1. Dynkin systems

Definition 1 *A* **Dynkin system** *on a set* Ω *is a subset* $\mathcal{D}$ *of the power set* $\mathcal{P}(\Omega)$*, with the following properties:*

$$
\begin{array}{ll}
(i) & \Omega \in \mathcal{D} \\
(ii) & A, B \in \mathcal{D}, A \subseteq B \;\Rightarrow\; B \setminus A \in \mathcal{D} \\
(iii) & A_n \in \mathcal{D}, A_n \subseteq A_{n+1}, n \geq 1 \;\Rightarrow\; \bigcup_{n=1}^{+\infty} A_n \in \mathcal{D}
\end{array}
$$

Definition 2 *A* **σ-algebra** *on a set* Ω *is a subset* $\mathcal{F}$ *of the power set* $\mathcal{P}(\Omega)$ *with the following properties:*

$$
\begin{array}{ll}
(i) & \Omega \in \mathcal{F} \\
(ii) & A \in \mathcal{F} \;\Rightarrow\; A^c \stackrel{\triangle}{=} \Omega \setminus A \in \mathcal{F} \\
(iii) & A_n \in \mathcal{F}, n \geq 1 \;\Rightarrow\; \bigcup_{n=1}^{+\infty} A_n \in \mathcal{F}
\end{array}
$$

Exercise 1. Let $\mathcal{F}$ be a σ-algebra on Ω. Show that $\emptyset \in \mathcal{F}$, that if $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$ and also $A \cap B \in \mathcal{F}$. Recall that $B \setminus A = B \cap A^c$ and conclude that $\mathcal{F}$ is also a Dynkin system on Ω.

Exercise 2. Let $(\mathcal{D}_i)_{i\in I}$ be an arbitrary family of Dynkin systems on Ω, with $I \neq \emptyset$. Show that $\mathcal{D} \stackrel{\triangle}{=} \cap_{i\in I}\mathcal{D}_i$ is also a Dynkin system on Ω.

Exercise 3. Let $(\mathcal{F}_i)_{i\in I}$ be an arbitrary family of σ-algebras on Ω, with $I \neq \emptyset$. Show that $\mathcal{F} \stackrel{\triangle}{=} \cap_{i\in I}\mathcal{F}_i$ is also a σ-algebra on Ω.

Exercise 4. Let $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. Define:

$$D(\mathcal{A}) \stackrel{\triangle}{=} \{\mathcal{D} \text{ Dynkin system on } \Omega : \ \mathcal{A} \subseteq \mathcal{D}\}$$

Show that $\mathcal{P}(\Omega)$ is a Dynkin system on Ω, and that $D(\mathcal{A})$ is not empty. Define:

$$\mathcal{D}(\mathcal{A}) \stackrel{\triangle}{=} \bigcap_{\mathcal{D}\in D(\mathcal{A})} \mathcal{D}$$

Show that $\mathcal{D}(\mathcal{A})$ is a Dynkin system on Ω such that $\mathcal{A} \subseteq \mathcal{D}(\mathcal{A})$, and that it is the smallest Dynkin system on Ω with such property, (i.e. if $\mathcal{D}$ is a Dynkin system on Ω with $\mathcal{A} \subseteq \mathcal{D}$, then $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{D}$).

Definition 3 *Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. We call* **Dynkin system generated** *by $\mathcal{A}$, the Dynkin system on Ω, denoted $\mathcal{D}(\mathcal{A})$, equal to the intersection of all Dynkin systems on Ω, which contain $\mathcal{A}$.*

EXERCISE 5. Do exactly as before, but replacing Dynkin systems by σ-algebras.

Definition 4 *Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. We call* **$\sigma$-algebra generated** *by $\mathcal{A}$, the σ-algebra on Ω, denoted $\sigma(\mathcal{A})$, equal to the intersection of all σ-algebras on Ω, which contain $\mathcal{A}$.*

Definition 5 *A subset $\mathcal{A}$ of the power set $\mathcal{P}(\Omega)$ is called a* **π-system** *on Ω, if and only if it is closed under finite intersection, i.e. if it has the property:*

$$A, B \in \mathcal{A} \ \Rightarrow \ A \cap B \in \mathcal{A}$$

Exercise 6. Let $\mathcal{A}$ be a π-system on Ω. For all $A \in \mathcal{D}(\mathcal{A})$, we define:

$$\Gamma(A) \stackrel{\triangle}{=} \{B \in \mathcal{D}(\mathcal{A}) : \ A \cap B \in \mathcal{D}(\mathcal{A})\}$$

1. If $A \in \mathcal{A}$, show that $\mathcal{A} \subseteq \Gamma(A)$
2. Show that for all $A \in \mathcal{D}(\mathcal{A})$, $\Gamma(A)$ is a Dynkin system on Ω.
3. Show that if $A \in \mathcal{A}$, then $\mathcal{D}(\mathcal{A}) \subseteq \Gamma(A)$.
4. Show that if $B \in \mathcal{D}(\mathcal{A})$, then $\mathcal{A} \subseteq \Gamma(B)$.
5. Show that for all $B \in \mathcal{D}(\mathcal{A})$, $\mathcal{D}(\mathcal{A}) \subseteq \Gamma(B)$.
6. Conclude that $\mathcal{D}(\mathcal{A})$ is also a π-system on Ω.

Exercise 7. Let $\mathcal{D}$ be a Dynkin system on Ω which is also a π-system.

1. Show that if $A, B \in \mathcal{D}$ then $A \cup B \in \mathcal{D}$.

2. Let $A_n \in \mathcal{D}, n \geq 1$. Consider $B_n \stackrel{\triangle}{=} \cup_{i=1}^{n} A_i$. Show that $\cup_{n=1}^{+\infty} A_n = \cup_{n=1}^{+\infty} B_n$.

3. Show that $\mathcal{D}$ is a σ-algebra on Ω.

Exercise 8. Let $\mathcal{A}$ be a π-system on Ω. Explain why $\mathcal{D}(\mathcal{A})$ is a σ-algebra on Ω, and $\sigma(\mathcal{A})$ is a Dynkin system on Ω. Conclude that $\mathcal{D}(\mathcal{A}) = \sigma(\mathcal{A})$. Prove the theorem:

Theorem 1 (Dynkin system) *Let $\mathcal{C}$ be a collection of subsets of Ω which is closed under pairwise intersection. If $\mathcal{D}$ is a Dynkin system containing $\mathcal{C}$ then $\mathcal{D}$ also contains the σ-algebra $\sigma(\mathcal{C})$ generated by $\mathcal{C}$.*

Solutions to Exercises

Exercise 1.

1. From (i) of definition (2), $\Omega \in \mathcal{F}$. Hence, from (ii), $\emptyset = \Omega^c \in \mathcal{F}$.

2. If $A, B \in \mathcal{F}$, we can construct a sequence $(A_n)_{n\geq 1}$ of elements of $\mathcal{F}$ by setting $A_1 = A$, $A_2 = B$ and $A_k = \emptyset$ for all $k \geq 3$. Then, using (iii) of definition (2):

$$A \cup B = \bigcup_{n=1}^{+\infty} A_n \in \mathcal{F}$$

3. From (ii), A^c and B^c are also elements of $\mathcal{F}$. So $A^c \cup B^c \in \mathcal{F}$. Finally, again from (ii):

$$A \cap B = (A^c \cup B^c)^c \in \mathcal{F}$$

4. B and A^c are both elements of $\mathcal{F}$, so:

$$B \setminus A \stackrel{\triangle}{=} B \cap A^c \in \mathcal{F}$$

5. $\mathcal{F}$ being a σ-algebra, conditions (i) and (iii) of definition (1) are immediately satisfied. But we have just proved that if $A, B \in \mathcal{F}$, then $B \setminus A \in \mathcal{F}$. Hence, condition (ii) of definition (1) is also satisfied, and $\mathcal{F}$ is therefore a Dynkin system.

Exercise 1

Exercise 2.

1. Each $\mathcal{D}_i$ is a Dynkin system. From (i) of definition (1), $\Omega \in \mathcal{D}_i$. This being true for all $i \in I$, $\Omega \in \cap_{i\in I}\mathcal{D}_i = \mathcal{D}$. Hence (i) of definition (1) is satisfied for $\mathcal{D}$.

2. Let $A, B \in \mathcal{D}$ with $A \subseteq B$. Then for all $i \in I$, $A, B \in \mathcal{D}_i$, with $A \subseteq B$. Since each $\mathcal{D}_i$ is a Dynkin system, from (ii) of definition (1) we see that $B \setminus A \in \mathcal{D}_i$. This being true for all $i \in I$, $B \setminus A \in \cap_{i\in I}\mathcal{D}_i = \mathcal{D}$. Hence, (ii) of definition (1) is satisfied for $\mathcal{D}$.

3. Let $(A_n)_{n\geq 1}$ be a sequence of elements of $\mathcal{D}$ with $A_n \subseteq A_{n+1}$. Then, for all $i \in I$, $(A_n)_{n\geq 1}$ is a sequence of elements of $\mathcal{D}_i$ with $A_n \subseteq A_{n+1}$. Since each $\mathcal{D}_i$ is a Dynkin system, from (iii) of definition (1) we see that $\cup_{n=1}^{+\infty} A_n \in \mathcal{D}_i$. This being true for all $i \in I$, $\cup_{n=1}^{+\infty} A_n \in \cap_{i\in I}\mathcal{D}_i = \mathcal{D}$. Hence, (iii) of definition (1) is satisfied for $\mathcal{D}$.

4. Having checked $(i), (ii), (iii)$ of definition (1), we conclude that $\mathcal{D}$ is indeed a Dynkin system on Ω.

Exercise 2

Exercise 3.

1. Each $\mathcal{F}_i$ is a σ-algebra. From (i) of definition (2), $\Omega \in \mathcal{F}_i$. This being true for all $i \in I$, $\Omega \in \cap_{i \in I} \mathcal{F}_i = \mathcal{F}$. Hence (i) of definition (2) is satisfied for $\mathcal{F}$.
2. Let $A \in \mathcal{F}$. Then for all $i \in I$, $A \in \mathcal{F}_i$. Since each $\mathcal{F}_i$ is a σ-algebra, from (ii) of definition (2) we see that $A^c \in \mathcal{F}_i$. This being true for all $i \in I$, $A^c \in \cap_{i \in I} \mathcal{F}_i = \mathcal{F}$. Hence, (ii) of definition (2) is satisfied for $\mathcal{F}$.
3. Let $(A_n)_{n \geq 1}$ be a sequence of elements of $\mathcal{F}$. Then, for all $i \in I$, $(A_n)_{n \geq 1}$ is a sequence of elements of $\mathcal{F}_i$. Since each $\mathcal{F}_i$ is a σ-algebra, from (iii) of definition (2) we see that $\cup_{n=1}^{+\infty} A_n \in \mathcal{F}_i$. This being true for all $i \in I$, $\cup_{n=1}^{+\infty} A_n \in \cap_{i \in I} \mathcal{F}_i = \mathcal{F}$. Hence, (iii) of definition (2) is satisfied for $\mathcal{F}$.
4. Having checked $(i), (ii), (iii)$ of definition (2), we conclude that $\mathcal{F}$ is indeed a σ-algebra on Ω.

Exercise 3

Exercise 4.

1. Ω is obviously a subset of Ω, so $\Omega \in \mathcal{P}(\Omega)$, and (i) of definition (1) is satisfied for $\mathcal{P}(\Omega)$. If $A, B \in \mathcal{P}(\Omega)$, whether or not $A \subseteq B$, $B \setminus A$ is still a subset of Ω, i.e. $B \setminus A \in \mathcal{P}(\Omega)$. So (ii) of definition (1) is also satisfied for $\mathcal{P}(\Omega)$. If $(A_n)_{n\geq 1}$ is a sequence of subsets of Ω, whether or not this sequence is increasing (i.e. $A_n \subseteq A_{n+1}$), $\cup_{n=1}^{+\infty} A_n$ is still a subset of Ω, i.e. belongs to $\mathcal{P}(\Omega)$. So (iii) of definition (1) is satisfied for $\mathcal{P}(\Omega)$, and finally, $\mathcal{P}(\Omega)$ is a Dynkin system on Ω.

2. By assumption, $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. Since $\mathcal{P}(\Omega)$ is also a Dynkin system on Ω, we see that $\mathcal{P}(\Omega) \in D(\mathcal{A})$. In particular, $D(\mathcal{A})$ is not empty.

3. Take $I = D(\mathcal{A})$, and for all $i \in I$, define $\mathcal{D}_i = i$. Then $(\mathcal{D}_i)_{i\in I}$ is a family of Dynkin systems on Ω (where $I \neq \emptyset$) and since:

$$\mathcal{D}(\mathcal{A}) \stackrel{\triangle}{=} \bigcap_{\mathcal{D}\in D(\mathcal{A})} \mathcal{D} = \bigcap_{i\in I} \mathcal{D}_i$$

using exercise (2), we conclude that $\mathcal{D}(\mathcal{A})$ is a Dynkin system on Ω.

4. Let $A \in \mathcal{A}$. For all $\mathcal{D} \in D(\mathcal{A})$, we have $\mathcal{A} \subseteq \mathcal{D}$. Hence, for all $\mathcal{D} \in D(\mathcal{A})$, $A \in \mathcal{D}$. So:

$$A \in \bigcap_{\mathcal{D} \in D(\mathcal{A})} \mathcal{D} \stackrel{\triangle}{=} \mathcal{D}(\mathcal{A})$$

It follows that $\mathcal{A} \subseteq \mathcal{D}(\mathcal{A})$.

5. Suppose $\mathcal{D}$ is another Dynkin system on Ω such that $\mathcal{A} \subseteq \mathcal{D}$. Then $\mathcal{D} \in D(\mathcal{A})$, from which we conclude that:

$$\mathcal{D}(\mathcal{A}) \stackrel{\triangle}{=} \bigcap_{\mathcal{D}' \in D(\mathcal{A})} \mathcal{D}' \subseteq \mathcal{D}$$

Exercise 4

Exercise 5.

1. We define similarly: $F(\mathcal{A}) \stackrel{\triangle}{=} \{\mathcal{F}\ \sigma\text{-algebra on } \Omega :\ \mathcal{A} \subseteq \mathcal{F}\}$ and:
$$\sigma(\mathcal{A}) \stackrel{\triangle}{=} \bigcap_{\mathcal{F} \in F(\mathcal{A})} \mathcal{F}$$

2. $\Omega \in \mathcal{P}(\Omega)$, and (i) of definition (2) is satisfied for $\mathcal{P}(\Omega)$. If $A \in \mathcal{P}(\Omega)$, then $A^c \in \mathcal{P}(\Omega)$ and (ii) of definition (2) is also satisfied for $\mathcal{P}(\Omega)$. If $(A_n)_{n\geq 1}$ is a sequence of subsets of Ω, $\cup_{n=1}^{+\infty} A_n$ is still a subset of Ω, and (iii) of definition (2) is satisfied for $\mathcal{P}(\Omega)$. Finally, $\mathcal{P}(\Omega)$ is a σ-algebra on Ω.

3. By assumption, $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. Since $\mathcal{P}(\Omega)$ is also a σ-algebra on Ω, we see that $\mathcal{P}(\Omega) \in F(\mathcal{A})$. In particular, $F(\mathcal{A})$ is not empty.

4. Take $I = F(\mathcal{A})$, and for all $i \in I$, define $\mathcal{F}_i = i$. Then $(\mathcal{F}_i)_{i\in I}$

is a family of σ-algebra on Ω (where $I \neq \emptyset$) and since:

$$\sigma(\mathcal{A}) \stackrel{\triangle}{=} \bigcap_{\mathcal{F} \in F(\mathcal{A})} \mathcal{F} = \bigcap_{i \in I} \mathcal{F}_i$$

using exercise (3), we conclude that $\sigma(\mathcal{A})$ is a σ-algebra on Ω.

5. Let $A \in \mathcal{A}$. For all $\mathcal{F} \in F(\mathcal{A})$, we have $\mathcal{A} \subseteq \mathcal{F}$. Hence, for all $\mathcal{F} \in F(\mathcal{A})$, $A \in \mathcal{F}$. So:

$$A \in \bigcap_{\mathcal{F} \in F(\mathcal{A})} \mathcal{F} \stackrel{\triangle}{=} \sigma(\mathcal{A})$$

It follows that $\mathcal{A} \subseteq \sigma(\mathcal{A})$.

6. Suppose $\mathcal{F}$ is another σ-algebra on Ω such that $\mathcal{A} \subseteq \mathcal{F}$. Then $\mathcal{F} \in F(\mathcal{A})$, from which we conclude that:

$$\sigma(\mathcal{A}) \stackrel{\triangle}{=} \bigcap_{\mathcal{F}' \in F(\mathcal{A})} \mathcal{F}' \subseteq \mathcal{F}$$

Exercise 5

Exercise 6.

1. Suppose $A \in \mathcal{A}$, and let $B \in \mathcal{A}$. $\mathcal{D}(\mathcal{A})$ being the Dynkin system generated by $\mathcal{A}$, $\mathcal{A} \subseteq \mathcal{D}(\mathcal{A})$ (see exercise (4)). In particular, $B \in \mathcal{D}(\mathcal{A})$. Since both A, B lie in $\mathcal{A}$, $\mathcal{A}$ being a π-system, we have $A \cap B \in \mathcal{A} \subseteq \mathcal{D}(\mathcal{A})$. Hence, we see that $B \in \Gamma(A)$. We have proved that for all $A \in \mathcal{A}$, $\mathcal{A} \subseteq \Gamma(A)$.

2. Let $A \in \mathcal{D}(\mathcal{A})$. $\mathcal{D}(\mathcal{A})$ being a Dynkin system, $\Omega \in \mathcal{D}(\mathcal{A})$. Moreover, $A \cap \Omega = A \in \mathcal{D}(\mathcal{A})$. So $\Omega \in \Gamma(A)$, and condition (i) of definition (1) is satisfied for $\Gamma(A)$. Let $B, C \in \Gamma(A)$ with $B \subseteq C$. Both B, C belong to $\mathcal{D}(\mathcal{A})$, with $B \subseteq C$. $\mathcal{D}(\mathcal{A})$ being a Dynkin system on Ω, $C \setminus B \in \mathcal{D}(\mathcal{A})$. Also:

$$A \cap (C \setminus B) = (A \cap C) \setminus (A \cap B)$$

Since $B, C \in \Gamma(A)$, both $A \cap B$ and $A \cap C$ belong to $\mathcal{D}(\mathcal{A})$, with $A \cap B \subseteq A \cap C$. It follows that $(A \cap C) \setminus (A \cap B) \in \mathcal{D}(\mathcal{A})$, i.e. $A \cap (C \setminus B) \in \mathcal{D}(\mathcal{A})$. Hence, we see that $C \setminus B \in \Gamma(A)$, and (ii) of definition (1) is satisfied for $\Gamma(A)$. Let $(B_n)_{n \geq 1}$ be a sequence of elements of $\Gamma(A)$, with $B_n \subseteq B_{n+1}$. Let $B = \cup_{n=1}^{+\infty} B_n$. Then $(B_n)_{n\geq 1}$ is a sequence of elements of $\mathcal{D}(\mathcal{A})$ with $B_n \subseteq B_{n+1}$. $\mathcal{D}(\mathcal{A})$ being a Dynkin system, we see that $B \in \mathcal{D}(\mathcal{A})$. Moreover, $(A \cap B_n)_{n\geq 1}$ is a sequence of elements of $\mathcal{D}(\mathcal{A})$, with $A \cap B_n \subseteq A \cap B_{n+1}$. It follows that $\cup_{n=1}^{+\infty} A \cap B_n = A \cap B \in \mathcal{D}(\mathcal{A})$. Hence we see that $B \in \Gamma(A)$, and condition (iii) of definition (1) is satisfied for $\Gamma(A)$. We have proved that for all $A \in \mathcal{D}(\mathcal{A})$, $\Gamma(A)$ is a Dynkin system on Ω.

3. If $A \in \mathcal{A}$, we saw in 1. that $\mathcal{A} \subseteq \Gamma(A)$. But $\Gamma(A)$ being a Dynkin system on Ω, using exercise (4), we conclude that $\mathcal{D}(\mathcal{A}) \subseteq \Gamma(A)$.

4. From the previous point, it follows that if $A \in \mathcal{A}$ and $B \in \mathcal{D}(\mathcal{A})$, then $A \cap B \in \mathcal{D}(\mathcal{A})$. In other words, if $B \in \mathcal{D}(\mathcal{A})$ and $A \in \mathcal{A}$, then $A \in \Gamma(B)$. We have proved that for all $B \in \mathcal{D}(\mathcal{A})$, we have $\mathcal{A} \subseteq \Gamma(B)$.

5. From 2., we know that $\Gamma(B)$ is a Dynkin system on Ω. Using exercise (4), it follows from the previous point that $\mathcal{D}(\mathcal{A}) \subseteq$

$\Gamma(B)$, for all $B \in \mathcal{D}(\mathcal{A})$.

6. Another way of writing the previous property, is that for all $B \in \mathcal{D}(\mathcal{A})$, and $A \in \mathcal{D}(\mathcal{A})$, we have $A \in \Gamma(B)$, i.e. $A \cap B \in \mathcal{D}(\mathcal{A})$. Hence, we see that $\mathcal{D}(\mathcal{A})$ is *closed under finite intersection*, i.e. it is a π-system on Ω. The purpose of this exercise is to show that whenever $\mathcal{A}$ is a π-system on Ω, its *generated Dynkin system* $\mathcal{D}(\mathcal{A})$ is also a π-system on Ω.

Exercise 6

Exercise 7.

1. Let $A \in \mathcal{D}$. Since $\mathcal{D}$ is a Dynkin system on Ω, $\Omega \in \mathcal{D}$. We obviously have $A \subseteq \Omega$. it follows that $\Omega \setminus A \in \mathcal{D}$, i.e. $A^c \in \mathcal{D}$. Hence we see that $\mathcal{D}$ is *closed under complementation*. Now, if $A, B \in \mathcal{D}$, then $A^c, B^c \in \mathcal{D}$. Since $\mathcal{D}$ is also assumed to be a π-system on Ω, we have $A^c \cap B^c \in \mathcal{D}$, and finally:
$$A \cup B = (A^c \cap B^c)^c \in \mathcal{D}$$

2. Let $(A_n)_{n\geq 1}$ be a sequence of elements of $\mathcal{D}$. Having defined $B_n = \cup_{i=1}^n A_i$ for all $n \geq 1$, we put $A = \cup_{n=1}^{+\infty} A_n$ and $B = \cup_{n=1}^{+\infty} B_n$. Let $x \in A$. There exists $n \geq 1$ such that $x \in A_n \subseteq B_n$. So $x \in B$, and $A \subseteq B$. Let $x \in B$. There exists $n \geq 1$ such that $x \in B_n = \cup_{i=1}^n A_i$. Hence, there exists $i \in \{1, \ldots, n\}$ such that $x \in A_i$. So $x \in A$ and $B \subseteq A$. We have proved that $A = B$.

3. $\mathcal{D}$ being a Dynkin system, $\Omega \in \mathcal{D}$ and condition (i) of definition (2) is satisfied for $\mathcal{D}$. If $A \in \mathcal{D}$, we saw in 1. that $A^c \in \mathcal{D}$. So

condition (ii) of definition (2) is also satisfied for $\mathcal{D}$. Let $(A_n)_{n\geq 1}$ be a sequence of elements of $\mathcal{D}$. Having defined $B_n = \cup_{i=1}^{n} A_i$ for all $n \geq 1$, we saw in 1. that $\mathcal{D}$ was *closed under finite union*, i.e. B_n is an element of $\mathcal{D}$ for all $n \geq 1$. Moreover, $B_n \subseteq B_{n+1}$ for all $n \geq 1$. $\mathcal{D}$ being a Dynkin system, $\cup_{n=1}^{+\infty} B_n$ is an element of $\mathcal{D}$. But from 2., $\cup_{n=1}^{+\infty} B_n = \cup_{n=1}^{+\infty} A_n$. Hence, we see that $\cup_{n=1}^{+\infty} A_n$ is also an element of $\mathcal{D}$, and condition (iii) of definition (2) is satisfied for $\mathcal{D}$. We have proved that $\mathcal{D}$ is indeed a σ-algebra on Ω. The purpose of this exercise is to show that whenever a Dynkin system $\mathcal{D}$ is also a π-system, then it is in fact a σ-algebra on Ω.

Exercise 7

Exercise 8.

1. From exercise (6), we know that since $\mathcal{A}$ is a π-system on Ω, its generated Dynkin system $\mathcal{D}(\mathcal{A})$ is also a π-system on Ω. However, from exercise (7), we know that any Dynkin system which is also a π-system, is in fact a σ-algebra on Ω. Hence, $\mathcal{D}(\mathcal{A})$ is a σ-algebra on Ω.

2. The σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$ being a σ-algebra, it is also a Dynkin system on Ω (see exercise (1)).

3. From $\mathcal{A} \subseteq \sigma(\mathcal{A})$ and the fact that $\sigma(\mathcal{A})$ is a Dynkin system on Ω, we conclude that $\mathcal{D}(\mathcal{A}) \subseteq \sigma(\mathcal{A})$ (see exercise (4)). From $\mathcal{A} \subseteq \mathcal{D}(\mathcal{A})$ and the fact that $\mathcal{D}(\mathcal{A})$ is also a σ-algebra on Ω, we conclude that $\sigma(\mathcal{A}) \subseteq \mathcal{D}(\mathcal{A})$ (see exercise (5)). Finally, $\sigma(\mathcal{A}) = \mathcal{D}(\mathcal{A})$. The purpose of this exercise is to show that for any π-system $\mathcal{A}$ on Ω, its generated σ-algebra $\sigma(\mathcal{A})$ and Dynkin system $\mathcal{D}(\mathcal{A})$ coincide.

4. If $\mathcal{C}$ is a π-system and $\mathcal{D}$ is a Dynkin system with $\mathcal{C} \subseteq \mathcal{D}$, then

$\mathcal{D}(\mathcal{C}) \subseteq \mathcal{D}$ (see exercise (4)). But we have just seen that $\mathcal{C}$ being a π-system, $\mathcal{D}(\mathcal{C}) = \sigma(\mathcal{C})$. Hence $\sigma(\mathcal{C}) \subseteq \mathcal{D}$, which proves theorem (1).

Exercise 8