\epsilon_2[} (T_u h)(x)$$

Hence, $\epsilon_1 > 0$ being given, we see that the left-hand-side of this inequality is a lower-bound of all $\sup_{u\in]0,\epsilon_2[}(T_u h)(x)$'s, as $\epsilon_2 > 0$. Since $(Th)(x)$ is the greatest of such lower-bounds, we obtain:

$$(Tf)(x) - \sup_{u\in]0,\epsilon_1[} (T_u g)(x) \leq (Th)(x)$$

or equivalently:

$$(Tf)(x) \leq \sup_{u\in]0,\epsilon_1[} (T_u g)(x) + (Th)(x)$$

which is still valid when $\sup_{u\in]0,\epsilon_1[}(T_u g)(x) = +\infty$. Suppose now that $(Th)(x) < +\infty$. Then $(Th)(x)$ can be safely subtracted from both sides of the previous inequality, to obtain:

$$(Tf)(x) - (Th)(x) \leq \sup_{u\in]0,\epsilon_1[} (T_u g)(x)$$

This being established for all $\epsilon_1 > 0$, $(Tf)(x) - (Th)(x)$ is a lower-bound of all $\sup_{u\in]0,\epsilon_1[}(T_u g)(x)$'s, as $\epsilon_1 > 0$. Since $(Tg)(x)$ is the greatest of such lower-bounds, we obtain:

$$(Tf)(x) - (Th)(x) \leq (Tg)(x)$$

or equivalently:

$$(Tf)(x) \leq (Tg)(x) + (Th)(x)$$

This being true for all $x \in \mathbf{R}^n$, $Tf \leq Tg + Th$.

6. Let $x \in \mathbf{R}^n$. Since $g \in C^c_{\mathbf{C}}(\mathbf{R}^n)$, g is a continuous element of L^1. From exercise (17), x is therefore a Lebesgue point of g. Hence, from definition (121):

$$\lim_{\epsilon\downarrow\downarrow 0}(T_\epsilon g)(x) = \lim_{\epsilon\downarrow\downarrow 0}\frac{1}{dx(B(x,\epsilon))}\int_{B(x,\epsilon)}|g(y) - g(x)|dy = 0$$

Let $\delta > 0$. There exists $\epsilon > 0$ such that:

$$u \in]0,\epsilon[\ \Rightarrow\ (T_u g)(x) \leq \delta$$

So δ is an upper-bound of all $(T_u g)(x)$'s as $u \in]0, \epsilon[$, and consequently $\sup_{u\in]0,\epsilon[}(T_u g)(x) \leq \delta$. Hence:

$$\begin{aligned}(Tg)(x) &= \inf_{\epsilon'>0} \sup_{u\in]0,\epsilon'[} (T_u g)(x) \\ &\leq \sup_{u\in]0,\epsilon[} (T_u g)(x) \\ &\leq \delta\end{aligned}$$

This being true for all $\delta > 0$, we conclude that $(Tg)(x) = 0$. This being true for all $x \in \mathbf{R}^n$, we have proved that $Tg = 0$.

7. Using 3. and 5. together with $Tg = 0$, we obtain:

$$Tf \leq Tg + Th = Th \leq Mh + |h|$$

8. Let $\alpha > 0$. Let $x \in \mathbf{R}^n$ and suppose that $(Mh)(x) \leq \alpha$ together with $|h|(x) \leq \alpha$. Using 7. we obtain:

$$(Tf)(x) \leq (Mh)(x) + |h|(x) \leq 2\alpha$$

Hence, we have shown the inclusion:

$$\{Mh \leq \alpha\} \cap \{|h| \leq \alpha\} \subseteq \{Tf \leq 2\alpha\}$$

from which we conclude that:

$$\{2\alpha < Tf\} \subseteq \{\alpha < Mh\} \cup \{\alpha < |h|\}$$

9. We have:

$$\begin{aligned} dx(\{\alpha < |h|\}) &= \alpha^{-1} \int \alpha 1_{\{\alpha < |h|\}} dx \\ &\leq \alpha^{-1} \int |h| 1_{\{\alpha < |h|\}} dx \\ &\leq \alpha^{-1} \int |h| dx \\ &= \alpha^{-1} \|h\|_1 \end{aligned}$$

10. Let $\alpha > 0$ and $\eta > 0$. From 1. we have the existence of $g \in C^c_{\mathbf{C}}(\mathbf{R}^n)$ such that $\|h\|_1 \leq \eta$ where $h = f - g$. Define $M_{\alpha,\eta} =$

$\{\alpha < Mh\} \cup \{\alpha < |h|\}$. From exercise (13) applied to the complex measure $\mu = \int h dx$, Mh is a Borel measurable map. Since $|h|$ is also Borel measurable, we see that $M_{\alpha,\eta} \in \mathcal{B}(\mathbf{R}^n)$. Furthermore from 8. we have $\{2\alpha < Tf\} \subseteq M_{\alpha,\eta}$. Finally, using 9. and exercise (16), we obtain:

$$\begin{aligned} dx(M_{\alpha,\eta}) &= dx(\{\alpha < Mh\} \cup \{\alpha < |h|\}) \\ &\leq dx(\{\alpha < Mh\}) + dx(\{\alpha < |h|\}) \\ &\leq 3^n \alpha^{-1} \|h\|_1 + \alpha^{-1} \|h\|_1 \\ &= (3^n + 1)\alpha^{-1} \|h\|_1 \\ &\leq (3^n + 1)\alpha^{-1} \eta \end{aligned}$$

Hence, given $\alpha > 0$ and $\eta > 0$, we have found $M_{\alpha,\eta} \in \mathcal{B}(\mathbf{R}^n)$ such that $\{2\alpha < Tf\} \subseteq M_{\alpha,\eta}$ and $dx(M_{\alpha,\eta}) \leq (3^n + 1)\alpha^{-1}\eta$. Take $N_{\alpha,\eta} = M_{\alpha,\eta^*}$ where $\eta^* = (3^n + 1)^{-1}\alpha\eta$. Then $N_{\alpha,\eta} \in B(\mathbf{R}^n)$, $\{2\alpha < Tf\} \subseteq N_{\alpha,\eta}$ and $dx(N_{\alpha,\eta}) \leq \eta$, which is exactly what we want.

11. Let $\alpha > 0$. With an obvious change of notation, given $n \geq 1$, from 10. there exists $N_{\alpha,n} \in \mathcal{B}(\mathbf{R}^n)$ such that we have $\{2\alpha < Tf\} \subseteq N_{\alpha,n}$ and $dx(N_{\alpha,n}) \leq 1/n$. Let $N_\alpha = \cap_{n\geq 1} N_{\alpha,n}$. Then $N_\alpha \in \mathcal{B}(\mathbf{R}^n)$, $\{2\alpha < Tf\} \subseteq N_\alpha$ and furthermore for all $n \geq 1$:

$$dx(N_\alpha) = dx(\cap_{n\geq 1} N_{\alpha,n}) \leq dx(N_{\alpha,n}) \leq \frac{1}{n}$$

So $dx(N_\alpha) = 0$.

12. Let $n \geq 1$. With an obvious change of notation, from 11. there exists $N_n \in \mathcal{B}(\mathbf{R}^n)$ such that $\{2/n < Tf\} \subseteq N_n$ together with $dx(N_n) = 0$. Define $N = \cup_{n\geq 1} N_n$. Then $N \in \mathcal{B}(\mathbf{R}^n)$ and $dx(N) = 0$. Furthermore:

$$\begin{aligned} \{Tf > 0\} &= \bigcup_{n\geq 1} \{2/n < Tf\} \\ &\subseteq \bigcup_{n\geq 1} N_n = N \end{aligned}$$

13. From 12. there exists $N \in \mathcal{B}(\mathbf{R}^n)$ with $dx(N) = 0$ such that $\{Tf > 0\} \subseteq N$. Hence, for all $x \in \mathbf{R}^n$, we have $x \in N^c \Rightarrow (Tf)(x) = 0$. We conclude that $Tf = 0$ dx-a.s.

14. Let $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$. Let $x \in \mathbf{R}^n$ and suppose that $(Tf)(x) = 0$. Let $\delta > 0$. Then $(Tf)(x) < \delta$. Since $(Tf)(x)$ is the greatest lower-bound of all $\sup_{u\in]0,\epsilon'[}(T_u f)(x)$'s as $\epsilon' > 0$, δ cannot be such a lower-bound. There exists $\epsilon' > 0$ such that $\sup_{u\in]0,\epsilon'[}(T_u f)(x) < \delta$. Hence for all $\epsilon \in]0, \epsilon'[$, we have:

$$\begin{aligned} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |f(y) - f(x)| dy &= (T_\epsilon f)(x) \\ &\leq \sup_{u\in]0,\epsilon'[} (T_u f)(x) < \delta \end{aligned}$$

We have proved that:

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |f(y) - f(x)| dy = 0$$

i.e. that x is a Lebesgue point of f. So every $x \in \mathbf{R}^n$ such that $(Tf)(x) = 0$ is a Lebesgue point of f. Since $Tf = 0$ dx-almost surely, we conclude that dx-almost all $x \in \mathbf{R}^n$ are Lebesgue points of f. This completes the proof of theorem (101).

Exercise 18

Exercise 19.

1. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\Omega' \in \mathcal{F}$. Let $\mathcal{F}' = \mathcal{F}_{|\Omega'}$ and $\mu' = \mu_{|\mathcal{F}'}$. Let $A \in \mathcal{F}'$. Since $\mathcal{F}'$ is the trace of $\mathcal{F}$ on Ω', from definition (22) there exists $A \in \mathcal{F}$ such that $A' = A \cap \Omega'$. Since $\Omega' \in \mathcal{F}$, we see that $A' \in \mathcal{F}$. This shows that $\mathcal{F}' \subseteq \mathcal{F}$ and the restriction $\mu' = \mu_{|\mathcal{F}'}$ is a well-defined measure on $(\Omega', \mathcal{F}')$.

2. For all maps f defined on Ω' with values in $\mathbf{C}$ or $[0, +\infty]$, we define an extension of f on Ω, denoted $\tilde{f}$, by setting $\tilde{f}(\omega) = 0$ for all $\omega \in \Omega \setminus \Omega'$. Let $A \in \mathcal{F}'$ and $1'_A$ be the indicator function of A on Ω'. A is also a subset of Ω, and we denote 1_A its indicator function on Ω. Let $\omega \in \Omega$. If $\omega \in A \subseteq \Omega'$, then:

$$\tilde{1}'_A(\omega) \stackrel{\triangle}{=} 1'_A(\omega) = 1 = 1_A(\omega)$$

If $\omega \in \Omega' \setminus A$, then:

$$\tilde{1}'_A(\omega) \stackrel{\triangle}{=} 1'_A(\omega) = 0 = 1_A(\omega)$$

if $\omega \in \Omega \setminus \Omega'$, then:

$$\tilde{1}'_A(\omega) \stackrel{\triangle}{=} 0 = 1_A(\omega)$$

In any case we have $\tilde{1}'_A(\omega) = 1_A(\omega)$. So $\tilde{1}'_A = 1_A$.

3. Let $f : (\Omega', \mathcal{F}') \to [0, +\infty]$ be a non-negative and measurable map. For all $B \in \mathcal{B}([0, +\infty])$ we have:

$$\begin{aligned} \{\tilde{f} \in B\} &= (\{\tilde{f} \in B\} \cap \Omega') \uplus (\{\tilde{f} \in B\} \cap (\Omega \setminus \Omega')) \\ &= \{f \in B\} \uplus (\{0 \in B\} \cap (\Omega \setminus \Omega')) \end{aligned}$$

where $\{0 \in B\}$ denotes Ω if $0 \in B$ and $\emptyset$ if $0 \notin B$. Since f is measurable, we have $\{f \in B\} \in \mathcal{F}' \subseteq \mathcal{F}$. Since $\Omega' \in \mathcal{F}$, it is clear that $\{0 \in B\} \cap (\Omega \setminus \Omega') \in \mathcal{F}$. It follows that $\{\tilde{f} \in B\} \in \mathcal{F}$, and we have proved that $\tilde{f}$ is a non-negative and measurable map. Suppose f is of the form $1'_A$ for some $A \in \mathcal{F}'$. Then:

$$\int_{\Omega'} 1'_A d\mu' = \mu'(A) = \mu(A) = \int_{\Omega} 1_A d\mu = \int_{\Omega} \tilde{1}'_A d\mu$$

Suppose now that $f = \sum_{i=1}^n \alpha_i 1'_{A_i}$ is a simple function on $(\Omega', \mathcal{F}')$. To make our proof clearer, let us denote $\phi(g)$ the extension $\tilde{g}$ of any map g defined on Ω'. Then:

$$
\begin{aligned}
\int_{\Omega'} f d\mu' &= \int_{\Omega'} \left(\sum_{i=1}^n \alpha_i 1'_{A_i} \right) d\mu' \\
&= \sum_{i=1}^n \alpha_i \int_{\Omega'} 1'_{A_i} d\mu' \\
&= \sum_{i=1}^n \alpha_i \int_{\Omega} \phi(1'_{A_i}) d\mu \\
&= \int_{\Omega} \left(\sum_{i=1}^n \alpha_i \phi(1'_{A_i}) \right) d\mu \\
&= \int_{\Omega} \phi \left(\sum_{i=1}^n \alpha_i 1'_{A_i} \right) d\mu
\end{aligned}
$$

$$\begin{aligned} &= \int_\Omega \phi(f)d\mu \\ &= \int_\Omega \tilde{f}d\mu \end{aligned}$$

Finally, if $f : (\Omega', \mathcal{F}') \to [0, +\infty]$ is an arbitrary non-negative and measurable map, from theorem (18) there exists a sequence $(s_n)_{n\geq 1}$ of simple functions on $(\Omega', \mathcal{F}')$ such that $s_n \uparrow f$, i.e. for all $\omega \in \Omega'$, $s_n(\omega) \leq s_{n+1}(\omega)$ for all $n \geq 1$, and $s_n(\omega) \to f(\omega)$. It is clear that $\tilde{s_n} \uparrow \tilde{f}$, and from the monotone convergence theorem (19) we obtain:

$$\begin{aligned} \int_{\Omega'} f d\mu' &= \lim_{n\to+\infty} \int_{\Omega'} s_n d\mu' \\ &= \lim_{n\to+\infty} \int_\Omega \tilde{s_n} d\mu \\ &= \int_\Omega \tilde{f} d\mu \end{aligned}$$

4. Let $f \in L^1_{\mathbf{C}}(\Omega', \mathcal{F}', \mu')$. Let $u = Re(f)$ and $v = Im(f)$. To make our proof clearer, we shall denote $\phi(g)$ the extension $\tilde{g}$ of any map g defined on Ω'. From $f = u^+ - u^- + i(v^+ - v^-)$ we obtain $\phi(f) = \phi(u^+) - \phi(u^-) + i(\phi(v^+) - \phi(v^-))$. From 3. each $\phi(u^\pm)$ and $\phi(v^\pm)$ is measurable, and consequently $\phi(f)$ is itself measurable. Note that given $B \in \mathcal{B}(\mathbf{C})$, it is not difficult to show directly that $\{\tilde{f} \in B\} \in \mathcal{F}$ just like we did in 3. with $B \in \mathcal{B}([0, +\infty])$. It is clear that $|\phi(f)| = \phi(|f|)$, and applying 3. to the non-negative and measurable map $|f|$ we obtain:

$$\int_\Omega |\phi(f)| d\mu = \int_\Omega \phi(|f|) d\mu = \int_{\Omega'} |f| d\mu' < +\infty$$

Hence, we have proved that $\tilde{f} = \phi(f) \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Finally, using 3. once more together with the linearity of the integral:

$$\int_{\Omega'} f d\mu' \ = \ \int_{\Omega'} u^+ d\mu' - \int_{\Omega'} u^- d\mu'$$

$$\begin{aligned}
&+ \quad i\left(\int_{\Omega'} v^+ d\mu' - \int_{\Omega'} v^- d\mu'\right) \\
&= \quad \int_\Omega \phi(u^+) d\mu - \int_\Omega \phi(u^-) d\mu \\
&+ \quad i\left(\int_\Omega \phi(v^+) d\mu - \int_\Omega \phi(v^-) d\mu\right) \\
&= \quad \int_\Omega [\phi(u^+) - \phi(u^-) + i(\phi(v^+) - \phi(v^-))] d\mu \\
&= \quad \int_\Omega \phi(f) d\mu = \int_\Omega \tilde{f} d\mu
\end{aligned}$$

Exercise 19

Exercise 20.

1. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map. Suppose b is absolutely continuous. From definition (122), b is right-continuous of finite variation, and furthermore it is absolutely continuous with respect to the right-continuous and non-decreasing map $a : \mathbf{R}^+ \to \mathbf{R}^+$ with $a(0) \geq 0$, defined by $a(t) = t$. From theorem (89), there exists $f \in L^{1,\text{loc}}_{\mathbf{C}}(t)$ such that $b(t) = \int_0^t f(s)ds$ for all $t \in \mathbf{R}^+$. Conversely, suppose such an f exists. From theorem (88), $b = f.a$ is a right-continuous map of finite variation, and from theorem (89), it is in fact absolutely continuous with respect to $a(t) = t$. So b is absolutely continuous. We have proved that b is absolutely continuous, if and only if there exists $f \in L^{1,\text{loc}}_{\mathbf{C}}(t)$ such that $b(t) = \int_0^t f(s)ds$ for all $t \in \mathbf{R}^+$.

2. Suppose b is absolutely continuous and let $f \in L^{1,\text{loc}}_{\mathbf{C}}(t)$ be such that $b(t) = \int_0^t f(s)ds$ for all $t \in \mathbf{R}^+$. From theorem (88), we have $\Delta b = f\Delta t = 0$. Since b is right-continuous of finite varia-

tion, in particular it is cadlag. We conclude from exercise (29) (part 1) of Tutorial 14 that b is in fact continuous with $b(0) = 0$.

Exercise 20

Exercise 21.

1. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be absolutely continuous. Let $f \in L^{1,\text{loc}}_{\mathbf{C}}(t)$ be such that $b(t) = \int_0^t f(s)ds$ for all $t \in \mathbf{R}^+$. For all $n \geq 1$, we define $f_n : \mathbf{R} \to \mathbf{C}$ by:

$$f_n(t) \stackrel{\triangle}{=} \begin{cases} f(t)1_{[0,n]}(t) & \text{if} \quad t \in \mathbf{R}^+ \\ 0 & \text{if} \quad t < 0 \end{cases}$$

Applying exercise (19) to $(\Omega, \Omega') = (\mathbf{R}, \mathbf{R}^+)$, bearing in mind that $\mathcal{B}(\mathbf{R}^+) = \mathcal{B}(\mathbf{R})_{|\mathbf{R}^+}$, we have $f_n = \phi(f1_{[0,n]})$ where $\phi(g)$ denotes the extension $\tilde{g}$ on $\mathbf{R}$, of any map g defined on $\mathbf{R}^+$. Since $f \in L^{1,\text{loc}}_{\mathbf{C}}(t)$, we have $f1_{[0,n]} \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), dx)$ and consequently $f_n = \phi(f1_{[0,n]}) \in L^1_{\mathbf{C}}(\mathbf{R}, \mathcal{B}(\mathbf{R}), dx)$. Note that we are using the same notation dx to denote successively the Lebesgue measure on $\mathbf{R}^+$ and the Lebesgue measure on $\mathbf{R}$, the former being the restriction of the latter to $\mathcal{B}(\mathbf{R}^+) \subseteq \mathcal{B}(\mathbf{R})$. Let

$n \geq 1$ and $t \in [0, n]$. Using exercise (19) once more:

$$\begin{aligned}
\int_0^t f_n dx &= \int_{\mathbf{R}} f_n 1_{[0,t]} dx \\
&= \int_{\mathbf{R}} \phi(f 1_{[0,n]} 1_{[0,t]}) dx \\
&= \int_{\mathbf{R}^+} f 1_{[0,n]} 1_{[0,t]} dx \\
&= \int_{\mathbf{R}^+} f 1_{[0,t]} dx \\
&= \int_0^t f(s) ds = b(t)
\end{aligned}$$

Note that we use the same notations $1_{[0,t]}$ and $1_{[0,n]}$ to denote characteristic functions defined successively on $\mathbf{R}$ and $\mathbf{R}^+$.

2. Since $f_n \in L^1_{\mathbf{C}}(\mathbf{R}, \mathcal{B}(\mathbf{R}), dx)$, from theorem (101), dx-almost every $t \in \mathbf{R}$ is a Lebesgue point of f_n. Hence, there exists

$N_n \in \mathcal{B}(\mathbf{R})$ with $dx(N_n) = 0$ such that for all $t \in N_n^c$, t is a Lebesgue point of f_n.

3. Let $t \in \mathbf{R}$ and $\epsilon > 0$. Since $B(t,\epsilon) =]t-\epsilon, t+\epsilon[$, we have:

$$\begin{aligned}
\frac{1}{\epsilon}\int_t^{t+\epsilon}|f_n(s) - f_n(t)|ds &= \frac{2}{dx(B(t,\epsilon))}\int_t^{t+\epsilon}|f_n(s) - f_n(t)|ds \\
&\leq \frac{2}{dx(B(t,\epsilon))}\int_{t-\epsilon}^{t+\epsilon}|f_n(s) - f_n(t)|ds \\
&= \frac{2}{dx(B(t,\epsilon))}\int_{B(t,\epsilon)}|f_n(s) - f_n(t)|ds
\end{aligned}$$

4. Let $t \in N_n^c$. Then t is a Lebesgue point of f_n. From the inequality obtained in 3. we have:

$$\lim_{\epsilon\downarrow\downarrow 0}\frac{1}{\epsilon}\int_t^{t+\epsilon}|f_n(s) - f_n(t)|ds = 0$$

Furthermore, since:

$$\begin{aligned}\left|\frac{1}{\epsilon}\int_t^{t+\epsilon} f_n(s)ds - f_n(t)\right| &= \frac{1}{\epsilon}\left|\int_t^{t+\epsilon}(f_n(s)-f_n(t))ds\right| \\ &\leq \frac{1}{\epsilon}\int_t^{t+\epsilon}|f_n(s)-f_n(t)|ds\end{aligned}$$

We conclude that:

$$\lim_{\epsilon\downarrow\downarrow 0}\frac{1}{\epsilon}\int_t^{t+\epsilon} f_n(s)ds = f_n(t)$$

5. Similarly to 3. and 4. we have:

$$\begin{aligned}\left|\frac{1}{\epsilon}\int_{t-\epsilon}^{t} f_n(s)ds - f_n(t)\right| &= \frac{1}{\epsilon}\left|\int_{t-\epsilon}^{t}(f_n(s)-f_n(t))ds\right| \\ &\leq \frac{1}{\epsilon}\int_{t-\epsilon}^{t}|f_n(s)-f_n(t)|ds\end{aligned}$$

$$\leq \frac{2}{dx(B(t,\epsilon))} \int_{B(t,\epsilon)} |f_n(s) - f_n(t)| ds$$

Hence for all $t \in N_n^c$, t being a Lebesgue point of f_n:

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{1}{\epsilon} \int_{t-\epsilon}^{t} f_n(s) ds = f_n(t)$$

6. Let $t \in N_n^c \cap [0, n[$. From 1. we have $b(t) = \int_0^t f_n(s) ds$. Furthermore, for $\epsilon > 0$ small enough we have $t + \epsilon \in [0, n]$, and consequently $b(t + \epsilon) = \int_0^{t+\epsilon} f_n(s) ds$. Hence:

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{b(t+\epsilon) - b(t)}{\epsilon} = \lim_{\epsilon \downarrow\downarrow 0} \frac{1}{\epsilon} \int_{t}^{t+\epsilon} f_n(s) ds = f_n(t)$$

Moreover, assuming $t > 0$, $t - \epsilon \in [0, n]$ for $\epsilon > 0$ small enough, and consequently $b(t - \epsilon) = \int_0^{t-\epsilon} f_n(s) ds$. Hence:

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{b(t) - b(t-\epsilon)}{\epsilon} = \lim_{\epsilon \downarrow\downarrow 0} \frac{1}{\epsilon} \int_{t-\epsilon}^{t} f_n(s) ds = f_n(t)$$

We conclude that for all $t \in N_n^c \cap [0,n[$, if $t=0$, the right-hand-side derivative $b'(0)$ exists and is equal to $f_n(0)$. If $t>0$, the derivative $b'(t)$ exists and is equal to $f_n(t)$. However if $t \in [0,n[$, $f_n(t) = f(t)$. So for all $t \in N_n^c \cap [0,n[$, $b'(t) = f(t)$.

7. Define $N = (\cup_{n\geq 1} N_n) \cap \mathbf{R}^+$. Then $N \in \mathcal{B}(\mathbf{R}^+)$ and $dx(N) = 0$. Let $t \in N^c$. Choosing $n \geq 1$ such that $t \in [0,n[$, from $t \notin N$ we obtain $t \notin N_n$ and consequently $t \in N_n^c \cap [0,n[$. From 6. it follows that $b'(t)$ exists and is equal to $f(t)$. We have found $N \in \mathcal{B}(\mathbf{R}^+)$ with $dx(N) = 0$, such that for all $t \in N^c$, $b'(t)$ exists and is equal to $f(t)$.

8. We have shown in exercise (20) that a map b is absolutely continuous, if and only if there exists $f \in L^{1,\text{loc}}_{\mathbf{C}}(t)$ such that $b = f.t$. Furthermore, it follows from 7. that if b is absolutely continuous, it is almost surely differentiable with $b' = f$ dx-almost surely. This completes the proof of theorem (102).

Exercise 21