# 14. Maps of Finite Variation

**Definition 108** *We call* **total variation** *of a map* $b : \mathbf{R}^+ \to \mathbf{C}$ *the map* $|b| : \mathbf{R}^+ \to [0, +\infty]$ *defined as:*

$$\forall t \in \mathbf{R}^+ \ , \ |b|(t) \stackrel{\triangle}{=} |b(0)| + \sup \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$$

*where the 'sup' is taken over all finite* $t_0 \leq \ldots \leq t_n$ *in* $[0, t]$, $n \geq 1$. *We say that* $b$ *is of* **finite variation**, *if and only if:*

$$\forall t \in \mathbf{R}^+ \ , \ |b|(t) < +\infty$$

*We say that* $b$ *is of* **bounded variation**, *if and only if:*

$$\sup_{t \in \mathbf{R}^+} |b|(t) < +\infty$$

Warning: The notation $|b|$ can be misleading: it can refer to the map $t \to |b(t)|$(the modulus), or to the map $t \to |b|(t)$ (the total variation).

Exercise 1. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be non-decreasing with $a(0) \geq 0$.

1. Show that $|a| = a$ and $a$ is of finite variation.

2. Show that the limit $\lim_{t\uparrow+\infty} a(t)$, denoted $a(\infty)$, exists in $\bar{\mathbf{R}}$.

3. Show that $a$ is of bounded variation if and only if $a(\infty) < +\infty$.

EXERCISE 2. Let $b = b_1 + ib_2 : \mathbf{R}^+ \to \mathbf{C}$ be a map, $b_1, b_2$ real-valued.

1. Show that $|b_1| \leq |b|$ and $|b_2| \leq |b|$.

2. Show that $|b| \leq |b_1| + |b_2|$.

3. Show that $b$ is of finite variation if and only if $b_1$, $b_2$ are.

4. Show that $b$ is of bounded variation if and only if $b_1$, $b_2$ are.

5. Show that $|b|(0) = |b(0)|$.

Exercise 3. Let $b : \mathbf{R}^+ \to \mathbf{R}$ be differentiable, such that $b'$ is bounded on each compact interval of $\mathbf{R}^+$. Show that $b$ is of finite variation.

Exercise 4. Show that if $b : \mathbf{R}^+ \to \mathbf{C}$ is of class $C^1$, i.e. continuous and differentiable with continuous derivative, then $b$ is of finite variation.

Exercise 5. Let $f : (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ be a measurable map, with $\int_0^t |f(s)|ds < +\infty$ for all $t \in \mathbf{R}^+$. Let $b : \mathbf{R}^+ \to \mathbf{C}$ defined by:

$$\forall t \in \mathbf{R}^+ \ , \ b(t) \stackrel{\triangle}{=} \int_{\mathbf{R}^+} f 1_{[0,t]} ds$$

1. Show that $b$ is of finite variation and:

$$\forall t \in \mathbf{R}^+ \ , \ |b|(t) \leq \int_0^t |f(s)| ds$$

2. Show that $f \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), ds) \Rightarrow b$ is of bounded variation.

Exercise 6. Show that if $b, b' : \mathbf{R}^+ \to \mathbf{C}$ are maps of finite variation, and $\alpha \in \mathbf{C}$, then $b + \alpha b'$ is also a map of finite variation. Prove the same result when the word 'finite' is replaced by 'bounded'.

Exercise 7. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map. For all $t \in \mathbf{R}^+$, let $\mathcal{S}(t)$ be the set of all finite subsets $A$ of $[0, t]$, with $\mathrm{card} A \geq 2$. For all $A \in \mathcal{S}(t)$, we define:

$$S(A) \stackrel{\triangle}{=} \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$$

where it is understood that $t_0, \ldots, t_n$ are such that:

$$t_0 < t_1 < \ldots < t_n \text{ and } A = \{t_0, \ldots, t_n\} \subseteq [0, t]$$

1. Show that for all $t \in \mathbf{R}^+$, if $s_0 \leq \ldots \leq s_p$ $(p \geq 1)$ is a finite

sequence in $[0, t]$, then if:

$$S \stackrel{\triangle}{=} \sum_{j=1}^{p} |b(s_j) - b(s_{j-1})|$$

either $S = 0$ or $S = S(A)$ for some $A \in \mathcal{S}(t)$.

2. Conclude that:

$$\forall t \in \mathbf{R}^+ \ , \ |b|(t) = |b(0)| + \sup\{S(A) : \ A \in \mathcal{S}(t)\}$$

3. Let $A \in \mathcal{S}(t)$ and $s \in [0, t]$. Show that $S(A) \leq S(A \cup \{s\})$.

4. Let $A, B \in \mathcal{S}(t)$. Show that:

$$A \subseteq B \ \Rightarrow \ S(A) \leq S(B)$$

5. Show that if $t_0 \leq \ldots \leq t_n$, $n \geq 1$, and $s_0 \leq \ldots \leq s_p$, $p \geq 1$, are finite sequences in $[0, t]$ such that:

$$\{t_0, \ldots, t_n\} \subseteq \{s_0, \ldots, s_p\}$$

then:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq \sum_{j=1}^{p} |b(s_j) - b(s_{j-1})|$$

Exercise 8. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be of finite variation. Let $s, t \in \mathbf{R}^+$, with $s \leq t$. We define:

$$\delta \stackrel{\triangle}{=} \sup \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$$

where the 'sup' is taken over all finite $t_0 \leq \ldots \leq t_n$, $n \geq 1$, in $[s, t]$.

1. Let $s_0 \leq \ldots \leq s_p$ and $t_0 \leq \ldots \leq t_n$ be finite sequences in $[0, s]$ and $[s, t]$ respectively, where $n, p \geq 1$. Show that:

$$\sum_{j=1}^{p} |b(s_j) - b(s_{j-1})| + \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(t) - |b(0)|$$

2. Show that $\delta \leq |b|(t) - |b|(s)$.

3. Let $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[0, t]$, where $n \geq 1$, and suppose that $s = t_j$ for some $0 < j < n$. Show that:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(s) - |b(0)| + \delta \tag{1}$$

4. Show that inequality (1) holds, for all $t_0 \leq \ldots \leq t_n$ in $[0, t]$.

5. Prove the following:

**Theorem 80** *Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map of finite variation. Then, for all $s, t \in \mathbf{R}^+$, $s \leq t$, we have:*

$$|b|(t) - |b|(s) = \sup \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$$

*where the 'sup' is taken over all finite $t_0 \leq \ldots \leq t_n$, $n \geq 1$, in $[s, t]$.*

**Exercise 9.** Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map of finite variation. Show that $|b|$ is non-decreasing with $|b|(0) \geq 0$, and $||b|| = |b|$.

**Definition 109** *Let $b : \mathbf{R}^+ \to \mathbf{R}$ be a map of finite variation. Let:*

$$|b|^+ \stackrel{\triangle}{=} \frac{1}{2}(|b| + b)$$

$$|b|^- \stackrel{\triangle}{=} \frac{1}{2}(|b| - b)$$

*$|b|^+$, $|b|^-$ are respectively the* **positive**, **negative variation** *of $b$.*

**Exercise 10.** Let $b : \mathbf{R}^+ \to \mathbf{R}$ be a map of finite variation.

1. Show that $|b| = |b|^+ + |b|^-$ and $b = |b|^+ - |b|^-$.

2. Show that $|b|^+(0) = b^+(0) \geq 0$ and $|b|^-(0) = b^-(0) \geq 0$.

3. Show that for all $s, t \in \mathbf{R}^+$, $s \leq t$, we have:

$$|b(t) - b(s)| \leq |b|(t) - |b|(s)$$

4. Show that $|b|^+$ and $|b|^-$ are non-decreasing.

Exercise 11. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be of finite variation. Show the existence of $b_1, b_2, b_3, b_4 : \mathbf{R}^+ \to \mathbf{R}^+$, non-decreasing with $b_i(0) \geq 0$, such that $b = b_1 - b_2 + i(b_3 - b_4)$. Show conversely that if $b : \mathbf{R}^+ \to \mathbf{C}$ is a map with such decomposition, then it is of finite variation.

Exercise 12. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of finite variation, and $x_0 \in \mathbf{R}^+$.

1. Show that the limit $|b|(x_0+) = \lim_{t\downarrow\downarrow x_0} |b|(t)$ exists in $\mathbf{R}$ and is equal to $\inf_{x_0<t} |b|(t)$.

2. Show that $|b|(x_0) \leq |b|(x_0+)$.

3. Given $\epsilon > 0$, show the existence of $y_0 \in \mathbf{R}^+$, $x_0 < y_0$, such that:

$$u \in ]x_0, y_0] \quad \Rightarrow \quad |b(u) - b(x_0)| \leq \epsilon/2$$

$$u \in ]x_0, y_0] \quad \Rightarrow \quad |b|(y_0) - |b|(u) \leq \epsilon/2$$

Exercise 13. Further to exercise (12), let $t_0 \leq \ldots \leq t_n$, $n \geq 1$, be a finite sequence in $[0, y_0]$, for which there exists $j$ with $0 < j < n-1$, $x_0 = t_j$ and $x_0 < t_{j+1}$.

1. Show that $\sum_{i=1}^{j} |b(t_i) - b(t_{i-1})| \leq |b|(x_0) - |b(0)|$.
2. Show that $|b(t_{j+1}) - b(t_j)| \leq \epsilon/2$.
3. Show that $\sum_{i=j+2}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(y_0) - |b|(t_{j+1}) \leq \epsilon/2$.
4. Show that for all finite sequences $t_0 \leq \ldots \leq t_n$, $n \geq 1$, in $[0, y_0]$:
$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(x_0) - |b(0)| + \epsilon$$
5. Show that $|b|(y_0) \leq |b|(x_0) + \epsilon$.
6. Show that $|b|(x_0+) \leq |b|(x_0)$ and that $|b|$ is right-continuous.

**Exercise 14.** Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a left-continuous map of finite variation, and let $x_0 \in \mathbf{R}^+ \setminus \{0\}$.

1. Show that the limit $|b|(x_0-) = \lim_{t\uparrow\uparrow x_0} |b|(t)$ exists in $\mathbf{R}$, and is equal to $\sup_{t<x_0} |b|(t)$.

2. Show that $|b|(x_0-) \leq |b|(x_0)$.

3. Given $\epsilon > 0$, show the existence of $y_0 \in [0, x_0[$, such that:

$$u \in [y_0, x_0[ \quad \Rightarrow \quad |b(x_0) - b(u)| \leq \epsilon/2$$
$$u \in [y_0, x_0[ \quad \Rightarrow \quad |b|(u) - |b|(y_0) \leq \epsilon/2$$

**Exercise 15.** Further to exercise (14), let $t_0 \leq \ldots \leq t_n$, $n \geq 1$, be a finite sequence in $[0, x_0]$, such that $y_0 = t_j$ for some $0 < j < n-1$, and $x_0 = t_n$. We denote $k = \max\{i : \ j \leq i \ , \ t_i < x_0\}$.

1. Show that $j \leq k \leq n-1$ and $t_k \in [y_0, x_0[$.

2. Show that $\sum_{i=1}^{j} |b(t_i) - b(t_{i-1})| \leq |b|(y_0) - |b(0)|$.

3. Show that $\sum_{i=j+1}^{k} |b(t_i) - b(t_{i-1})| \leq |b|(t_k) - |b|(y_0) \leq \epsilon/2$, where if $j = k$, the corresponding sum is zero.

4. Show that $\sum_{i=k+1}^{n} |b(t_i) - b(t_{i-1})| = |b(x_0) - b(t_k)| \leq \epsilon/2$.

5. Show that for all finite sequences $t_0 \leq \ldots \leq t_n$, $n \geq 1$, in $[0, x_0]$:
$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(y_0) - |b(0)| + \epsilon$$

6. Show that $|b|(x_0) \leq |b|(y_0) + \epsilon$.

7. Show that $|b|(x_0) \leq |b|(x_0-)$ and that $|b|$ is left-continuous.

8. Prove the following:

**Theorem 81** *Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map of finite variation. Then:*

$$\begin{array}{rcl} b \text{ is right-continuous} & \Rightarrow & |b| \text{ is right-continuous} \\ b \text{ is left-continuous} & \Rightarrow & |b| \text{ is left-continuous} \\ b \text{ is continuous} & \Rightarrow & |b| \text{ is continuous} \end{array}$$

Exercise 16. Let $b : \mathbf{R}^+ \to \mathbf{R}$ be an $\mathbf{R}$-valued map of finite variation.

1. Show that if $b$ is right-continuous, then so are $|b|^+$ and $|b|^-$.
2. State and prove similar results for left-continuity and continuity.

Exercise 17. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of finite variation. Show the existence of $b_1, b_2, b_3, b_4 : \mathbf{R}^+ \to \mathbf{R}^+$, right-continuous and non-decreasing maps with $b_i(0) \geq 0$, such that:

$$b = b_1 - b_2 + i(b_3 - b_4)$$

**Exercise 18.** Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map. Let $t \in \mathbf{R}^+$. For all $p \geq 1$, we define:

$$S_p \stackrel{\triangle}{=} |b(0)| + \sum_{k=1}^{2^p} |b(kt/2^p) - b((k-1)t/2^p)|$$

1. Show that for all $p \geq 1$, $S_p \leq S_{p+1}$ and define $S = \sup_{p\geq 1} S_p$.

2. Show that $S \leq |b|(t)$.

**Exercise 19.** Further to exercise (18), let $t_0 < \ldots < t_n$ be a finite sequence of distinct elements of $[0, t]$. Let $\epsilon > 0$.

1. Show that for all $i = 0, \ldots, n$, there exists $p_i \geq 1$ and $q_i \in \{0, 1, \ldots, 2^{p_i}\}$ such that:

$$0 \leq t_0 \leq \frac{q_0 t}{2^{p_0}} < t_1 \leq \frac{q_1 t}{2^{p_1}} < \ldots < t_n \leq \frac{q_n t}{2^{p_n}} \leq t$$

and:

$$|b(t_i) - b(q_i t/2^{p_i})| \leq \epsilon \ , \ \forall i = 0, \ldots, n$$

2. Show the existence of $p \geq 1$, and $k_0, \ldots, k_n \in \{0, \ldots, 2^p\}$ with:

$$0 \leq t_0 \leq \frac{k_0 t}{2^p} < t_1 \leq \frac{k_1 t}{2^p} < \ldots < t_n \leq \frac{k_n t}{2^p} \leq t$$

and:

$$|b(t_i) - b(k_i t/2^p)| \leq \epsilon \ , \ \forall i = 0, \ldots, n$$

3. Show that:

$$\sum_{i=1}^{n} |b(k_i t/2^p) - b(k_{i-1} t/2^p)| \leq S_p - |b(0)|$$

4. Show that:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq S - |b(0)| + 2n\epsilon$$

5. Show that:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq S - |b(0)|$$

6. Conclude that $|b|(t) \leq S$.

7. Prove the following:

**Theorem 82** *Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous or left-continuous. Then, for all $t \in \mathbf{R}^+$:*

$$|b|(t) = |b(0)| + \lim_{n \to +\infty} \sum_{k=1}^{2^n} |b(kt/2^n) - b((k-1)t/2^n)|$$

Exercise 20. Let $b : \mathbf{R}^+ \to \mathbf{R}^+$ be defined by $b = 1_{\mathbf{Q}^+}$. Show that:

$$+\infty = |b|(1) \neq \lim_{n \to +\infty} \sum_{k=1}^{2^n} |b(k/2^n) - b((k-1)/2^n)| = 0$$

EXERCISE 21. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of bounded variation.

1. Let $b_1 = Re(b)$ and $b_2 = Im(b)$. Explain why $d|b_1|^+$, $d|b_1|^-$, $d|b_2|^+$ and $d|b_2|^-$ are all well-defined measures on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$.

2. Is this still true, if $b$ is right-continuous of finite variation?

3. Show that $d|b_1|^+, d|b_1|^-, d|b_2|^+$ and $d|b_2|^-$ are finite measures.

4. Let $db = d|b_1|^+ - d|b_1|^- + i(d|b_2|^+ - d|b_2|^-)$. Show that $db$ is a well-defined complex measure on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$.

5. Show that $db(\{0\}) = b(0)$.

6. Show that for all $s, t \in \mathbf{R}^+$, $s \leq t$, $db(]s,t]) = b(t) - b(s)$.

7. Show that $\lim_{t \to +\infty} b(t)$ exists in $\mathbf{C}$. We denote $b(\infty)$ this limit.

8. Show that $db(\mathbf{R}^+) = b(\infty)$.

9. Proving the uniqueness of $db$, justify the following:

**Definition 110** *Let* $b : \mathbf{R}^+ \to \mathbf{C}$ *be a right-continuous map of bounded variation. There exists a unique complex measure* $db$ *on* $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$*, such that:*

$$(i) \qquad db(\{0\}) = b(0)$$

$$(ii) \qquad \forall s, t \in \mathbf{R}^+ \; s \leq t \;, \; db(]s, t]) = b(t) - b(s)$$

$db$ *is called the* **complex Stieltjes measure** *associated with* $b$*.*

Exercise 22. Show that if $a : \mathbf{R}^+ \to \mathbf{R}^+$ is right-continuous, non-decreasing with $a(0) \geq 0$ and $a(\infty) < +\infty$, then definition (110) of $da$ coincides with the already known definition (24).

Exercise 23. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation.

1. Let $b_1 = Re(b)$ and $b_2 = Im(b)$. Explain why $d|b_1|^+$, $d|b_1|^-$, $d|b_2|^+$ and $d|b_2|^-$ are all well-defined measures on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$.

2. Why is it in general impossible to define:

$$db \stackrel{\triangle}{=} d|b_1|^+ - d|b_1|^- + i(d|b_2|^+ - d|b_2|^-)$$

Warning: it does not make sense to write something like '$db$', unless $b$ is either right-continuous, non-decreasing and $b(0) \geq 0$, or $b$ is a right-continuous map of bounded variation.

EXERCISE 24. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map. For all $T \in \mathbf{R}^+$, we define $b^T : \mathbf{R}^+ \to \mathbf{C}$ as $b^T(t) = b(T \wedge t)$ for all $t \in \mathbf{R}^+$.

1. Show that for all $T \in \mathbf{R}^+$, $|b^T| = |b|^T$.
2. Show that if $b$ is of finite variation, then for all $T \in \mathbf{R}^+$, $b^T$ is of bounded variation, and we have $|b^T|(\infty) = |b|(T) < +\infty$.
3. Show that if $b$ is right-continuous and of finite variation, for all $T \in \mathbf{R}^+$, $db^T$ is the unique complex measure on $\mathbf{R}^+$, with:

$$(i) \qquad db^T(\{0\}) = b(0)$$

$$(ii) \qquad \forall s,t \in \mathbf{R}^+, s \leq t \ , \ db^T(]s,t]) = b(T \wedge t) - b(T \wedge s)$$

4. Show that if $b$ is $\mathbf{R}$-valued and of finite variation, for all $T \in \mathbf{R}^+$, we have $|b^T|^+ = (|b|^+)^T$ and $|b^T|^- = (|b|^-)^T$.

5. Show that if $b$ is right-continuous and of bounded variation, for all $T \in \mathbf{R}^+$, we have $db^T = db^{[0,T]} = db([0,T] \cap \cdot\,)$

6. Show that if $b$ is right-continuous, non-decreasing with $b(0) \geq 0$, for all $T \in \mathbf{R}^+$, we have $db^T = db^{[0,T]} = db([0,T] \cap \cdot\,)$

Exercise 25. Let $\mu, \nu$ be two finite measures on $\mathbf{R}^+$, such that:

$$(i) \qquad \mu(\{0\}) \leq \nu(\{0\})$$

$$(ii) \qquad \forall s,t \in \mathbf{R}^+, s \leq t \ , \ \mu(]s,t]) \leq \nu(]s,t])$$

We define $a, c : \mathbf{R}^+ \to \mathbf{R}^+$ by $a(t) = \mu([0,t])$ and $c(t) = \nu([0,t])$.

1. Show that $a$ and $c$ are right-continuous, non-decreasing with $a(0) \geq 0$ and $c(0) \geq 0$.

2. Show that $da = \mu$ and $dc = \nu$.

3. Show that $a \leq c$.

4. Define $b : \mathbf{R}^+ \to \mathbf{R}^+$ by $b = c - a$. Show that $b$ is right-continuous, non-decreasing with $b(0) \geq 0$.

5. Show that $da + db = dc$.

6. Conclude with the following:

**Theorem 83** *Let $\mu, \nu$ be two finite measures on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$ with:*

$$
\begin{array}{ll}
(i) & \mu(\{0\}) \leq \nu(\{0\}) \\
(ii) & \forall s, t \in \mathbf{R}^+, s \leq t\ ,\ \mu(]s,t]) \leq \nu(]s,t])
\end{array}
$$

*Then $\mu \leq \nu$, i.e. for all $B \in \mathcal{B}(\mathbf{R}^+)$, $\mu(B) \leq \nu(B)$.*

Exercise 26. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of bounded variation.

1. Show that $|db|(\{0\}) = |b(0)| = d|b|(\{0\})$.

2. Let $s, t \in \mathbf{R}^+$, $s \leq t$. Let $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[s,t]$, $n \geq 1$. Show that:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq |db|(]s,t])$$

3. Show that $|b|(t) - |b|(s) \leq |db|(]s,t])$.

4. Show that $d|b| \leq |db|$.

5. Show that $L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|) \subseteq L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), d|b|)$.

6. Show that $\mathbf{R}^+$ is metrizable and strongly $\sigma$-compact.

7. Show that $C^c_{\mathbf{C}}(\mathbf{R}^+)$, $C^b_{\mathbf{C}}(\mathbf{R}^+)$ are dense in $L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$.

8. Let $h \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$. Given $\epsilon > 0$, show the existence of $\phi \in C^b_{\mathbf{C}}(\mathbf{R}^+)$ such that $\int |\phi - h||db| \leq \epsilon$.

9. Show that $|\int h db| \leq |\int \phi db| + \epsilon$.

10. Show that:

$$\left|\int |\phi| d|b| - \int |h| d|b|\right| \leq \int |\phi - h| d|b| \leq \int |\phi - h||db|$$

11. Show that $\int |\phi| d|b| \leq \int |h| d|b| + \epsilon$.

12. For all $n \geq 1$, we define:

$$\phi_n \stackrel{\triangle}{=} \phi(0) 1_{\{0\}} + \sum_{k=0}^{n2^n-1} \phi(k/2^n) 1_{]k/2^n,(k+1)/2^n]}$$

Show there is $M \in \mathbf{R}^+$, such that $|\phi_n(x)| \leq M$ for all $x$ and $n$.

13. Using the continuity of $\phi$, show that $\phi_n \to \phi$.

14. Show that $\lim \int \phi_n db = \int \phi db$.

15. Show that $\lim \int |\phi_n| d|b| = \int |\phi| d|b|$.

16. Show that for all $n \geq 1$:

$$\int \phi_n db = \phi(0)b(0) + \sum_{k=0}^{n2^n-1} \phi(k/2^n)(b((k+1)/2^n) - b(k/2^n))$$

17. Show that $|\int \phi_n db| \leq \int |\phi_n| d|b|$ for all $n \geq 1$.

18. Show that $|\int \phi db| \leq \int |\phi| d|b|$.

19. Show that $|\int h db| \leq \int |h| d|b| + 2\epsilon$.

20. Show that $|\int h db| \leq \int |h| d|b|$ for all $h \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$.

21. Let $B \in \mathcal{B}(\mathbf{R}^+)$ and $h \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$ be such that $|h| = 1$ and $db = \int h|db|$. Show that $|db|(B) = \int_B \bar{h} db$.

22. Conclude that $|db| \leq d|b|$.

Exercise 27. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation.

1. Show that for all $T \in \mathbf{R}^+$, $|db^T| = d|b^T| = d|b|^T$.
2. Show that $d|b|^T = d|b|^{[0,T]} = d|b|([0,T] \cap \cdot\,)$, and conclude:

**Theorem 84** *If $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of bounded variation, the total variation of its associated complex Stieltjes measure, is equal to the Stieltjes measure associated with its total variation, i.e.*

$$|db| = d|b|$$

*If $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation, then for all $T \in \mathbf{R}^+$, $b^T$ defined by $b^T(t) = b(T \wedge t)$, is right-continuous of bounded variation, and we have $|db^T| = d|b|([0,T] \cap \cdot\,) = d|b|^T$.*

**Definition 111** *Let $b : \mathbf{R}^+ \to E$ be a map, where $E$ is a Hausdorff topological space. We say that $b$ is* **cadlag** *with respect to $E$, if and only if $b$ is right-continuous, and the limit:*

$$b(t-) = \lim_{s \uparrow\uparrow t} b(s)$$

*exists in $E$, for all $t \in \mathbf{R}^+ \setminus \{0\}$. In the case when $E = \mathbf{C}$ or $E = \mathbf{R}$, given $b$ cadlag, we define $b(0-) = 0$, and for all $t \in \mathbf{R}^+$:*

$$\Delta b(t) \stackrel{\triangle}{=} b(t) - b(t-)$$

Exercise 28. Let $b : \mathbf{R}^+ \to E$ be cadlag, where $E$ is a Hausdorff topological space. Suppose $b$ has values in $E' \subseteq E$.

1. Show that for all $t > 0$, the limit $b(t-)$ is unique.

2. Show that $E'$ is Hausdorff.

3. Explain why $b$ may not be cadlag with respect to $E'$.

4. Show that $b$ is cadlag with respect to $\bar{E}'$.

5. Show that $b : \mathbf{R}^+ \to \mathbf{R}$ is cadlag $\Leftrightarrow$ it is cadlag w.r. to $\mathbf{C}$.

**Exercise 29.**

1. Show that if $b : \mathbf{R}^+ \to \mathbf{C}$ is cadlag, then $b$ is continuous with $b(0) = 0$ if and only if $\Delta b(t) = 0$ for all $t \in \mathbf{R}^+$.

2. Show that if $a : \mathbf{R}^+ \to \mathbf{R}^+$ is right-continuous, non-decreasing with $a(0) \geq 0$, then $a$ is cadlag (w.r. to $\mathbf{R}$ and $\mathbf{R}^+$) with $\Delta a \geq 0$.

3. Show that any linear combination of cadlag maps is itself cadlag.

4. Show that if $b : \mathbf{R}^+ \to \mathbf{C}$ is a right-continuous map of finite variation, then $b$ is cadlag.

5. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Show that $da(\{t\}) = \Delta a(t)$ for all $t \in \mathbf{R}^+$.

6. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of bounded variation. Show that $db(\{t\}) = \Delta b(t)$ for all $t \in \mathbf{R}^+$.

7. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of finite variation. Let $T \in \mathbf{R}^+$. Show that:

$$\forall t \in \mathbf{R}^+ \ , \ b^T(t-) = \left\{ \begin{array}{lll} b(t-) & \text{if} & t \leq T \\ b(T) & \text{if} & T < t \end{array} \right.$$

Show that $\Delta b^T = (\Delta b)1_{[0,T]}$, and $db^T(\{t\}) = \Delta b(t)1_{[0,T]}(t)$.

**Exercise 30.** Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a cadlag map and $T \in \mathbf{R}^+$.

1. Show that if $t \to b(t-)$ is not bounded on $[0,T]$, there exists a sequence $(t_n)_{n\geq 1}$ in $[0,T]$ such that $|b(t_n)| \to +\infty$.

2. Suppose from now on that $b$ is not bounded on $[0,T]$. Show the existence of a sequence $(t_n)_{n\geq 1}$ in $[0,T]$, such that $t_n \to t$ for some $t \in [0,T]$, and $|b(t_n)| \to +\infty$.

3. Define $R = \{n \geq 1 : \ t \leq t_n\}$ and $L = \{n \geq 1 : \ t_n < t\}$. Show that $R$ and $L$ cannot be both finite.

4. Suppose that $R$ is infinite. Show the existence of $n_1 \geq 1$, with:

$$t_{n_1} \in [t, t+1[\cap[0,T]$$

5. If $R$ is infinite, show there is $n_1 < n_2 < \ldots$ such that:

$$t_{n_k} \in [t, t+\frac{1}{k}[\cap[0,T] \ , \ \forall k \geq 1$$

6. Show that $|b(t_{n_k})| \not\to +\infty$.

7. Show that if $L$ is infinite, then $t > 0$ and there is an increasing sequence $n_1 < n_2 < \ldots$, such that:

$$t_{n_k} \in ]t - \frac{1}{k}, t[\cap[0,T] \ , \ \forall k \geq 1$$

8. Show that: $|b(t_{n_k})| \not\to +\infty$.

9. Prove the following:

**Theorem 85** *Let* $b : \mathbf{R}^+ \to \mathbf{C}$ *be a cadlag map. Let* $T \in \mathbf{R}^+$*. Then* $b$ *and the map* $t \to b(t-)$ *are bounded on* $[0, T]$*, i.e. there exists* $M \in \mathbf{R}^+$ *such that:*

$$|b(t)| \vee |b(t-)| \leq M \ , \ \forall t \in [0, T]$$

# Solutions to Exercises

**Exercise 1.**

1. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be non-decreasing with $a(0) \geq 0$. Let $t \in \mathbf{R}^+$. Taking $t_0 = 0$ and $t_1 = t$, from definition (108), we have:

$$|a(t_1) - a(t_0)| \leq |a|(t) - |a(0)|$$

Since $a$ is non-decreasing and $a(0) \geq 0$, we obtain $a(t) \leq |a|(t)$. Let $n \geq 1$ and $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[0, t]$:

$$\sum_{i=1}^{n} |a(t_i) - a(t_{i-1})| = a(t_n) - a(t_0) \leq a(t) - a(0)$$

So $a(t) - a(0)$ is an upper-bound of all sums $\sum_{i=1}^{n} |a(t_i) - a(t_{i-1})|$ as $t_0 \leq \ldots \leq t_n$ runs through all finite sequences in $[0, t]$. From definition (108), $|a|(t) - |a(0)|$ is the smallest of such upper-bounds. Hence:

$$|a|(t) - |a(0)| \leq a(t) - a(0)$$

and since $a(0) \geq 0$, we obtain $|a|(t) \leq a(t)$. We have proved that $|a|(t) = a(t)$ for all $t \in \mathbf{R}^+$, i.e. $|a| = a$.

2. Let $l = \sup_{t\in\mathbf{R}^+} a(t) \in \bar{\mathbf{R}}$. We claim that $a(t)$ converges to $l$ as $t \to +\infty$. Suppose $l = +\infty$. $l$ being the smallest upper-bound of all $a(t)$'s, for all $A \in \mathbf{R}^+$ $A$ cannot be such an upper-bound. Hence, there exists $t_A \in \mathbf{R}^+$ such that $A < a(t_A)$. Since $a$ is non-decreasing, for all $t \in \mathbf{R}^+$:

$$t_A \leq t \;\Rightarrow\; A < a(t_A) \leq a(t)$$

This shows that $\lim_{t\to+\infty} a(t) = +\infty = l$. Suppose $l < +\infty$. Then, given $\epsilon > 0$ we have $l - \epsilon < l$. Again, $l - \epsilon$ cannot be an upper-bound of all $a(t)$'s. There exists $t_\epsilon \in \mathbf{R}^+$ such that $l - \epsilon < a(t_\epsilon)$. Since $a$ is non-decreasing we obtain, for all $t \in \mathbf{R}^+$:

$$t_\epsilon \leq t \;\Rightarrow\; l - \epsilon < a(t_\epsilon) \leq a(t) \leq l$$

This shows that $\lim_{t\to+\infty} a(t) = l$. We have proved that $a(t)$ has a limit in $\bar{\mathbf{R}}$ as $t \to +\infty$. This limit is denoted $a(\infty)$.

3. The proof of 2. together with 1. shows that:

$$a(\infty) = \sup_{t \in \mathbf{R}^+} a(t) = \sup_{t \in \mathbf{R}^+} |a|(t)$$

It follows from definition (108) that $a$ is of bounded variation if and only if $a(\infty) < +\infty$.

Exercise 1

**Exercise 2.**

1. Let $b = b_1 + ib_2 : \mathbf{R}^+ \to \mathbf{C}$ be a map where $b_1 = Re(b)$ and $b_2 = Im(b)$. Let $t \in \mathbf{R}^+$. Let $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[0, t]$. Since $|Re(z)| \leq |z|$ for all $z \in \mathbf{C}$ and by virtue of definition (108):

$$\sum_{i=1}^{n} |b_1(t_i) - b_1(t_{i-1})| \leq \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(t) - |b(0)|$$

It follows that $|b|(t) - |b(0)|$ is an upper-bound of all sums $\sum_{i=1}^{n} |b_1(t_i) - b_1(t_{i-1})|$ as $t_0 \leq \ldots \leq t_n$ runs through all finite sequences in $[0, t]$. $|b_1|(t) - |b_1(0)|$ being the smallest of such upper-bounds, we obtain:

$$|b_1|(t) - |b_1(0)| \leq |b|(t) - |b(0)|$$

and from $|b_1(0)| \leq |b(0)|$ we conclude that $|b_1|(t) \leq |b|(t)$. This being true for all $t \in \mathbf{R}^+$, we have proved that $|b_1| \leq |b|$. Since

$|Im(z)| \leq |z|$ for all $z \in \mathbf{C}$, we obtain $|b_2| \leq |b|$ with a strictly identical argument.

2. Let $t \in \mathbf{R}^+$ and $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[0, t]$:

$$\begin{aligned}\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| &\leq \sum_{i=1}^{n} |b_1(t_i) - b_1(t_{i-1})| + \sum_{i=1}^{n} |b_2(t_i) - b_2(t_{i-1})| \\ &\leq |b_1|(t) - |b_1(0)| + |b_2|(t) - |b_2(0)|\end{aligned}$$

It follows that the r.h.s of this last inequality is an upper-bound of all sums $\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$ as $t_0 \leq \ldots \leq t_n$ runs through all finite sequences in $[0, t]$. $|b|(t) - |b(0)|$ being the smallest of such upper-bounds, we obtain:

$$|b|(t) - |b(0)| \leq |b_1|(t) - |b_1(0)| + |b_2|(t) - |b_2(0)|$$

and from $|b(0)| \leq |b_1(0)| + |b_2(0)|$ we conclude that:

$$|b|(t) \leq |b_1|(t) + |b_2|(t)$$

This being true for all $t \in \mathbf{R}^+$, we have proved $|b| \leq |b_1| + |b_2|$.

3. Suppose $b$ is of finite variation. Then $|b|(t) < +\infty$ for all $t \in \mathbf{R}^+$. It follows from 1. that $|b_1|(t) < +\infty$ and $|b_2|(t) < +\infty$ for all $t \in \mathbf{R}^+$. So $b_1$ and $b_2$ are also of finite variation. Suppose conversely that $b_1$ and $b_2$ are of finite variation. Then $|b_1|(t) < +\infty$ and $|b_2|(t) < +\infty$ for all $t \in \mathbf{R}^+$. It follows from 2. that $|b|(t) < +\infty$ for all $t \in \mathbf{R}^+$. So $b$ is also of finite variation. We have proved that $b$ is of finite variation if and only if $b_1$ and $b_2$ are of finite variation.

4. From 1. we have:

$$\sup_{t\in\mathbf{R}^+} |b_1|(t) \leq \sup_{t\in\mathbf{R}^+} |b|(t)$$

together with:

$$\sup_{t\in\mathbf{R}^+} |b_2|(t) \leq \sup_{t\in\mathbf{R}^+} |b|(t)$$

Furthermore, from 2. we obtain:

$$\sup_{t\in\mathbf{R}^+} |b|(t) \leq \sup_{t\in\mathbf{R}^+} |b_1|(t) + \sup_{t\in\mathbf{R}^+} |b_2|(t)$$

We conclude from definition (108) that $b$ is of bounded variation if and only if both $b_1$ and $b_2$ are also of bounded variation.

5. Take $t = t_0 = t_1 = 0$. From definition (108), we have:

$$|b(t_1) - b(t_0)| \leq |b|(t) - |b(0)|$$

i.e. $|b(0)| \leq |b|(0)$. Furthermore, let $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[0, t] = \{0\}$. Then $t_0 = \ldots = t_n = 0$ and:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| = 0$$

So 0 is an upper-bound of all sums $\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$ as $t_0 \leq \ldots \leq t_n$ runs through all finite sequences in $[0, t] = \{0\}$. $|b|(0) - |b(0)|$ being the smallest of such upper-bounds, we have $|b|(0) - |b(0)| \leq 0$. We have proved that $|b|(0) = |b(0)|$.

Exercise 2

**Exercise 3.** Let $b : \mathbf{R}^+ \to \mathbf{R}$ be differentiable, such that $b'$ is bounded on each compact interval of $\mathbf{R}^+$. In particular $b'$ is bounded on $[0,t]$ for all $t \in \mathbf{R}^+$ and consequently:

$$\sup_{u\in[0,t]} |b'(u)| < +\infty$$

Let $t \in \mathbf{R}^+$ be given and $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[0,t]$. Let $i \in \mathbf{N}_n$ and suppose $t_{i-1} < t_i$. $b$ being differentiable on $\mathbf{R}^+$ is in particular continuous. In particular $b$ is continuous on $[t_{i-1}, t_i]$ and differentiable on $]t_{i-1}, t_i[$. From Taylor's theorem (39), there exists $c_i \in ]t_{i-1}, t_i[$ such that:

$$b(t_i) - b(t_{i-1}) = b'(c_i) \cdot (t_i - t_{i-1})$$

and in particular:

$$|b(t_i) - b(t_{i-1})| \leq m_t \cdot (t_i - t_{i-1})$$

where $m_t = \sup_{u\in[0,t]} |b'(u)| < +\infty$. It is clear that this last inequality is still valid when $t_{i-1} = t_i$. Hence:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq m_t \cdot (t_n - t_0) \leq m_t \cdot t$$

It follows that $m_t \cdot t$ is an upper-bound of all sums $\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$ as $t_0 \leq \ldots \leq t_n$ runs through all finite sequences in $[0, t]$. $|b|(t) - |b(0)|$ being the smallest of such upper-bounds, we obtain $|b|(t) - |b(0)| \leq m_t \cdot t$ and finally $|b|(t) \leq |b(0)| + m_t \cdot t < +\infty$. We have proved that $b$ is of finite variation.

Exercise 3

**Exercise 4.** Let $b : \mathbf{R}^+ \to \mathbf{C}$ be of class $C^1$. Then both $Re(b)$ and $Im(b)$ are of class $C^1$. In particular, they are differentiable, and from theorem (37) their derivatives are bounded on any compact subset of $\mathbf{R}^+$. From exercise (3), $Re(b)$ and $Im(b)$ are both of finite variation. It follows from exercise (2) that $b$ is also of finite variation.

Exercise 4

**Exercise 5.**

1. Let $f : (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ be a measurable map such that $\int_0^t |f(s)|ds < +\infty$ for all $t \in \mathbf{R}^+$. Let $b(t) = \int_0^t f(s)ds$. Let $t \in \mathbf{R}^+$ be given and $t_0 \le \ldots \le t_n$ be a finite sequence in $[0, t]$:

$$\begin{aligned}\sum_{i=1}^n |b(t_i) - b(t_{i-1})| &= \sum_{i=1}^n \left| \int f(s) 1_{]t_{i-1}, t_i]}(s) ds \right| \\ &\le \sum_{i=1}^n \int |f(s)| 1_{]t_{i-1}, t_i]}(s) ds \\ &= \int |f(s)| 1_{]t_0, t_n]}(s) ds \\ &\le \int_0^t |f(s)| ds\end{aligned}$$

So $\int_0^t |f(s)|ds$ is an upper-bound of all sums $\sum_{i=1}^n |b(t_i) - b(t_{i-1})|$ as $t_0 \le \ldots \le t_n$ runs through all finite sequences in $[0, t]$. Since

$|b|(t) - |b(0)|$ is the smallest of such upper-bounds, we obtain $|b|(t) - |b(0)| \leq \int_0^t |f(s)|ds$ and since $b(0) = 0$ we have proved that for all $t \in \mathbf{R}^+$:

$$|b|(t) \leq \int_0^t |f(s)|ds < +\infty$$

In particular, $b$ is a map of finite variation.

2. Suppose $f \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), ds)$. Then $\int_{\mathbf{R}^+} |f|ds < +\infty$, and from 1. we have for all $t \in \mathbf{R}^+$:

$$|b|(t) \leq \int_0^t |f(s)|ds \leq \int_{\mathbf{R}^+} |f(s)|ds$$

In particular:

$$\sup_{t \in \mathbf{R}^+} |b|(t) \leq \int_{\mathbf{R}^+} |f(s)|ds < +\infty$$

We conclude from definition (108) that $b$ is of bounded variation.

Exercise 5

**Exercise 6.** Let $b, b' : \mathbf{R}^+ \to \mathbf{C}$ be two maps and $\alpha \in \mathbf{C}$. Define $c = b + \alpha b'$. Let $t \in \mathbf{R}^+$ and $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[0, t]$. Then:

$$\begin{aligned}\sum_{i=1}^{n} |c(t_i) - c(t_{i-1})| &\leq \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| + |\alpha| \sum_{i=1}^{n} |b'(t_i) - b'(t_{i-1})| \\ &\leq |b|(t) - |b(0)| + |\alpha| \cdot (|b'|(t) - |b'(0)|)\end{aligned}$$

It follows that the r.h.s of this last inequality is an upper-bound of all sums $\sum_{i=1}^{n} |c(t_i) - c(t_{i-1})|$ as $t_0 \leq \ldots \leq t_n$ runs through all finite sequences in $[0, t]$. $|c|(t) - |c(0)|$ being the smallest of such upper-bounds, we obtain:

$$|c|(t) - |c(0)| \leq |b|(t) - |b(0)| + |\alpha| \cdot (|b'|(t) - |b'(0)|)$$

Since $|c(0)| \leq |b(0)| + |\alpha| \cdot |b'(0)|$, we conclude that for all $t \in \mathbf{R}^+$:

$$|c|(t) \leq |b|(t) + |\alpha| \cdot |b'|(t)$$

Hence, if $b$ and $b'$ are of finite variation, $c = b + \alpha b'$ is also of finite variation. Furthermore, we have:

$$\sup_{t \in \mathbf{R}^+} |c|(t) \leq \sup_{t \in \mathbf{R}^+} |b|(t) + |\alpha| \cdot \sup_{t \in \mathbf{R}^+} |b'|(t)$$

So $b, b'$ of bounded variation $\Rightarrow$ $c$ of bounded variation.

Exercise 6

**Exercise 7.**

1. Let $t \in \mathbf{R}^+$ and $s_0 \leq \ldots \leq s_p$, $p \geq 1$, be a finite sequence in $[0, t]$. We define:

$$S \stackrel{\triangle}{=} \sum_{j=1}^{p} |b(s_j) - b(s_{j-1})| \tag{2}$$

Let $A = \{s_0, \ldots, s_p\}$. If $\text{card}A = 1$, then $s_0 = \ldots = s_p$, and it is clear from (2) that $S = 0$. We assume that $\text{card}A \geq 2$. Then $A$ is a subset of $[0, t]$ with $\text{card}A \geq 2$, and consequently $A \in \mathcal{S}(t)$. We shall prove that $S = S(A)$. Let $t_0 < \ldots < t_n$ be distinct in $[0, t]$ such that $A = \{t_0, \ldots, t_n\}$. By definition:

$$S(A) \stackrel{\triangle}{=} \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \tag{3}$$

Since $A = \{t_0, \ldots, t_n\} = \{s_0, \ldots, s_p\}$, it is intuitively fairly obvious from (2) and (3) that $S = S(A)$. After all, the only difference between the $t_i$'s and the $s_j$'s (both are ordered, i.e. $t_0 < \ldots < t_n$ and $s_0 \leq \ldots \leq s_p$) is that the former are assumed to be distinct and not the latter, and any 'repetition' in the $s_j$'s will not affect the sum in (2) as the corresponding term $|b(s_j) - b(s_{j-1})|$ is nil. We may choose to go no further and rely solely on intuition to conclude that $S = S(A)$. To manufacture a more formal proof of the fact that $S = S(A)$ (which may not be that pointless for a student in search of more technical strength), one may proceed with an induction argument based on the difference $p-n$. Since $\text{card}A = n+1$ and $A = \{s_0, \ldots, s_p\}$, we have $n \leq p$. If $n = p$, then $s_k = t_k$ for all $k = 0, \ldots, n$ (the $t_k$'s and $s_k$'s are ordered), and it is clear from (2) and (3) that $S = S(A)$. So $S = S(A)$ is proved when $p - n = 0$. Suppose that $S = S(A)$ is proved when $p-n = k$ for $k \geq 0$, and assume that $p-n = k+1$. In particular $p > n$. Since $A = \{t_0, \ldots, t_n\} = \{s_0, \ldots, s_p\}$, it is impossible that all $s_j$'s be distinct, and consequently the integer:

$$j_0 = \min\{j : \ j \in \{1, \ldots, p\} \ , \ s_j = s_{j-1}\}$$

as the smallest element of an non-empty subset of $\mathbf{N}$ is well-defined. Let $s'_0 \leq \ldots \leq s'_{p-1}$ be defined as:

$$s'_k = \begin{cases} s_k & \text{if } k \leq j_0 - 1 \\ s_{k+1} & \text{if } k \geq j_0 \end{cases}$$

for $k = 0, \ldots, p-1$. Informally, the finite sequence $s'_0 \leq \ldots \leq s'_{p-1}$ is nothing but the sequence $s_0 \leq \ldots \leq s_p$ where the 'duplicated point' $s_{j_0}$ has been 'taken out'. The sum $S'$ associated with $s'_0 \leq \ldots \leq s'_{p-1}$ is given similarly to (2) as:

$$S' = \sum_{j=1}^{p-1} |b(s'_j) - b(s'_{j-1})| \tag{4}$$

We shall prove formally that $S = S'$ (which is also intuitively obvious in the light of (2) and (4)) by distinguishing three possible cases. Suppose $j_0 = p$. Then (4) can be re-expressed as:

$$\begin{aligned} S' &= \sum_{j=1}^{j_0-1} |b(s_j) - b(s_{j-1})| \\ &= \sum_{j=1}^{j_0} |b(s_j) - b(s_{j-1})| \\ &= \sum_{j=1}^{p} |b(s_j) - b(s_{j-1})| = S \end{aligned}$$

where the fact that $s_{j_0} = s_{j_0-1}$ was used for the second equality. Suppose that $j_0 = p - 1$. Then (4) can be re-expressed as:

$$S' = \sum_{j=1}^{j_0-1} |b(s_j) - b(s_{j-1})| + |b(s_{j_0+1}) - b(s_{j_0-1})|$$

$$
\begin{aligned}
&= \sum_{j=1}^{j_0-1} |b(s_j) - b(s_{j-1})| + |b(s_{j_0+1}) - b(s_{j_0})| \\
&= \sum_{j=1}^{j_0+1} |b(s_j) - b(s_{j-1})| \\
&= \sum_{j=1}^{p} |b(s_j) - b(s_{j-1})| = S
\end{aligned}
$$

where the fact that $s_{j_0} = s_{j_0-1}$ was used for the third equality. Suppose lastly that $j_0 < p-1$. Then (4) can be split in three:

$$
\sum_{j=1}^{j_0-1} |b(s_j)-b(s_{j-1})|+|b(s_{j_0+1})-b(s_{j_0-1})|+\sum_{j=j_0+1}^{p-1} |b(s_{j+1})-b(s_j)|
$$

which can be re-expressed as:

$$\sum_{j=1}^{j_0-1} |b(s_j)-b(s_{j-1})|+|b(s_{j_0+1})-b(s_{j_0})|+\sum_{j=j_0+2}^{p} |b(s_j)-b(s_{j-1})|$$

and finally from $|b(s_{j_0}) - b(s_{j_0-1})| = 0$, we obtain:

$$S' = \sum_{j=1}^{p} |b(s_j) - b(s_{j-1})| = S$$

In all cases, we have proved that $S = S'$. However, it is clear that $\{s'_0, \ldots, s'_{p-1}\} = A = \{t_0, \ldots, t_n\}$ and from our induction hypothesis, since $p-1-n = k$, we have $S' = S(A)$. We conclude that $S = S(A)$ and our induction hypothesis is proved for $p-n = k + 1$. This completes the induction argument and we have showed that $S = S(A)$. For all $t \in \mathbf{R}^+$ and $s_0 \leq \ldots \leq s_p$ finite sequences in $[0, t]$ (with $p \geq 1$ as always, in line with definition (108)), then if $S$ is defined by (2), either $S = 0$ or $S = S(A)$ for some $A \in \mathcal{S}(t)$.

2. Let $t \in \mathbf{R}^+$ and $a(t) = \sup\{S(A) : A \in \mathcal{S}(t)\}$. Let $s_0 \leq \ldots \leq s_p$, $p \geq 1$, be a finite sequence in $[0, t]$. Define:

$$S = \sum_{j=1}^{p} |b(s_j) - b(s_{j-1})|$$

From 1. either $S = 0$ or $S = S(A)$ for some $A \in \mathcal{S}(t)$. In any case, $a(t)$ being an upper-bound of all $S(A)$'s, we have $S \leq a(t)$. So $a(t)$ is an upper-bound of all sums $\sum_{j=1}^{p} |b(s_j) - b(s_{j-1})|$ as $s_0 \leq \ldots \leq s_p$ runs through all finite sequences (with $p \geq 1$) in $[0, t]$. Since $|b|(t) - |b(0)|$ is the smallest of such upper-bounds, we have $|b|(t) - |b(0)| \leq a(t)$. Let $A \in \mathcal{S}(t)$. Let $t_0 < \ldots < t_n$ be distinct in $[0, t]$ such that $A = \{t_0, \ldots, t_n\}$. Then, by definition, the sum $S(A)$ is given by:

$$S(A) = \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$$

In particular, $|b|(t) - |b(0)|$ being an upper-bound of all sums

$\sum_{i=1}^n |b(t_i) - b(t_{i-1})|$ as $t_0 \leq \ldots \leq t_n$ runs through all finite sequences (with $n \geq 1$) in $[0, t]$, we have $S(A) \leq |b|(t) - |b(0)|$. It follows that $|b|(t) - |b(0)|$ is an upper-bound of all $S(A)$'s with $A \in \mathcal{S}(t)$. Since $a(t)$ is the smallest of such upper-bounds, we obtain $a(t) \leq |b|(t) - |b(0)|$. We have proved that $a(t) = |b|(t) - |b(0)|$ for all $t \in \mathbf{R}^+$, or equivalently:

$$|b|(t) = |b(0)| + \sup\{S(A) : A \in \mathcal{S}(t)\}$$

3. Let $A \in \mathcal{S}(t)$ and $s \in [0, t]$. Then $A \cup \{s\}$ is a subset of $[0, t]$ with $\text{card}(A \cup \{s\}) \geq 2$. So $S(A \cup \{s\})$ is well-defined. Let $t_0 < \ldots < t_n$ be distinct in $[0, t]$ such that $A = \{t_0, \ldots, t_n\}$. Then, by definition:

$$S(A) = \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$$

If $s \in A$ then $A \cup \{s\} = A$ and $S(A) = S(A \cup \{s\})$. We assume that $s \notin A$. There are three possible cases to consider: firstly

$s < t_0$, secondly $t_{j-1} < s < t_j$ for some $j = 1, \ldots, n$ and thirdly $t_n < s$. In the first case we have:

$$S(A \cup \{s\}) = |b(t_0) - b(s)| + S(A)$$

and in the third case:

$$S(A \cup \{s\}) = S(A) + |b(s) - b(t_n)|$$

In the second case, $S(A \cup \{s\})$ can be split into four parts:

$$\begin{aligned} \sum_{i=1}^{j-1} |b(t_i) - b(t_{i-1})| \quad &+ \quad |b(s) - b(t_{j-1})| + |b(t_j) - b(s)| \\ &+ \quad \sum_{i=j+1}^{n} |b(t_i) - b(t_{i-1})| \end{aligned}$$

and from $|b(t_j) - b(t_{j-1})| \leq |b(s) - b(t_{j-1})| + |b(t_j) - b(s)|$ we conclude that $S(A) \leq S(A \cup \{s\})$. In any case, we have proved that $S(A) \leq S(A \cup \{s\})$.

4. Let $A, B \in \mathcal{S}(t)$ such that $A \subseteq B$. We shall prove that $S(A) \leq S(B)$ using an induction argument based on the cardinality of $B \setminus A$. If $\mathrm{card}(B \setminus A) = 0$, then $A = B$ and $S(A) = S(B)$. We assume that $S(A) \leq S(B)$ is true when $\mathrm{card}(B \setminus A) = k$ for $k \geq 0$, and that $\mathrm{card}(B \setminus A) = k + 1$. In particular $B \setminus A \neq \emptyset$ and there exists $s \in B \setminus A$. From 3. we have $S(A) \leq S(A \cup \{s\})$. Furthermore, $A \cup \{s\}$ is an element of $\mathcal{S}(t)$ with $A \cup \{s\} \subseteq B$ and $\mathrm{card}(B \setminus (A \cup \{s\})) = k$. From our induction hypothesis, it follows that $S(A \cup \{s\}) \leq S(B)$. We conclude that $S(A) \leq S(B)$ and the induction hypothesis is proved for $\mathrm{card}(B \setminus A) = k + 1$. This completes the induction argument, and we have proved that $S(A) \leq S(B)$ for all $A, B \in \mathcal{S}(t)$ with $A \subseteq B$.

5. Let $t_0 \leq \ldots \leq t_n$, $n \geq 1$, and $s_0 \leq \ldots \leq s_p$, $p \geq 1$, be finite sequences in $[0, t]$ such that $\{t_0, \ldots, t_n\} \subseteq \{s_0, \ldots, s_p\}$. Define:

$$S = \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$$

and:

$$S' = \sum_{j=1}^{p} |b(s_j) - b(s_{j-1})|$$

Let $A = \{t_0, \dots, t_n\}$ and $B = \{s_0, \dots, s_p\}$. If $\mathrm{card}A = 1$ then $S = 0$ and in particular $S \leq S'$. We assume that $\mathrm{card}A \geq 2$. Then $\mathrm{card}B \geq 2$ and looking back at the proof of 1. we have $S = S(A)$ and $S' = S(B)$. Since $A \subseteq B$, it follows from 4. that $S(A) \leq S(B)$. We conclude that $S \leq S'$.

Exercise 7

**Exercise 8.**

1. Let $s_0 \leq \ldots \leq s_p$ and $t_0 \leq \ldots \leq t_n$ be finite sequences in $[0, s]$ and $[s, t]$ respectively, $n, p \geq 1$. $s_0 \leq \ldots\ s_p \leq t_0 \leq \ldots \leq t_n$ is a finite sequence in $[0, t]$, with $n + p + 2$ terms and associated sum:

$$\sum_{j=1}^{p} |b(s_j) - b(s_{j-1})| + |b(t_0) - b(s_p)| + \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \quad (5)$$

From definition (108), $|b|(t) - |b(0)|$ is an upper-bound of all sums $\sum_{k=1}^{m} |b(u_k) - b(u_{k-1})|$ as $u_0 \leq \ldots \leq u_m$ runs through all finite sequences in $[0, t]$, $m \geq 1$. So $|b|(t) - |b(0)|$ is greater than or equal to (5). In particular, we have:

$$\sum_{j=1}^{p} |b(s_j) - b(s_{j-1})| + \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(t) - |b(0)|$$

2. Let $s_0 \leq \ldots \leq s_p$ be a finite sequence in $[0, s]$, $p \geq 1$. It follows from 1. that $|b|(t) - |b(0)| - \sum_{j=1}^{p} |b(s_j) - b(s_{j-1})|$ is an

upper-bound of all sums $\sum_{i=1}^{n}|b(t_i)-b(t_{i-1})|$ as $t_0\leq\ldots\leq t_n$ runs through all finite sequences in $[s,t]$, $n\geq 1$. Since $\delta$ is the smallest of such upper-bounds, we obtain:

$$\delta\leq |b|(t)-|b(0)|-\sum_{j=1}^{p}|b(s_j)-b(s_{j-1})|$$

$b$ being of finite variation we have $|b|(t)<+\infty$ and consequently $\delta<+\infty$. The previous inequality can be re-arranged as:

$$\sum_{j=1}^{p}|b(s_j)-b(s_{j-1})|\leq |b|(t)-|b(0)|-\delta$$

It follows that $|b|(t)-|b(0)|-\delta$ is an upper-bound of all sums $\sum_{j=1}^{p}|b(s_j)-b(s_{j-1})|$ as $s_0\leq\ldots\leq s_p$ runs through all finite sequences in $[0,s]$, $p\geq 1$. Since $|b|(s)-|b(0)|$ is the smallest of such upper-bounds, we obtain:

$$|b|(s)-|b(0)|\leq |b|(t)-|b(0)|-\delta$$

and all terms being finite, we conclude that:

$$\delta \leq |b|(t) - |b|(s)$$

3. Let $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[0,t]$, $n \geq 1$. We assume that $s = t_j$ for some $j$ with $0 < j < n$. Then $t_0 \leq \ldots \leq t_j$ is a finite sequence in $[0,s]$, $j \geq 1$, and consequently:

$$\sum_{i=1}^{j} |b(t_i) - b(t_{i-1})| \leq |b|(s) - |b(0)| \tag{6}$$

Furthermore, $t_j \leq \ldots \leq t_n$ is a finite sequence in $[s,t]$ (with $n-j+1 \geq 2$ terms) and consequently:

$$\sum_{i=j+1}^{n} |b(t_i) - b(t_{i-1})| \leq \delta \tag{7}$$

From (6) and (7) we conclude that:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(s) - |b(0)| + \delta \tag{8}$$

4. Let $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[0,t]$, $n \geq 1$. We claim that inequality (8) still holds, despite not having made the assumption that $s = t_j$ for some $j$ with $0 < j < n$. Consider the finite sequence $0 \leq t_0 \leq \ldots \leq t_n \leq t$ in $[0,t]$ (with $n+3$ terms), which we may denote $s'_0 \leq \ldots \leq s'_{n+2}$ (how each $s'_k$ is defined is obvious). Since $s \in [0,t]$ we claim that there exists $p \in \{1,\ldots,n+2\}$ such that $s'_{p-1} \leq s \leq s'_p$. A formal proof of this (intuitively obvious) fact can be obtained as follows: If $s = s'_0 = 0$, then in particular $s'_0 \leq s \leq s'_1$. We assume that $s'_0 < s$. Since $s \leq t = s'_{n+2}$, the set $\{j : s \leq s'_j, j = 0,\ldots,n+2\}$ is a non-empty subset of $\mathbf{N}$, and therefore has a smallest element, say $p$. Since $s'_0 < s$ we have $p \geq 1$, and furthermore $s'_{p-1} < s \leq s'_p$. In particular, we have been able to find $p \in \{1,\ldots,n+2\}$ such

that $s'_{p-1} \leq s \leq s'_p$. Consider the finite sequence $s'_0 \leq \ldots \leq s'_{p-1} \leq s \leq s'_p \leq \ldots \leq s'_{n+2}$ in $[0,t]$ (with $n+4$ terms), which we may denote $s_0 \leq \ldots \leq s_{n+3}$. This finite sequence in $[0,t]$ is such that there exists $j$ with $s = s_j$ and $0 < j < n+3$. From 3. we obtain:

$$\sum_{i=1}^{n+3} |b(s_i) - b(s_{i-1})| \leq |b|(s) - |b(0)| + \delta$$

However, it is clear that $\{t_0, \ldots, t_n\} \subseteq \{s_0, \ldots, s_{n+3}\}$, and it follows from 5. of exercise (7) that:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq \sum_{i=1}^{n+3} |b(s_i) - b(s_{i-1})|$$

We conclude that:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(s) - |b(0)| + \delta$$

5. It follows from 4. that $|b|(s) - |b(0)| + \delta$ is an upper-bound of all sums $\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$ as $t_0 \leq \ldots \leq t_n$ runs through all finite sequences in $[0, t]$, $n \geq 1$. Since $|b|(t) - |b(0)|$ is the smallest of such upper-bounds, we obtain:

$$|b|(t) - |b(0)| \leq |b|(s) - |b(0)| + \delta$$

Equivalently, since $|b|(s) < +\infty$, $|b|(t) - |b|(s) \leq \delta$. Having proved the reverse inequality in 2. we conclude that:

$$|b|(t) - |b|(s) = \delta = \sup \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$$

where the supremum is taken over all finite sequences $t_0 \leq \ldots \leq t_n$ in $[s, t]$, $n \geq 1$. This completes the proof of theorem (80).

Exercise 8

**Exercise 9.** Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map of finite variation. Let $s, t \in \mathbf{R}^+$, $s \leq t$. A consequence of theorem (80) is that $|b|(s) \leq |b|(t)$. So $|b|$ is non-decreasing. From 5. of exercise (2), we have $|b|(0) = |b(0)|$ and in particular $|b|(0) \geq 0$. From exercise (1), it follows that the total variation $||b||$ of $|b|$ is nothing but itself, i.e. $||b|| = |b|$.

Exercise 9

**Exercise 10.**

1. Let $b : \mathbf{R}^+ \to \mathbf{R}$ be a map of finite variation. From definition (109), we have:

$$|b|^+ + |b|^- = \frac{1}{2}(|b| + b) + \frac{1}{2}(|b| - b) = |b|$$

and furthermore:

$$|b|^+ - |b|^- = \frac{1}{2}(|b| + b) - \frac{1}{2}(|b| - b) = b$$

2. Since $|b|(0) = |b(0)|$, we have:

$$|b|^+(0) = \frac{1}{2}(|b(0)| + b(0)) \stackrel{\triangle}{=} b^+(0)$$

and:

$$|b|^-(0) = \frac{1}{2}(|b(0)| - b(0)) \stackrel{\triangle}{=} b^-(0)$$

In particular, $|b|^+(0) \geq 0$ and $|b|^-(0) \geq 0$.

3. Let $s, t \in \mathbf{R}^+$, $s \leq t$. Then $s \leq t$ is a finite sequence in $[s, t]$ (with 2 terms). It follows from theorem (80) that:

$$|b(t) - b(s)| \leq |b|(t) - |b|(s) \tag{9}$$

4. Let $s, t \in \mathbf{R}^+$, $s \leq t$. It follows from (9) that:

$$b(s) - b(t) \leq |b|(t) - |b|(s)$$

and:

$$b(t) - b(s) \leq |b|(t) - |b|(s)$$

we conclude that $|b|^+(s) \leq |b|^+(t)$ and $|b|^-(s) \leq |b|^-(t)$. So $|b|^+$ and $|b|^-$ are non-decreasing.

Exercise 10

**Exercise 11.** Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map of finite variation. Let $u = Re(b)$ and $v = Im(b)$. From exercise (2), $u, v : \mathbf{R}^+ \to \mathbf{R}$ are both of finite variation. Let $b_1 = |u|^+$, $b_2 = |u|^-$, $b_3 = |v|^+$ and $b_4 = |v|^-$. From exercise (10), $b_1$, $b_2$, $b_3$ and $b_4$ are all non-decreasing maps with $b_i(0) \geq 0$, $i = 1, \ldots, 4$. Furthermore, since $u = b_1 - b_2$ and $v = b_3 - b_4$, we have $b = b_1 - b_2 + i(b_3 - b_4)$. Conversely, suppose $b = b_1 - b_2 + i(b_3 - b_4)$ where each $b_i$, $i = 1, \ldots, 4$ is non-decreasing with $b_i(0) \geq 0$. From exercise (1), each $b_i$ is a map of finite variation. From exercise (6), it follows that $b$ is also a map of finite variation. We have proved that a map $b : \mathbf{R}^+ \to \mathbf{C}$ is of finite variation, if and only if there exist $b_1$, $b_2$, $b_3$ and $b_4$ non-decreasing with $b_i(0) \geq 0$, $i = 1, \ldots, 4$, such that $b = b_1 - b_2 + i(b_3 - b_4)$.

Exercise 11

**Exercise 12.**

1. Let $b : \mathbf{R}^+ \rightarrow \mathbf{C}$ be a right-continuous map of finite variation. Let $x_0 \in \mathbf{R}^+$. From exercise (9), $|b| : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is non-decreasing with $|b|(0) \geq 0$. In particular, for all $t \in \mathbf{R}^+$, $x_0 < t$, we have $|b|(x_0) \leq |b|(t)$. So $|b|(x_0)$ is a lower-bound of all $|b|(t)$'s as $t \in \mathbf{R}^+$, $x_0 < t$. If we define $l = \inf_{x_0<t} |b|(t)$, then $l$ is the greatest of such lower-bounds, and consequently $|b|(x_0) \leq l$. In particular $-\infty < l$. Furthermore, $t$ being an arbitrary element of $\mathbf{R}^+$ with $x_0 < t$, we have $l \leq |b|(t)$ and in particular, since $b$ is of finite variation, $l < +\infty$. So $l$ is a well-defined element of $\mathbf{R}$. We claim that $|b|(t) \rightarrow l$ as $t \rightarrow x_0$ with $x_0 < t$. Let $\epsilon > 0$. Since $l < l + \epsilon$, $l + \epsilon$ cannot be a lower-bound of all $|b|(t)$'s as $x_0 < t$. Hence, there exists $t_1 \in \mathbf{R}^+$, $x_0 < t_1$, such that $|b|(t_1) < l + \epsilon$. $|b|$ being non-decreasing, we have:

$$t \in ]x_0, t_1[ \; \Rightarrow \; l \leq |b|(t) \leq l + \epsilon$$

This shows that the limit $\lim_{t\downarrow\downarrow x_0} |b|(t)$ exists and is equal to $l$.

This limit is denoted $|b|(x_0+)$. We have proved that for all $x_0 \in \mathbf{R}^+$, the limit $|b|(x_0+)$ exists in $\mathbf{R}$, and $|b|(x_0+) = \inf_{t<x_0} |b|(t)$.

2. From 1. we have $|b|(x_0+) = \inf_{x_0<t} |b|(t)$. However, since $|b|$ is non-decreasing, for all $t \in \mathbf{R}^+$, $x_0 < t$, we have $|b|(x_0) \leq |b|(t)$. It follows that $|b|(x_0)$ is a lower-bound of all $|b|(t)$'s as $t \in \mathbf{R}^+$, $x_0 < t$. Since $|b|(x_0+)$ is the greatest of such lower-bounds, we conclude that $|b|(x_0) \leq |b|(x_0+)$.

3. Let $\epsilon > 0$. Since $|b|(x_0+) = \lim_{t\downarrow\downarrow x_0} |b|(t)$ exists in $\mathbf{R}$, there exists $y_1 \in \mathbf{R}^+$, $x_0 < y_1$, such that:

$$u \in ]x_0, y_1] \;\Rightarrow\; \left|\, |b|(u) - |b|(x_0+) \,\right| \leq \frac{\epsilon}{4}$$

In particular, from the triangle inequality:

$$u, v \in ]x_0, y_1] \;\Rightarrow\; \left|\, |b|(v) - |b|(u) \,\right| \leq \frac{\epsilon}{2} \qquad (10)$$

Furthermore, since $b$ is right-continuous, in particular it is right-continuous at $x_0$. There exists $y_2 \in \mathbf{R}^+$, $x_0 < y_2$, such that:

$$u \in ]x_0, y_2] \;\Rightarrow\; |b(u) - b(x_0)| \leq \frac{\epsilon}{2} \tag{11}$$

Taking $y_0 = \min(y_1, y_2)$, $y_0 \in \mathbf{R}^+$, $x_0 < y_0$, and from (11):

$$u \in ]x_0, y_0] \;\Rightarrow\; |b(u) - b(x_0)| \leq \frac{\epsilon}{2}$$

Furthermore, $y_0 \in ]x_0, y_1]$ and from (10) we have:

$$u \in ]x_0, y_0] \;\Rightarrow\; |b|(y_0) - |b|(u) \leq \frac{\epsilon}{2}$$

Exercise 12

**Exercise 13.**

1. Let $t_0 \leq \ldots \leq t_n$, $n \geq 1$, be a finite sequence in $[0, y_0]$, for which there exists $j$ with $0 < j < n-1$ (so in particular $n \geq 3$), $x_0 = t_j$ and $x_0 < t_{j+1}$. Then $t_0 \leq \ldots \leq t_j$ is a finite sequence in $[0, x_0]$ with $j \geq 1$. From definition (108), we have:

$$\sum_{i=1}^{j} |b(t_i) - b(t_{i-1})| \leq |b|(x_0) - |b(0)| \tag{12}$$

2. Since $t_j = x_0$ and $t_{j+1} \in ]x_0, y_0]$, it follows from exercise (12):

$$|b(t_{j+1}) - b(t_j)| \leq \frac{\epsilon}{2} \tag{13}$$

3. Since $t_{j+1} \leq \ldots \leq t_n$ is a finite sequence in $[t_{j+1}, y_0]$ (with $n - j \geq 2$ terms), from theorem (80) we have:

$$\sum_{i=j+2}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(y_0) - |b|(t_{j+1})$$

and furthermore, since $t_{j+1} \in ]x_0, y_0]$, from exercise (12) we have:

$$|b|(y_0) - |b|(t_{j+1}) \leq \frac{\epsilon}{2}$$

We conclude that:

$$\sum_{i=j+2}^{n} |b(t_i) - b(t_{i-1})| \leq \frac{\epsilon}{2} \tag{14}$$

4. Let $t_0 \leq \ldots \leq t_n$, $n \geq 1$, be a finite sequence in $[0, y_0]$. We claim that:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(x_0) - |b(0)| + \epsilon \tag{15}$$

In the case when there exists an index $j$ with $0 < j < n-1$, $x_0 = t_j$ and $x_0 < t_{j+1}$, we can apply 1. 2. 3. and adding (12), (13) and (14) together, we obtain (15). Our task is to extend (15) to the general case where there may not exists such an index $j$. However, since $x_0 < y_0$, it is always possible to 'add points' to the sequence $t_0 \leq \ldots \leq t_n$ so as to obtain $s_0 \leq \ldots \leq s_p$ in $[0, y_0]$ with $\{t_0, \ldots t_n\} \subseteq \{s_0, \ldots, s_p\}$ and $x_0 = s_j$, $x_0 < s_{j+1}$ for some $0 < j < p-1$. Applying (15) to the $s_i$'s, we obtain:

$$\sum_{i=1}^{p} |b(s_i) - b(s_{i-1})| \leq |b|(x_0) - |b(0)| + \epsilon$$

However, from exercise (7), since $\{t_0, \ldots t_n\} \subseteq \{s_0, \ldots, s_p\}$:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq \sum_{i=1}^{p} |b(s_i) - b(s_{i-1})|$$

and we conclude that (15) is true.

5. It follows from (15) that $|b|(x_0) - |b(0)| + \epsilon$ is an upper-bound of all sums $\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$ as $t_0 \leq \ldots \leq t_n$ runs through the set of all finite sequences in $[0, y_0]$, $n \geq 1$. Since $|b|(y_0) - |b(0)|$ is the smallest of such upper-bounds, we obtain:

$$|b|(y_0) - |b(0)| \leq |b|(x_0) - |b(0)| + \epsilon$$

and finally $|b|(y_0) \leq |b|(x_0) + \epsilon$.

6. Given $\epsilon > 0$, in the light of 5. and exercise (12), we have found $y_0 \in \mathbf{R}^+$, $x_0 < y_0$, such that $|b|(y_0) \leq |b|(x_0) + \epsilon$. However, still from exercise (12), we have $|b|(x_0+) = \inf_{x_0<t} |b|(t)$. In particular, $|b|(x_0+)$ is a lower-bound of all $|b|(t)$'s with $t \in \mathbf{R}^+$, $x_0 < t$. So $|b|(x_0+) \leq |b|(y_0)$, and we have proved that $|b|(x_0+) \leq |b|(x_0) + \epsilon$. This being true for all $\epsilon > 0$, we obtain $|b|(x_0+) \leq |b|(x_0)$. Having proved in exercise (12) that $|b|(x_0) \leq |b|(x_0+)$, we conclude that $|b|(x_0) = |b|(x_0+)$, i.e.

$$|b|(x_0) = \lim_{t\downarrow\downarrow x_0} |b|(t)$$

It follows that $|b|$ is right-continuous at $x_0$. This being true for all $x_0 \in \mathbf{R}^+$, the map $|b| : \mathbf{R}^+ \to \mathbf{R}^+$ is right-continuous.

Exercise 13

**Exercise 14.**

1. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a left-continuous map of finite variation. Let $x_0 \in \mathbf{R}^+ \setminus \{0\}$. Let $l = \sup_{t<x_0} |b|(t)$. Since $|b|$ is non-decreasing, for all $t \in \mathbf{R}^+$, $t < x_0$, we have $|b|(t) \leq |b|(x_0)$. It follows that $|b|(x_0)$ is an upper-bound of all $|b|(t)$'s as $t \in \mathbf{R}^+$, $t < x_0$. Since $l$ is the smallest of such upper-bounds, we obtain $l \leq |b|(x_0)$. In particular, since $b$ is of finite variation, $l < +\infty$. Furthermore, since $0 < x_0$, there exists some $t \in \mathbf{R}^+$ with $t < x_0$. For any such $t$ we have $|b|(t) \leq l$ and it follows in particular that $-\infty < l$. So $l$ is a well-defined element of $\mathbf{R}$. We claim that $|b|(t) \to l$ as $t \to x_0$ with $t < x_0$. Let $\epsilon > 0$. Since $l - \epsilon < l$, $l - \epsilon$ cannot be an upper-bound of all $|b|(t)$'s as $t < x_0$. There exists $t_1 \in \mathbf{R}^+$, $t_1 < x_0$, such that $l - \epsilon < |b|(t_1)$. Since $|b|$ is non-decreasing, we obtain:
$$t \in [t_1, x_0[ \;\Rightarrow\; l - \epsilon < |b|(t) \leq l$$
This shows that the limit $\lim_{t\uparrow\uparrow x_0} |b|(t)$ exists in $\mathbf{R}$ and is equal to $l$. This limit is denoted $|b|(x_0-)$. We have proved that for

all $x_0 \in \mathbf{R}^+ \setminus \{0\}$, the limit $|b|(x_0-)$ exists in $\mathbf{R}$ and is equal to $\sup_{t<x_0} |b|(t)$.

2. From 1. we have $|b|(x_0-) = \sup_{t<x_0} |b|(t)$. However, since $|b|$ is non-decreasing, for all $t \in \mathbf{R}^+$, $t < x_0$, $|b|(t) \leq |b|(x_0)$. So $|b|(x_0)$ is an upper-bound of all $|b|(t)$'s as $t \in \mathbf{R}^+$, $t < x_0$. Since $|b|(x_0-)$ is the smallest of such upper-bounds, we obtain $|b|(x_0-) \leq |b|(x_0)$.

3. Let $\epsilon > 0$. By definition of the left-hand limit $|b|(x_0-)$, there exists $y_1 \in [0, x_0[$ such that:

$$u \in [y_1, x_0[ \;\Rightarrow\; |\,|b|(u) - |b|(x_0-)\,| \leq \frac{\epsilon}{4}$$

In particular, from the triangle inequality:

$$u, v \in [y_1, x_0[ \;\Rightarrow\; |\,|b|(u) - |b|(v)\,| \leq \frac{\epsilon}{2} \tag{16}$$

Furthermore, from the left-continuity of $b$ at $x_0$, there exists

$y_2 \in [0, x_0[$, such that:

$$u \in [y_2, x_0[\ \Rightarrow\ |b(x_0) - b(u)| \leq \frac{\epsilon}{2} \tag{17}$$

Taking $y_0 = \max(y_1, y_2)$, $y_0 \in [0, x_0[$ and from (17):

$$u \in [y_0, x_0[\ \Rightarrow\ |b(x_0) - b(u)| \leq \frac{\epsilon}{2}$$

Furthermore, since $y_0 \in [y_1, x_0[$, we have from (16):

$$u \in [y_0, x_0[\ \Rightarrow\ |b|(u) - |b|(y_0) \leq \frac{\epsilon}{2}$$

Exercise 14

**Exercise 15.**

1. By definition, $k = \max\{i : j \leq i, t_i < x_0\}$. Since $t_j = y_0$ and $y_0 \in [0, x_0[$, we have $t_j < x_0$. It follows that $j \leq k$. Furthermore, since $t_n = x_0$, we have $k \leq n-1$. So $j \leq k \leq n-1$. Since $j \leq k$, we have $t_j \leq t_k$ and from $t_j = y_0$ we obtain $y_0 \leq t_k$. Furthermore, it is clear that $t_k < x_0$. So $t_k \in [y_0, x_0[$.

2. $t_0 \leq \ldots \leq t_j$ being a sequence in $[0, y_0]$ (with $j \geq 1$):

$$\sum_{i=1}^{j} |b(t_i) - b(t_{i-1})| \leq |b|(y_0) - |b(0)| \tag{18}$$

3. If $j = k$, the sum $\sum_{i=j+1}^{k} |b(t_i) - b(t_{i-1})|$ is by convention set to zero. So there is nothing to prove. We assume that $j < k$. Then $t_j \leq \ldots \leq t_k$ is a finite sequence in $[y_0, t_k]$ (with $k - j + 1 \geq 2$

terms). From theorem (80), we have:

$$\sum_{i=j+1}^{k} |b(t_i) - b(t_{i-1})| \leq |b|(t_k) - |b|(y_0)$$

Furthermore from 1. we have $t_k \in [y_0, x_0[$ and consequently from exercise (14):

$$|b|(t_k) - |b|(y_0) \leq \frac{\epsilon}{2}$$

We conclude that:

$$\sum_{i=j+1}^{k} |b(t_i) - b(t_{i-1})| \leq \frac{\epsilon}{2} \qquad (19)$$

4. By definition, $k$ is the greatest index with $j \leq k$ and $t_k < x_0$. Hence, for all $i = k+1, \ldots, n$, we have $t_i = x_0$. It follows that:

$$\sum_{i=k+1}^{n} |b(t_i) - b(t_{i-1})| = |b(x_0) - b(t_k)|$$

Furthermore, since $t_k \in [y_0, x_0[$, from exercise (14):

$$|b(x_0) - b(t_k)| \leq \frac{\epsilon}{2}$$

We conclude that:

$$\sum_{i=k+1}^{n} |b(t_i) - b(t_{i-1})| \leq \frac{\epsilon}{2} \tag{20}$$

5. Let $t_0 \leq \ldots \leq t_n$, $n \geq 1$, be a finite sequence in $[0, x_0]$. In the case when $t_n = x_0$ and there exists an index $j$ with $0 < j < n-1$ and $t_j = y_0$, we obtain from (18), (19) and (20):

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(y_0) - |b(0)| + \epsilon \tag{21}$$

Our task is to extend (21) to the general case when $t_0 \leq \ldots \leq t_n$ may not satisfy this property. However, it is always possible to 'add points' to the finite sequence $t_0 \leq \ldots \leq t_n$, so as to obtain

$s_0 \leq \ldots \leq s_p$ in $[0, x_0]$ with $\{t_0, \ldots, t_n\} \subseteq \{s_0, \ldots, s_p\}$, such that $s_p = x_0$ and for which there exists $j$ with $0 < j < p-1$ and $s_j = y_0$. Applying (21) to the sequence $s_0 \leq \ldots \leq s_p$:

$$\sum_{i=1}^{p} |b(s_i) - b(s_{i-1})| \leq |b|(y_0) - |b(0)| + \epsilon$$

and since $\{t_0, \ldots, t_n\} \subseteq \{s_0, \ldots, s_p\}$, from exercise (7):

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq \sum_{i=1}^{p} |b(s_i) - b(s_{i-1})|$$

We conclude that (21) is true in the general case.

6. It follows from (21) that $|b|(y_0) - |b(0)| + \epsilon$ is an upper-bound of all sums $\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$ as $t_0 \leq \ldots \leq t_n$ runs through the set of all finite sequences in $[0, x_0]$, $n \geq 1$. Since $|b|(x_0) - |b(0)|$ is the smallest of such upper-bounds, we obtain:

$$|b|(x_0) - |b(0)| \leq |b|(y_0) - |b(0)| + \epsilon$$

and finally $|b|(x_0) \leq |b|(y_0) + \epsilon$.

7. Since $|b|(x_0-) = \sup_{t<x_0} |b|(t)$ and $y_0 \in [0, x_0[$, we have $|b|(y_0) \leq |b|(x_0-)$. It follows from 6. that $|b|(x_0) \leq |b|(x_0-) + \epsilon$. This being true for all $\epsilon > 0$, we obtain $|b|(x_0) \leq |b|(x_0-)$. Having proved in exercise (14) that $|b|(x_0-) \leq |b|(x_0)$, we conclude that $|b|(x_0) = |b|(x_0-)$, i.e.

$$|b|(x_0) = \lim_{t \uparrow\uparrow x_0} |b|(t)$$

This shows that $|b|$ is left-continuous at $x_0$. This being true for all $x_0 \in \mathbf{R}^+ \setminus \{0\}$, we have proved that $|b|$ is a left-continuous.

8. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map of finite variation. If $b$ is right-continuous, then $|b|$ is right-continuous by virtue of exercise (13). If $b$ is left-continuous, we have just proved that $|b|$ is also left-continuous. It follows that if $b$ is continuous then $|b|$ is also continuous. This completes the proof of theorem (81).

Exercise 15

**Exercise 16.**

1. Let $b : \mathbf{R}^+ \to \mathbf{R}$ be an $\mathbf{R}$-valued map of finite variation. We assume that $b$ is right-continuous. From definition (109), the positive variation of $b$ is given by $|b|^+ = (|b| + b)/2$. From theorem (81), $|b|$ is right-continuous. It follows that $|b|^+$ is right-continuous. Similarly, $|b|^- = (|b| - b)/2$ is right continuous.

2. It follows likewise from theorem (81) that if $b$ is left-continuous, then $|b|^+$ and $|b|^-$ are left-continuous. If $b$ is continuous, then $|b|^+$ and $|b|^-$ are continuous.

Exercise 16

**Exercise 17.** Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right continuous map of finite variation. Let $u = Re(b)$ and $v = Im(b)$. Define $b_1 = |u|^+$, $b_2 = |u|^-$, $b_3 = |v|^+$ and $b_4 = |v|^-$. Then $b = b_1 - b_2 + i(b_3 - b_4)$, and each $b_i$ is non-decreasing with $b_i(0) \geq 0$ (see proof of exercise (11)). Furthermore, since $u$ and $v$ are right-continuous maps of finite variation, from exercise (16) we conclude that each $b_i$ is right-continuous.

Exercise 17

**Exercise 18.**

1. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map. Let $t \in \mathbf{R}^+$. For all $p \geq 1$, we define:

$$S_p \stackrel{\triangle}{=} |b(0)| + \sum_{k=1}^{2^p} \left| b\left(\frac{kt}{2^p}\right) - b\left(\frac{(k-1)t}{2^p}\right) \right|$$

Then, given $p \geq 1$, we have:

$$\{kt/2^p \ : \ k = 0, \ldots, 2^p\} \subseteq \{kt/2^{p+1} \ : \ k = 0, \ldots, 2^{p+1}\}$$

and it follows from exercise (7) that:

$$\sum_{k=1}^{2^p} \left| b\left(\frac{kt}{2^p}\right) - b\left(\frac{(k-1)t}{2^p}\right) \right| \leq \sum_{k=1}^{2^{p+1}} \left| b\left(\frac{kt}{2^{p+1}}\right) - b\left(\frac{(k-1)t}{2^{p+1}}\right) \right|$$

We conclude that $S_p \leq S_{p+1}$.

2. It is clear from definition (108) that:

$$\sum_{k=1}^{2^p} \left| b\left(\frac{kt}{2^p}\right) - b\left(\frac{(k-1)t}{2^p}\right) \right| \leq |b|(t) - |b(0)|$$

or equivalently $S_p \leq |b|(t)$. It follows that $|b|(t)$ is an upper-bound of all $S_p$'s. Since $S = \sup_{p\geq 1} S_p$ is the smallest of such upper-bounds, we obtain $S \leq |b|(t)$.

Exercise 18

**Exercise 19.**

1. Let $t_0 < \ldots < t_n$ be a finite sequence of distinct elements of $[0, t]$. Let $\epsilon > 0$. Let $i \in \{0, \ldots, n-1\}$. We want to find an integer $p_i \geq 1$ and some $q_i \in \{0, \ldots, 2^{p_i}\}$, such that $t_i \leq q_i t/2^{p_i} < t_{i+1}$ and:
$$\left| b(t_i) - b\left(\frac{q_i t}{2^{p_i}}\right) \right| \leq \epsilon$$
When $i = n$, we want to find an integer $p_n \geq 1$ and some $q_n \in \{0, \ldots, 2^{p_n}\}$ such that $t_n \leq q_n t/2^{p_n} \leq t$ and:
$$\left| b(t_n) - b\left(\frac{q_n t}{2^{p_n}}\right) \right| \leq \epsilon$$
If $t_n = t$, then $p_n = 1$ and $q_n = 2$ will satisfy our requirements, and we only need to consider the case of $i \in \{0, \ldots, n-1\}$. If $t_n < t$, then we may set $t_{n+1} = t$ and we no longer need to treat the case of $i = n$ separately. Indeed, if we achieve the condition $t_i \leq q_i t/2^{p_i} < t_{i+1}$ for $i = n$, then in particular $t_n \leq q_n t/2^{p_n} \leq t$

will be satisfied. Now, from the right-continuity of $b$ at $t_i$, there exists $s_i > t_i$ such that:

$$u \in [t_i, s_i[ \;\Rightarrow\; |b(t_i) - b(u)| \leq \epsilon$$

Let $u_i = \min(s_i, t_{i+1})$. Then $t_i < u_i \leq t$, and:

$$u \in [t_i, u_i[ \;\Rightarrow\; t_i \leq u < t_{i+1} \text{ and } |b(t_i) - b(u)| \leq \epsilon$$

Hence, all we need to do is find $u \in [t_i, u_i[$ which can be written as some $q_i t/2^{p_i}$. Note that since $0 \leq t_0 < t_1 \leq t$, in particular $t > 0$ and finding $u \in [t_i, u_i[$ of the form $q_i t/2^{p_i}$ is equivalent to finding $u' \in [t_i/t, u_i/t[$ of the form $q_i/2^{p_i}$. In other words, since $0 \leq t_i/t < u_i/t \leq 1$, we are reduced to showing that any interval $[\alpha, \beta[$ where $0 \leq \alpha < \beta \leq 1$, contains a dyadic number of $[0, 1]$, i.e. a number of the form $q/2^p$ where $p \geq 1$ and $q \in \{0, \ldots, 2^p\}$. It is well-known that dyadic numbers are dense in $[0, 1]$ and some of us will be happy to conclude our proof here. For those who do not wish to take the density of dyadic numbers for granted, we may proceed as follows: We assume that $0 \leq \alpha < \beta \leq 1$.

Choose an integer $p \geq 1$ such that $2^{-p} \leq \beta - \alpha$, and consider the set:

$$J = \{r \ : \ r \in \{0, \ldots, 2^p\} \ , \ \beta \leq r/2^p\}$$

Since $r = 2^p \in J$, $J$ is a non-empty subset of $\mathbf{N}$, and therefore has a smallest element, say $q$. Since $\beta > 0$, we have $q \geq 1$ and furthermore:

$$\frac{q-1}{2^p} < \beta \leq \frac{q}{2^p}$$

However, since $\beta - \alpha \geq 1/2^p$, we have:

$$\alpha \leq \beta - \frac{1}{2^p} \leq \frac{q}{2^p} - \frac{1}{2^p} = \frac{q-1}{2^p}$$

It follows that $\alpha \leq (q-1)/2^p < \beta$ and we have proved that any non empty sub-interval $[\alpha, \beta[$ of $[0,1]$ contains a dyadic number. This completes our proof. Coming back to our original problem, we have proved that there exists integers $p_i \geq 1$ and $q_i \in \{0, \ldots, 2^{p_i}\}$, $i = 0, \ldots, n$, such that:

$$0 \leq t_0 \leq \frac{q_0 t}{2^{p_0}} < t_1 \leq \frac{q_1 t}{2^{p_1}} < \ldots < t_n \leq \frac{q_n t}{2^{p_n}} \leq t$$

and:

$$\left| b(t_i) - b\left(\frac{q_i t}{2^{p_i}}\right) \right| \leq \epsilon \ , \ \forall i = 0, \ldots, n$$

2. Define $p = \max_{i=0,\ldots,n} p_i$ and $k_i = q_i 2^{(p-p_i)}$. Then $p \geq 1$ and from $0 \leq q_i \leq 2^{p_i}$ we obtain $0 \leq k_i \leq 2^p$. Furthermore, for all $i = 0, \ldots, n$, we have:

$$\frac{k_i t}{2^p} = q_i 2^{(p-p_i)} \frac{t}{2^p} = \frac{q_i t}{2^{p_i}}$$

We conclude from 1. that:

$$0 \leq t_0 \leq \frac{k_0 t}{2^p} < t_1 \leq \frac{k_1 t}{2^p} < \ldots < t_n \leq \frac{k_n t}{2^p} \leq t$$

and:

$$\left| b(t_i) - b\left(\frac{k_i t}{2^p}\right) \right| \leq \epsilon \ , \ \forall i = 0, \ldots, n$$

3. It follows from the inclusion:

$$\left\{\frac{k_i t}{2^p}\ :\ i=0,\ldots,n\right\} \subseteq \left\{\frac{kt}{2^p}\ :\ k=0,\ldots,2^p\right\}$$

together with exercise (7), that:

$$\sum_{i=1}^{n}\left|b\left(\frac{k_i t}{2^p}\right)-b\left(\frac{k_{i-1}t}{2^p}\right)\right| \leq \sum_{k=1}^{2^p}\left|b\left(\frac{kt}{2^p}\right)-b\left(\frac{(k-1)t}{2^p}\right)\right|$$

or equivalently:

$$\sum_{i=1}^{n}\left|b\left(\frac{k_i t}{2^p}\right)-b\left(\frac{k_{i-1}t}{2^p}\right)\right| \leq S_p - |b(0)| \qquad (22)$$

4. Let $i\in\{1,\ldots,n\}$. Then:

$$\begin{aligned}|b(t_i)-b(t_{i-1})| \ \leq\ & \left|b(t_i)-b\left(\frac{k_i t}{2^p}\right)\right| + \left|b\left(\frac{k_i t}{2^p}\right)-b\left(\frac{k_{i-1}t}{2^p}\right)\right| \\ +\ & \left|b(t_{i-1})-b\left(\frac{k_{i-1}t}{2^p}\right)\right|\end{aligned}$$

$$\leq \; 2\epsilon + \left| b\left(\frac{k_i t}{2^p}\right) - b\left(\frac{k_{i-1} t}{2^p}\right) \right|$$

and consequently from (22):

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq S_p - |b(0)| + 2n\epsilon \tag{23}$$

Since $S = \sup_{p\geq 1} S_p$, in particular $S_p \leq S$, and we obtain:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq S - |b(0)| + 2n\epsilon \tag{24}$$

5. Having proved (24) for arbitrary $\epsilon > 0$, we conclude that:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq S - |b(0)| \tag{25}$$

6. Let $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[0, t]$, $n \geq 1$. If $\text{card}\{t_0, \ldots, t_n\} = 1$, then all $t_i$'s are equal and (25) is true.

We assume that card$\{t_0, \ldots, t_n\} \geq 2$. Let $s_0 < \ldots < s_p$ be distinct in $[0, t]$ such that $\{s_0, \ldots, s_p\} = \{t_0, \ldots, t_n\}$. Then, inequality (25) holds for the $s_j$'s, i.e.:

$$\sum_{j=1}^{p} |b(s_j) - b(s_{j-1})| \leq S - |b(0)| \tag{26}$$

However, from exercise (7), since $\{t_0, \ldots, t_n\} \subseteq \{s_0, \ldots, s_p\}$:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq \sum_{j=1}^{p} |b(s_j) - b(s_{j-1})|$$

and it follows that (25) is true for the $t_i$'s. Hence, we have proved that (25) holds for all finite sequences $t_0 \leq \ldots \leq t_n$ in $[0, t]$, $n \geq 1$. In other words, $S - |b(0)|$ is an upper-bound of all sums $\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$ as $t_0 \leq \ldots \leq t_n$ runs through the set of all finite sequences in $[0, t]$, $n \geq 1$. Since $|b|(t) - |b(0)|$ is the smallest of such upper-bounds, we obtain $|b|(t) - |b(0)| \leq S - |b(0)|$, and finally $|b|(t) \leq S$.

7. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous. Let $t \in \mathbf{R}^+$. From 6. we have $|b|(t) \leq S$ and we have proved in exercise (18) that $S \leq |b|(t)$. It follows that $|b|(t) = S$. Furthermore, still from exercise (18), the sequence $(S_p)_{p\geq 1}$ is non-decreasing. Hence:

$$S = \sup_{p\geq 1} S_p = \lim_{p\to+\infty} S_p \in [0, +\infty]$$

we conclude that $|b|(t) = \lim_{p\to+\infty} S_p$, or equivalently:

$$|b|(t) = |b(0)| + \lim_{n\to+\infty} \sum_{k=1}^{2^n} \left| b\left(\frac{kt}{2^n}\right) - b\left(\frac{(k-1)t}{2^n}\right)\right| \qquad (27)$$

This completes the proof of theorem (82) in the case when $b$ is right-continuous. We now assume that $b$ is left-continuous instead of right-continuous. In order to prove (27), we need to show that given $t \in \mathbf{R}^+$, we have $|b|(t) = S$. It is clear that $S \leq |b|(t)$ still holds, so we need to prove the reverse inequality $|b|(t) \leq S$, which we shall do with a very similar argument to that contained in 1. to 6.. Let $\epsilon > 0$ be given, and suppose

$t_0 < \ldots < t_n$ is a finite sequence of distinct elements of $[0, t]$. From the left-continuity of $b$, there exists integers $p_i \geq 1$ and $q_i \in \{0, \ldots, 2^{p_i}\}$ such that:

$$0 \leq \frac{q_0 t}{2^{p_0}} \leq t_0 < \frac{q_1 t}{2^{p_1}} \leq t_1 < \ldots < \frac{q_n t}{2^{p_n}} \leq t_n \leq t$$

and:

$$\left| b(t_i) - b\left(\frac{q_i t}{2^{p_i}}\right) \right| \leq \epsilon \ , \ \forall i = 0, \ldots, n$$

Note that some extra care is required for $t_0$. Indeed, if $t_0 = 0$, then there is no such thing as the *left-continuity* of $b$ at $t_0$. However, $p_0 = 1$ and $q_0 = 0$ will satisfy our requirements. Having found the $p_i$'s and the $q_i$'s, we then define $p = \max_{i=0,\ldots,n} p_i$ and $k_i = q_i 2^{(p-p_i)}$. Then $p \geq 1$, $0 \leq k_i \leq 2^p$ and furthermore:

$$0 \leq \frac{k_0 t}{2^p} \leq t_0 < \frac{k_1 t}{2^p} \leq t_1 < \ldots < \frac{k_n t}{2^p} \leq t_n \leq t$$

and:

$$\left| b(t_i) - b\left(\frac{k_i t}{2^p}\right) \right| \leq \epsilon \ , \ \forall i = 0, \ldots, n$$

Using exercise (7), we then argue that:

$$\sum_{i=1}^{n} \left| b\left(\frac{k_i t}{2^p}\right) - b\left(\frac{k_{i-1} t}{2^p}\right) \right| \leq S_p - |b(0)|$$

from which we obtain, just like in 4. and 5.:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq S - |b(0)|$$

This being true when the $t_i$'s are distinct, is in fact true in general, and we conclude that $|b|(t) \leq S$. This completes the proof of theorem (82).

Exercise 19

**Exercise 20.** Let $b : \mathbf{R}^+ \to \mathbf{R}^+$ be defined by $b = 1_{\mathbf{Q}^+}$. Since for all $n \geq 1$ and $k = 0, \ldots, n$, the number $k/2^n$ is rational, we have:

$$\sum_{k=1}^{2^n} \left| b\left(\frac{k}{2^n}\right) - b\left(\frac{k-1}{2^n}\right) \right| = 0$$

However, we claim that $|b|(1) = +\infty$. Let $n \geq 1$. Define:

$$t_0 = 0, t_2 = \frac{1}{n}, t_4 = \frac{2}{n}, \ldots, t_{2n} = \frac{n}{n} = 1$$

and for all $k \in \{1, \ldots, n\}$, let $t_{2k-1}$ be an arbitrary irrational number in $]t_{2k-2}, t_{2k}[$. The fact that such irrational number exists, stems from the density of irrational numbers in $[0, 1]$, which we shall admit in this tutorial. Hence, we have a finite sequence $t_0 \leq t_1 \leq \ldots \leq t_{2n}$ in $[0, 1]$, such that:

$$\sum_{i=1}^{2n} |b(t_i) - b(t_{i-1})| = 2n$$

It follows that $2n \leq |b|(1)$, and this being true for all $n \geq 1$, we conclude that $|b|(1) = +\infty$. We have proved that:

$$+\infty = |b|(1) \neq \lim_{n \to +\infty} \sum_{k=1}^{2^n} |b(k/2^n) - b((k-1)/2^n)| = 0$$

The purpose of this exercise is to illustrate the fact that the conclusion of theorem (82) may not hold. This obviously does not contradict theorem (82), as the map $1_{\mathbf{Q}^+}$ is neither left, nor right-continuous.

Exercise 20

**Exercise 21.**

1. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of bounded variation. Let $b_1 = Re(b)$ and $b_2 = Im(b)$. Then, $b_1$ and $b_2$ are both right-continuous of bounded variations, and in particular right-continuous of finite variations. Their positive and negative variations $|b_1|^+$, $|b_1|^-$, $|b_2|^+$ and $|b_2|^-$ are right-continuous, non-decreasing with non-negative initial values (see exercises (10) and (16)). It follows from definition (24) that the Stieltjes measures $d|b_1|^+$, $d|b_1|^-$, $d|b_2|^+$ and $d|b_2|^-$ are all well-defined measures on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$.

2. Yes. It is still true if $b$ is right-continuous of finite variation. The assumption that $b$ is in fact of bounded variation has not been used in 1.

3. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$, and let $da$ be its associated Stieltjes measure (see

definition (24)). Then for all $n \geq 1$, we have:

$$da([0,n]) = da(\{0\}) + da(]0,n]) = a(0) + a(n) - a(0) = a(n)$$

Furthermore, since $[0,n] \uparrow \mathbf{R}^+$, using theorem (7):

$$da(\mathbf{R}^+) = \lim_{n\to+\infty} da([0,n])$$

It follows that $da(\mathbf{R}^+) = \lim_{n\to+\infty} a(n) = a(\infty)$. So $da$ is a finite measure, if and only if $a(\infty) < +\infty$. Now, $b$ being of bounded variation, we have:

$$|b|(\infty) = \lim_{t\to+\infty} |b|(t) = \sup_{t\in\mathbf{R}^+} |b|(t) < +\infty$$

From exercise (2) we have $|b_1| \leq |b|$ and $|b_2| \leq |b|$. Furthermore from exercise (10), $|b_1| = |b_1|^+ + |b_1|^-$ and $|b_2| = |b_2|^+ + |b_2|^-$. In particular, it follows that $|b_1|^+ \leq |b|$ and consequently:

$$|b_1|^+(\infty) \leq |b|(\infty) < +\infty$$

We conclude that $d|b_1|^+$ is a finite measure on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$. Similarly, $d|b_1|^-$, $d|b_2|^+$ and $d|b_2|^-$ are all finite measures.

4. We define:

$$db = d|b_1|^+ - d|b_1|^- + i(d|b_2|^+ - d|b_2|^-) \tag{28}$$

Since $d|b_1|^+$, $d|b_1|^-$, $d|b_2|^+$ and $d|b_2|^-$ are all finite measures, in particular they are complex measures on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$, i.e. elements of the $\mathbf{C}$-vector space $M^1(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$. $db$ being defined by (28) as a linear combinations of elements of $M^1(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$, is a well-defined complex measure on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$.

5. From (28) and definition (24), we have:

$$\begin{aligned} db(\{0\}) &= d|b_1|^+(\{0\}) - d|b_1|^-(\{0\}) \\ &+ i(d|b_2|^+(\{0\}) - d|b_2|^-(\{0\})) \\ &= |b_1|^+(0) - |b_1|^-(0) + i(|b_2|^+(0) - |b_2|^-(0)) \\ &= b_1(0) + ib_2(0) = b(0) \end{aligned}$$

6. Let $s,t\in\mathbf{R}^+$, $s\leq t$. From (28) and definition (24):

$$\begin{aligned} db(]s,t]) &= d|b_1|^+(]s,t]) - d|b_1|^-(]s,t]) \\ &+ i(d|b_2|^+(]s,t]) - d|b_2|^-(]s,t])) \\ &= |b_1|^+(t) - |b_1|^+(s) - |b_1|^-(t) + |b_1|^-(s) \\ &+ i(|b_2|^+(t) - |b_2|^+(s) - |b_2|^-(t) + |b_2|^-(s)) \\ &= b_1(t) - b_1(s) + i(b_2(t) - b_2(s)) \\ &= b(t) - b(s) \end{aligned}$$

7. Since $b = |b_1|^+ - |b_1|^- + i(|b_2|^+ - |b_2|^-)$ and $|b_1|^+$, $|b_1|^-$, $|b_2|^+$ and $|b_2|^-$ all have finite limits as $t\to+\infty$ (see 3.), we conclude that $\lim_{t\to+\infty} b(t)$ exists in $\mathbf{C}$. This limit is denoted $b(\infty)$.

8. Since $[0,n]\uparrow\mathbf{R}^+$, in particular $1_{[0,n]}\to 1_{\mathbf{R}^+}=1$ and using exercise (13) of Tutorial 12, $db([0,n])\to db(\mathbf{R}^+)$. Hence:

$$\begin{aligned} db(\mathbf{R}^+) &= \lim_{n\to+\infty} db([0,n]) \\ &= \lim_{n\to+\infty} (db(\{0\}) + db(]0,n])) \end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to+\infty}(b(0)+b(n)-b(0)) \\
&= \lim_{n\to+\infty} b(n) = b(\infty)
\end{aligned}
$$

9. Given $b : \mathbf{R}^+ \to \mathbf{C}$ right-continuous of bounded variation, we have seen that $db$ is a complex measure on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$ with:

$$
\begin{array}{ll}
(i) & db(\{0\}) = b(0) \\
(ii) & \forall s,t\in\mathbf{R}^+\ s\leq t\ ,\ db(]s,t]) = b(t)-b(s)
\end{array}
$$

This proves the existence property stated in definition (110). To prove the uniqueness, we shall use a standard argument based on Dynkin systems. Suppose $\mu$ and $\nu$ are two complex measures on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$ such that $\mu(\{0\}) = \nu(\{0\})$ and $\mu(]s,t]) = \nu(]s,t])$ for all $s,t\in\mathbf{R}^+$, $s\leq t$. Define:

$$
\mathcal{D} = \{B\in\mathcal{B}(\mathbf{R}^+)\ :\ \mu(B)=\nu(B)\}
$$

and let:

$$
\mathcal{C} = \{\{0\}\}\cup\{]s,t]\ :\ s,t\in\mathbf{R}^+, s\leq t\}
$$

By assumption $\mathcal{C} \subseteq \mathcal{D}$, and since $\mathcal{C}$ is closed under finite intersection while $\mathcal{D}$ is a Dynkin system on $\mathbf{R}^+$, from the Dynkin system theorem (1) we obtain $\sigma(\mathcal{C}) \subseteq \mathcal{D}$. Finally, since $\sigma(\mathcal{C}) = \mathcal{B}(\mathbf{R}^+)$, we have $\mathcal{B}(\mathbf{R}^+) \subseteq \mathcal{D}$ which shows that $\mu = \nu$. This proves the uniqueness property stated in definition (110). It may be that some of us think this proof of the uniqueness property was a little bit short, as some of the key points have not been justified. The fact that $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$ was already proved in detail in Tutorial 3, and it is pretty straightforward anyway. The fact that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbf{R}^+)$ is the object of exercise (20) in Tutorial 3. As an alternative quick proof, it is by now known that $\mathcal{C}' = \{]s,t] : s,t \in \mathbf{R}\ s \leq t\}$ generates the $\sigma$-algebra on $\mathbf{R}$, i.e. $\sigma(\mathcal{C}') = \mathcal{B}(\mathbf{R})$. However, any element of $\mathcal{C}'_{|\mathbf{R}^+}$, the trace of $\mathcal{C}'$ on $\mathbf{R}^+$, is of the form $]s,t]$ or $\{0\}\cup]0,t]$ with $s,t \in \mathbf{R}^+$. Hence, it is a simple exercise to show that $\sigma(\mathcal{C}) = \sigma(\mathcal{C}'_{|\mathbf{R}^+})$. Using the trace theorem (10) we obtain:

$$\sigma(\mathcal{C}) = \sigma(\mathcal{C}'_{|\mathbf{R}^+}) = \sigma(\mathcal{C}')_{|\mathbf{R}^+} = \mathcal{B}(\mathbf{R})_{|\mathbf{R}^+} = \mathcal{B}(\mathbf{R}^+)$$

The fact that $\mathcal{D}$ is a Dynkin system on $\mathbf{R}^+$ can be seen as follows:

$$\mu(\mathbf{R}^+) = \lim_{n\to+\infty}\mu([0,n]) = \lim_{n\to+\infty}\nu([0,n]) = \nu(\mathbf{R}^+)$$

So $\mathbf{R}^+ \in \mathcal{D}$. Furthermore, if $A, B \in \mathcal{D}$, $A \subseteq B$, then:

$$\mu(B \setminus A) = \mu(B) - \mu(A) = \nu(B) - \nu(A) = \nu(B \setminus A)$$

So $B \setminus A \in \mathcal{D}$. Finally if $A_n \in \mathcal{D}$ and $A_n \uparrow A$, then in particular $1_{A_n} \to 1_A$ and from exercise (13) of Tutorial 12, we have:

$$\mu(A) = \lim_{n\to+\infty}\mu(A_n) = \lim_{n\to+\infty}\nu(A_n) = \nu(A)$$

which shows that $A \in \mathcal{D}$. This really completes our proof of the uniqueness property stated in definition (110).

Exercise 21

**Exercise 22.** Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. From definition (24), the Stieltjes measure $da$ associated with $a$ is well-defined. However, if we assume that $a(\infty) < +\infty$, from exercise (1) $|a| = a$, and $a$ is therefore right-continuous of bounded variation. According to definition (110), the notation '$da$' refers to the so-called *complex Stieltjes measure* associated with $a$. Hence, we are in a situation where because $a$ can be viewed both as *right-continuous, non-decreasing with* $a(0) \geq 0$ and *right-continuous of bounded variation*, the notation '$da$' is potentially ambiguous as its meaning is derived from two possibly conflicting definitions (24) and (110). The purpose of this exercise is to show that in fact, no conflict arises. Let $\mu$ be the Stieltjes measure on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$ associated with $a$, as per definition (24), and $\nu$ be the complex Stieltjes measure associated with $a$, as per definition (110). Then, we have:

$$(i) \qquad \mu(\{0\}) = \nu(\{0\}) = a(0)$$

$$(ii) \qquad \forall s,t \in \mathbf{R}^+ \ s \leq t \ , \ \mu(]s,t]) = \nu(]s,t]) = a(t) - a(s)$$

However, $\mu(\mathbf{R}^+) = a(\infty) < +\infty$ and $\mu$ is therefore a finite measure. In particular, it is a complex measure on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$. From the uniqueness property of definition (110), it follows that $\mu = \nu$. So the *Stieltjes measure* associated with $a$, coincides with its *complex Stieltjes measure*, and there is no conflict regarding the notation '$da$'.

Exercise 22

**Exercise 23.**

1. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. Let $b_1 = Re(b)$ and $b_2 = Im(b)$. Then $b_1$ and $b_2$ are right-continuous maps of finite variation, and their negative and positive variations $|b_1|^+$, $|b_1|^-$, $|b_2|^+$ and $|b_2|^-$ are all right-continuous, non-decreasing with non-negative initial values. By virtue of definition (24), $d|b_1|^+$, $d|b_1|^-$, $d|b_2|^+$ and $d|b_2|^-$ are all well-defined measures on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$.

2. It is impossible to define $db = d|b_1|^+ - d|b_1|^- + i(d|b_2|^+ - d|b_2|^-)$, because $d|b_1|^+$, $d|b_1|^-$, $d|b_2|^+$ and $d|b_2|^-$ are not necessarily finite measures, and any algebraic expression involving $+\infty - (+\infty)$ makes no sense. To ensure that $d|b_1|^+$, $d|b_1|^-$, $d|b_2|^+$ and $d|b_2|^-$ be finite measures, we have to assume that $b$ is not just of finite variation, but also of bounded variation.

Exercise 23

**Exercise 24.**

1. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map and $T \in \mathbf{R}^+$. Let $b^T : \mathbf{R}^+ \to \mathbf{C}$ be the map defined by $b^T(t) = b(T \wedge t)$ for all $t \in \mathbf{R}^+$. Let $t \in \mathbf{R}^+$ and $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[0, t]$, $n \geq 1$. Then $T \wedge t_0 \leq \ldots \leq T \wedge t_n$ is a finite sequence in $[0, T \wedge t]$, $n \geq 1$. Hence, from definition (108):

$$\sum_{i=1}^{n} |b(T \wedge t_i) - b(T \wedge t_{i-1})| \leq |b|(T \wedge t) - |b(0)|$$

or equivalently:

$$\sum_{i=1}^{n} |b^T(t_i) - b^T(t_{i-1})| \leq |b|^T(t) - |b(0)|$$

It follows that $|b|^T(t) - |b(0)|$ is an upper-bound of all sums $\sum_{i=1}^{n} |b^T(t_i) - b^T(t_{i-1})|$ as $t_0 \leq \ldots \leq t_n$ runs through the set of all finite sequences in $[0, t]$, $n \geq 1$. Since $|b^T|(t) - |b^T(0)|$ is the smallest of such upper-bounds, we obtain:

$$|b^T|(t) - |b^T(0)| \leq |b|^T(t) - |b(0)|$$

Since $b^T(0) = b(0)$ we finally have $|b^T|(t) \leq |b|^T(t)$. To show the reverse inequality, let $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[0, T \wedge t]$, $n \geq 1$. Then:

$$\begin{aligned} \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| &= \sum_{i=1}^{n} |b^T(t_i) - b^T(t_{i-1})| \\ &\leq |b^T|(T \wedge t) - |b^T(0)| \\ &\leq |b^T|(t) - |b(0)| \end{aligned}$$

It follows that $|b^T|(t) - |b(0)|$ is an upper-bound of all sums $\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$ as $t_0 \leq \ldots \leq t_n$ runs through the set of all finite sequences in $[0, T \wedge t]$, $n \geq 1$. Since $|b|(T \wedge t) - |b(0)|$ is the smallest of such upper-bounds, we obtain:

$$|b|(T \wedge t) - |b(0)| \leq |b^T|(t) - |b(0)|$$

i.e. $|b|^T(t) \leq |b^T|(t)$. Finally, we have proved that $|b^T|(t) = |b|^T(t)$. This being true for all $t \in \mathbf{R}^+$, we have $|b^T| = |b|^T$.

2. Suppose $b$ is of finite variation. Then $|b|(t) < +\infty$ for all $t \in \mathbf{R}^+$. Let $T \in \mathbf{R}^+$. Using 1. we obtain:

$$\begin{aligned} |b^T|(\infty) &= \lim_{t \to +\infty} |b^T|(t) \\ &= \lim_{t \to +\infty} |b|^T(t) \\ &= \lim_{t \to +\infty} |b|(T \wedge t) \\ &= |b|(T) < +\infty \end{aligned}$$

So $b^T$ is a map of bounded variation.

3. Suppose $b$ is right-continuous of finite variation. Let $T \in \mathbf{R}^+$. From 2. $b^T$ is right-continuous of bounded variation. From definition (110), its associated complex Stieltjes measure $db^T$ is well-defined, and is the unique complex measure on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$

such that:

$$(i) \qquad db^T(\{0\}) = b^T(0)$$
$$(ii) \qquad \forall s,t \in \mathbf{R}^+, s \leq t \ , \ db^T(]s,t]) = b^T(t) - b^T(s)$$

In other words, it is the unique complex measure such that:

$$(i) \qquad db^T(\{0\}) = b(0)$$
$$(ii) \qquad \forall s,t \in \mathbf{R}^+, s \leq t \ , \ db^T(]s,t]) = b(T\wedge t) - b(T\wedge s)$$

4. Suppose $b$ is **R**-valued of finite variation. Let $T \in \mathbf{R}^+$. Using 1. together with definition (109) we obtain for all $t \in \mathbf{R}^+$:

$$\begin{aligned} |b^T|^+(t) &= \frac{1}{2}(|b^T|(t) + b^T(t)) \\ &= \frac{1}{2}(|b|^T(t) + b^T(t)) \\ &= \frac{1}{2}(|b|(T\wedge t) + b(T\wedge t)) \\ &= |b|^+(T\wedge t) = (|b|^+)^T(t) \end{aligned}$$

So $|b^T|^+ = (|b|^+)^T$ and similarly, the negative variation $|b^T|^-$ of $b^T$ is given by $|b^T|^- = (|b|^-)^T$.

5. Suppose $b$ is right-continuous of bounded variation. Then its associated complex Stieltjes measure $db$ is well-defined as per definition (110). Let $db^{[0,T]}$ be the complex measure defined by:

$$\forall B \in \mathcal{B}(\mathbf{R}^+)\ ,\ db^{[0,T]} \stackrel{\triangle}{=} db([0,T] \cap B)$$

Then, we have:

$$db^{[0,T]}(\{0\}) = db([0,T] \cap \{0\}) = db(\{0\}) = b(0)$$

and for all $s, t \in \mathbf{R}^+$, $s \leq t$:

$$\begin{aligned} db^{[0,T]}(]s,t]) &= db([0,T] \cap ]s,t]) \\ &= db(]T \wedge s, T \wedge t]) \\ &= b(T \wedge t) - b(T \wedge s) \end{aligned}$$

Hence, from the uniqueness property of 3., $db^T = db^{[0,T]}$.

6. Suppose $b$ is right-continuous, non-decreasing with $b(0) \geq 0$. In particular, $b$ is right-continuous of finite variation, and from 3. $db^T$ is the unique complex measure on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$ such that:

$$(i) \qquad db^T(\{0\}) = b(0)$$
$$(ii) \qquad \forall s,t \in \mathbf{R}^+, s \leq t \ , \ db^T(]s,t]) = b(T \wedge t) - b(T \wedge s)$$

However, the Stieltjes measure $db$ is well-defined as per definition (24), and similarly to 5. we have $db^{[0,T]}(\{0\}) = b(0)$, with:

$$db^{[0,T]}(]s,t]) = b(T \wedge t) - b(T \wedge s)$$

Furthermore:

$$db^{[0,T]}(\mathbf{R}^+) = db([0,T]) = b(T) < +\infty$$

and consequently $db^{[0,T]}$ is a finite measure, and in particular a complex measure on $\mathbf{R}^+$. From the uniqueness property of 3. we conclude that $db^{[0,T]} = db^T$.

Exercise 24

**Exercise 25.**

1. Let $\mu, \nu$ be two finite measures on $\mathbf{R}^+$ such that:

$$\begin{array}{ll} (i) & \mu(\{0\}) \leq \nu(\{0\}) \\ (ii) & \forall s, t \in \mathbf{R}^+, s \leq t\ ,\ \mu(]s,t]) \leq \nu(]s,t]) \end{array}$$

   Let $a, c : \mathbf{R}^+ \to \mathbf{R}^+$ be defined by $a(t) = \mu([0,t])$ and $c(t) = \nu([0,t])$. Then $a(0) = \mu(\{0\}) \geq 0$ and similarly $c(0) \geq 0$. Let $s, t \in \mathbf{R}^+$, $s \leq t$. Then, we have:

$$\begin{aligned} a(t) &= \mu([0,t]) \\ &= \mu([0,s]) + \mu(]s,t]) \\ &\geq \mu([0,s]) = a(s) \end{aligned}$$

   So $a$ is non-decreasing, and similarly $c$ is non-decreasing. Let $t \in \mathbf{R}^+$ and $(t_n)_{n\geq 1}$ be an arbitrary sequence in $\mathbf{R}^+$ such that $t_n \downarrow\downarrow t$ (i.e. $t_n \to t$ and $t < t_{n+1} \leq t_n$ for all $n \geq 1$). Then, $[0,t_n] \downarrow [0,t]$, and since $\mu$ is a finite measure, from theorem (8) we have:

$$\mu([0,t]) = \lim_{n\to+\infty} \mu([0,t_n])$$

It follows that $a(t) = \lim_{n\to+\infty} a(t_n)$, which shows that $a$ is right-continuous. Similarly, $c$ is right-continuous.

2. Let $da$ be the Stieltjes measure associated with $a$ as per definition (24). We have $\mu(\{0\}) = a(0) = da(\{0\})$ and since $\mu$ is a finite measure, for all $s,t \in \mathbf{R}^+$, $s \leq t$:

$$\begin{aligned} \mu(]s,t]) &= \mu([0,t]) - \mu([0,s]) \\ &= a(t) - a(s) = da(]s,t]) \end{aligned}$$

From the uniqueness property of definition (24), we conclude that $da = \mu$. Similarly $dc = \nu$.

3. For all $t \in \mathbf{R}^+$, we have:

$$\begin{aligned} a(t) &= \mu([0,t]) \\ &= \mu(\{0\}) + \mu(]0,t]) \end{aligned}$$

$$\begin{aligned} &\leq \nu(\{0\}) + \nu(]0,t]) \\ &= \nu([0,t]) = c(t) \end{aligned}$$

which shows that $a \leq c$.

4. Let $b = c - a$. Since $a$ and $c$ are right-continuous, $b$ is also right-continuous. Since $a \leq c$, in particular $a(0) \leq c(0)$ and consequently $b(0) \geq 0$. Let $s, t \in \mathbf{R}^+$, $s \leq t$. We have:

$$\begin{aligned} b(t) &= c(t) - a(t) \\ &= \nu([0,t]) - \mu([0,t]) \\ &= \nu([0,s]) - \mu([0,s]) + \nu(]s,t]) - \mu(]s,t]) \\ &= c(s) - a(s) + \nu(]s,t]) - \mu(]s,t]) \\ &\geq c(s) - a(s) = b(s) \end{aligned}$$

which shows that $b$ is non-decreasing.

5. Let $db$ be the Stieltjes measure associated with $b$ as per defini-

tion (24). Then $da + db$ is a measure on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$, and:

$$(da + db)(\{0\}) = da(\{0\}) + db(\{0\}) = a(0) + b(0) = c(0)$$

Furthermore, for all $s, t \in \mathbf{R}^+$, $s \leq t$:

$$\begin{aligned}(da + db)(]s, t]) &= da(]s, t]) + db(]s, t]) \\ &= a(t) - a(s) + b(t) - b(s) \\ &= c(t) - c(s)\end{aligned}$$

From the uniqueness property of definition (24), we conclude that $da + db = dc$.

6. It follows from 5. that for all $B \in \mathcal{B}(\mathbf{R}^+)$, we have:

$$dc(B) = da(B) + db(B)$$

In particular $da(B) \leq dc(B)$, and since $da = \mu$ and $dc = \nu$, we have proved that $\mu(B) \leq \nu(B)$. This proves theorem (83).

Exercise 25

**Exercise 26.**

1. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of bounded variation. Let $db$ be its associated complex Stieltjes measure, as per definition (110). Let $d|b|$ be the Stieltjes measure associated with the total variation map $|b|$, as per definition (24). Then, we have $d|b|(\{0\}) = |b|(0) = |b(0)|$. Furthermore, since $E_1 = \{0\}$, $E_n = \emptyset$, $n \geq 1$, defines a measurable partition of $\{0\}$ (see definition (91)), we have from definition (94):

$$|b(0)| = |db(\{0\})| = \sum_{n=1}^{+\infty} |db(E_n)| \leq |db|(\{0\})$$

where $|db|$ denotes the total variation measure $|db|$ of the complex measure $db$. Furthermore, if $(E_n)_{n\geq 1}$ is an arbitrary measurable partition of $\{0\}$, then $\{0\} = E_n$ for some $n \geq 1$, and it is easy to see that $E_m = \emptyset$ for $m \neq n$. Hence:

$$\sum_{n=1}^{+\infty} |db(E_n)| = |db(\{0\})| = |b(0)|$$

In particular $|b(0)|$ is an upper-bound of all sums $\sum_{n=1}^{+\infty} |db(E_n)|$ as $(E_n)_{n\geq 1}$ runs through the set of all measurable partitions of $\{0\}$. From definition (94), $|db|(\{0\})$ is the smallest of such upper-bounds, and consequently $|db|(\{0\}) \leq |b(0)|$. Finally, we have proved that $|db|(\{0\}) = |b(0)| = d|b|(\{0\})$.

2. Let $s, t \in \mathbf{R}^+$, $s \leq t$. Let $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[s, t]$, $n \geq 1$. Then, the sequence $]s, t_0]$, $]t_0, t_1]$,$\ldots$,$]t_{n-1}, t_n]$, $]t_n, t]$, $\emptyset$,... constitutes a measurable partition of $]s, t]$. Hence:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| = \sum_{i=1}^{n} |db(]t_{i-1}, t_i])|$$

$$\leq |db(]s,t_0])| + \sum_{i=1}^{n} |db(]t_{i-1},t_i])| + |db(]t_n,t])|$$
$$\leq |db|(]s,t])$$

3. It follows from 2. that $|db|(]s,t])$ is an upper-bound of all sums $\sum_{i=1}^n |b(t_i) - b(t_{i-1})|$ as $t_0 \leq \ldots \leq t_n$ runs through the set of all finite sequences in $[s,t]$, $n \geq 1$. Since from theorem (80), $|b|(t) - |b|(s)$ is the smallest of such upper-bounds, we obtain:

$$|b|(t) - |b|(s) \leq |db|(]s,t])$$

4. From 3. we have for all $s,t \in \mathbf{R}^+$, $s \leq t$:

$$d|b|(]s,t]) = |b|(t) - |b|(s) \leq |db|(]s,t])$$

Furthermore from 1. $d|b|(\{0\}) = |db|(\{0\})$ and in particular $d|b|(\{0\}) \leq |db|(\{0\})$. Moreover, from theorem (57), the total variation $|db|$ is a finite measure on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$, and since

$b$ is of bounded variation:

$$d|b|(\mathbf{R}^+) = \lim_{n\to+\infty} d|b|([0,n]) = \lim_{n\to+\infty} |b|(n) = |b|(\infty) < +\infty$$

So $d|b|$ is also a finite measure on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$. Applying theorem (83), we conclude that $d|b| \leq |db|$.

5. Let $f \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$. Then $f$ is measurable, and using $d|b| \leq |db|$ together with exercise (18) of Tutorial 12, we obtain:

$$\int |f| d|b| \leq \int |f||db| < +\infty$$

So $f \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), d|b|)$.

6. From theorem (12), since $\mathbf{R}$ is metrizable, $\mathbf{R}^+$ is also metrizable. Furthermore, if $V_n = [0, n[$, $n \geq 1$, then $(V_n)_{n\geq 1}$ is a sequence of open subsets of $\mathbf{R}^+$ with compact closure, such that $V_n \uparrow \mathbf{R}^+$. From definition (104), it follows that $\mathbf{R}^+$ is strongly $\sigma$-compact.

7. Since $\mathbf{R}^+$ is metrizable and $|db|$ is a finite measure, from theorem (70) the set of continuous and bounded functions $C^b_{\mathbf{C}}(\mathbf{R}^+)$ is dense in $L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$. Since $\mathbf{R}^+$ is metrizable and strongly $\sigma$-compact, since $|db|$ is a finite measure, in particular $|db|$ is locally finite. From theorem (78) the space of continuous functions with compact support $C^c_{\mathbf{C}}(\mathbf{R}^+)$ is dense in $L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$.

8. Let $h \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$. Let $\epsilon > 0$. From the density of $C^b_{\mathbf{C}}(\mathbf{R}^+)$ obtained in 7. there exists $\phi \in C^b_{\mathbf{C}}(\mathbf{R}^+)$ such that:

$$\int |\phi - h||db| \leq \epsilon \tag{29}$$

9. Using (29) and exercise (16) of Tutorial 12:

$$\begin{aligned} \left|\int hdb\right| - \left|\int \phi db\right| &\leq \left|\left|\int \phi db\right| - \left|\int hdb\right|\right| \\ &\leq \left|\int \phi db - \int hdb\right| \end{aligned}$$

$$
\begin{aligned}
&= \left|\int (\phi - h)db\right| \\
&\leq \int |\phi - h||db| \leq \epsilon
\end{aligned}
$$

and we conclude that:

$$\left|\int hdb\right| \leq \left|\int \phi db\right| + \epsilon \tag{30}$$

10. Since $d|b| \leq |db|$, using exercise (18) of Tutorial 12:

$$
\begin{aligned}
\left|\int |\phi|d|b| - \int |h|d|b|\right| &= \left|\int (|\phi| - |h|)d|b|\right| \\
&\leq \int |\,|\phi| - |h|\,|d|b| \\
&\leq \int |\phi - h|d|b| \\
&\leq \int |\phi - h||db|
\end{aligned}
$$

11. Using 10 and (29) we obtain:

$$\begin{aligned}\int |\phi|d|b| - \int |h|d|b| &\leq \left|\int |\phi|d|b| - \int |h|d|b|\right| \\ &\leq \int |\phi - h||db| \leq \epsilon\end{aligned}$$

and consequently:

$$\int |\phi|d|b| \leq \int |h|d|b| + \epsilon \tag{31}$$

12. For all $n \geq 1$, we define:

$$\phi_n \stackrel{\triangle}{=} \phi(0)1_{\{0\}} + \sum_{k=0}^{n2^n-1} \phi(k/2^n)1_{]k/2^n,(k+1)/2^n]}$$

Since $\phi \in C^b_{\mathbf{C}}(\mathbf{R}^+)$, there exists $M \in \mathbf{R}^+$ such that $|\phi(x)| \leq M$ for all $x \in \mathbf{R}^+$. Then $|\phi_n(x)| \leq M$ for all $x \in \mathbf{R}^+$ and $n \geq 1$.

13. Let $t \in \mathbf{R}^+$. If $t = 0$ then $\phi_n(t) = \phi_n(0) = \phi(0) = \phi(t)$ for all $n \geq 1$, and it is clear that $\phi_n(t) \to \phi(t)$. We assume that $t > 0$. Since $\phi$ is continuous at $t$, given $\delta > 0$ there exists $\eta > 0$ with:

$$|t - t'| < \eta \;\Rightarrow\; |\phi(t) - \phi(t')| \leq \delta$$

Choose $N \geq 1$ large enough so that $2^{-N} < \eta$ and $t \in [0, N]$. Then, for all $n \geq N$, there exists $k \in \{0, \ldots, n2^n - 1\}$ such that $t \in ]k/2^n, (k+1)/2^n]$, and consequently:

$$|\phi_n(t) - \phi(t)| = |\phi(k/2^n) - \phi(t)| \leq \delta$$

We have found $N \geq 1$ such that:

$$n \geq N \;\Rightarrow\; |\phi_n(t) - \phi(t)| \leq \delta$$

This shows that $\phi_n(t) \to \phi(t)$. This being true for all $t \in \mathbf{R}^+$, we have proved that $\phi_n \to \phi$ pointwise.

14. Since $\phi$ and all the $\phi_n$'s are measurable and bounded, $|db|$ being a finite measure, the integrals $\int \phi db$ and $\int \phi_n db$ with

respect to the complex measure $db$, are well-defined, as per definition (97). Let $g \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$ be such that $|g| = 1$ and $db = \int g|db|$. Since $\phi_n \to \phi$ pointwise, we have $\phi_n g \to \phi g$. Furthermore from 12. $|\phi_n| \leq M$ and since $|db|$ is a finite measure, the constant $M$ can be viewed as an element of $L^1_{\mathbf{R}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$. Applying definition (97) together with the dominated convergence theorem (23), we obtain:

$$\begin{aligned} \int \phi db &= \int \phi g|db| \\ &= \lim_{n\to+\infty} \int \phi_n g|db| \\ &= \lim_{n\to+\infty} \int \phi_n db \end{aligned}$$

15. Since $|\phi_n| \to |\phi|$ pointwise and $|\phi_n| \leq M$ for all $n \geq 1$ while $d|b|$ is a finite measure, from the dominated convergence theo-

rem (23), we have:

$$\int |\phi|d|b| = \lim_{n\to+\infty}\int |\phi_n|d|b|$$

16. For all $n \geq 1$ we have:

$$\begin{aligned}\int \phi_n db &= \phi(0)\int 1_{\{0\}}db + \sum_{k=0}^{n2^n-1}\phi\left(\frac{k}{2^n}\right)\int 1_{]k/2^n,(k+1)/2^n]}db \\ &= \phi(0)db(\{0\}) + \sum_{k=0}^{n2^n-1}\phi\left(\frac{k}{2^n}\right)db(]k/2^n,(k+1)/2^n]) \\ &= \phi(0)b(0) + \sum_{k=0}^{n2^n-1}\phi\left(\frac{k}{2^n}\right)\left(b\left(\frac{k+1}{2^n}\right) - b\left(\frac{k}{2^n}\right)\right)\end{aligned}$$

17. Given $n \geq 1$ and $k \in \{0,\dots,n2^n-1\}$, from theorem (80):

$$\left|b\left(\frac{k+1}{2^n}\right) - b\left(\frac{k}{2^n}\right)\right| \leq |b|\left(\frac{k+1}{2^n}\right) - |b|\left(\frac{k}{2^n}\right)$$

Hence, from 16. we obtain:

$$\begin{aligned}
\left|\int \phi_n db\right| &\leq |\phi(0)||b(0)| + \sum_{k=0}^{n2^n-1} \left|\phi\left(\frac{k}{2^n}\right)\right| \left|b\left(\frac{k+1}{2^n}\right) - b\left(\frac{k}{2^n}\right)\right| \\
&\leq |\phi(0)||b(0)| + \sum_{k=0}^{n2^n-1} \left|\phi\left(\frac{k}{2^n}\right)\right| \left(|b|\left(\frac{k+1}{2^n}\right) - |b|\left(\frac{k}{2^n}\right)\right) \\
&= |\phi(0)| d|b|(\{0\}) + \sum_{k=0}^{n2^n-1} \left|\phi\left(\frac{k}{2^n}\right)\right| d|b|\left(\left]\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) \\
&= |\phi(0)| \int 1_{\{0\}} d|b| + \sum_{k=0}^{n2^n-1} \left|\phi\left(\frac{k}{2^n}\right)\right| \int 1_{]k/2^n,(k+1)/2^n]} d|b| \\
&= \int |\phi_n| d|b|
\end{aligned}$$

18. From 14. 15. and 17. taking the limit as $n \to +\infty$, we obtain:

$$\left|\int \phi db\right| \leq \int |\phi| d|b| \tag{32}$$

19. From (30), (32) and (31) we obtain:

$$\begin{aligned}\left|\int h db\right| &\leq \left|\int \phi db\right| + \epsilon \\ &\leq \int |\phi| d|b| + \epsilon \\ &\leq \int |h| d|b| + 2\epsilon\end{aligned}$$

20. Having proved that $|\int h db| \leq \int |h| d|b| + 2\epsilon$ for arbitrary $\epsilon > 0$, we conclude that:

$$\left|\int h db\right| \leq \int |h| d|b| \tag{33}$$

This has been proved for arbitrary $h \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$.

21. Let $B \in \mathcal{B}(\mathbf{R}^+)$ and $h \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$ be such that $|h| = 1$ and $db = \int h|db|$. Since $|db|$ is a finite measure, in particular it is a complex measure, and we can therefore apply theorem (65) to obtain:

$$\begin{aligned}\int_B \bar{h}db &= \int 1_B \bar{h}db \\ &= \int 1_B \bar{h} \cdot h|db| \\ &= \int 1_B |h|^2 |db| \\ &= \int 1_B |db| = |db|(B)\end{aligned}$$

22. Let $B \in \mathcal{B}(\mathbf{R}^+)$. We have:

$$|db|(B) = |\,|db|(B)\,|$$

$$
\begin{aligned}
&= \left|\int 1_B \bar{h} db\right| \\
&\leq \int 1_B |\bar{h}| d|b| \\
&= \int 1_B d|b| = d|b|(B)
\end{aligned}
$$

where the second equality stems from 21. and the inequality from (33) applied to the map $1_B\bar{h} \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$. Having proved that $|db|(B) \leq d|b|(B)$ for all $B \in \mathbf{R}^+$, we have proved that $|db| \leq d|b|$. From 4. we conclude that $|db| = d|b|$. The purpose of this exercise is to show that given a right-continuous map of bounded variation $b : \mathbf{R}^+ \to \mathbf{C}$, the total variation $|db|$ of its associated complex Stieltjes measure, is equal to the Stieltjes measure $d|b|$ associated with its total variation.

Exercise 26

**Exercise 27.**

1. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of finite variation. Let $T \in \mathbf{R}^+$. From exercise (24), $b^T$ is right-continuous of bounded variation and its complex Stieltjes measure $db^T$ is therefore well-defined, as per definition (110). From exercise (26), we have $|db^T| = d|b^T|$. Furthermore, we showed in exercise (24) that $|b^T| = |b|^T$. Hence, we have $d|b^T| = d|b|^T$ and we have proved that for all $T \in \mathbf{R}^+$:

$$|db^T| = d|b^T| = d|b|^T \tag{34}$$

2. Since $|b|$ is right-continuous, non-negative with $|b|(0) \geq 0$, from exercise (24) we have:

$$d|b|^T = d|b|^{[0,T]} \stackrel{\triangle}{=} d|b|([0,T] \cap \cdot) \tag{35}$$

Now, if $b$ is right-continuous of bounded variation, the fact that $|db| = d|b|$ was proved in exercise (26). If $b$ is right-continuous of finite variation and $T \in \mathbf{R}^+$, then $b^T$ is right-continuous of bounded variation, and from (34) and (35) we conclude that $|db^T| = d|b|([0,T] \cap \cdot) = d|b|^T$. This proves theorem (84).

Exercise 27

**Exercise 28.**

1. Let $t > 0$. Suppose the limit $b(t-)$ is not unique. There exist $l, l' \in E$, $l \neq l'$ such that $b(s)$ tends both to $l$ and $l'$ as $s \uparrow\uparrow t$ (i.e. $s \to t$, $s < t$). However, since $E$ is Hausdorff and $l \neq l'$, from definition (67) there exist $U$ and $U'$ open in $E$ such that $l \in U$, $l' \in U'$ and $U \cap U' = \emptyset$. From $b(s) \to l$ as $s \uparrow\uparrow t$ we see that there exists $t_1 \in \mathbf{R}^+$, $t_1 < t$, such that:

$$s \in ]t_1, t[ \Rightarrow \ b(s) \in U$$

   Similarly, there exists $t'_1 \in \mathbf{R}^+$, $t'_1 < t$, such that:

$$s \in ]t'_1, t[ \Rightarrow \ b(s) \in U'$$

   This contradicts the fact that $U \cap U' = \emptyset$. We have proved that the limit $b(t-)$ is unique. More generally, any limit (when it exists) in a Hausdorff topological space is unique.

2. Let $x, y \in E'$ with $x \neq y$. In particular $x, y \in E$ with $x \neq y$. Since $E$ is Hausdorff, there exist $U$ and $V$ open in $E$, such that

$x \in U$, $y \in V$ and $U \cap V = \emptyset$. It follows that $x \in U \cap E'$, $y \in V \cap E'$ and $(U \cap E') \cap (V \cap E') = \emptyset$. Since $U \cap E'$ and $V \cap E'$ are open subsets of $E'$, we conclude that the induced topological space $E'$ is Hausdorff.

3. Let $b : \mathbf{R}^+ \to E$ be cadlag with values in $E' \subseteq E$. By assumption, $b$ is right-continuous and for all $t > 0$, the limit:

$$b(t-) \stackrel{\triangle}{=} \lim_{s \uparrow\uparrow t} b(s)$$

exists in $E$. Since $b$ has values in $E'$, it can be viewed as a map $b : \mathbf{R}^+ \to E'$. Such a map is still right-continuous (see proof of 4.), but the limit $b(t-)$ for $t > 0$ may not be an element of $E'$. So $b : \mathbf{R}^+ \to E'$ may not be cadlag. In other words, $b$ may not be cadlag with respect to $E'$.

4. Consider $b : \mathbf{R}^+ \to \bar{E}'$, where $\bar{E}'$ is the closure of $E'$ in $E$. We claim that $b$ is cadlag (with respect to $\bar{E}'$). Since $b : \mathbf{R}^+ \to E$ is right-continuous, for all $t_0 \in \mathbf{R}^+$, for all $U$ open subsets of $E$

with $b(t_0) \in U$, there exists $t_1 \in \mathbf{R}^+$, $t_0 < t_1$, such that:

$$t \in [t_0, t_1[ \ \Rightarrow \ b(t) \in U \tag{36}$$

Let $U'$ be open in $\bar{E}'$ with $b(t_0) \in U'$. Then $U' = U \cap \bar{E}'$ for some $U$ open in $E$ with $b(t_0) \in U$. Let $t_1 \in \mathbf{R}^+$, $t_0 < t_1$ be such that (36) holds. Since $b$ has values in $E' \subseteq \bar{E}'$, $b(t) \in U$ is equivalent to $b(t) \in U \cap \bar{E}' = U'$. Hence, we have:

$$t \in [t_0, t_1[ \ \Rightarrow \ b(t) \in U'$$

which shows that $b : \mathbf{R}^+ \to \bar{E}'$ is indeed right-continuous (the fact that $\bar{E}'$ is the closure of $E'$ has not be used so far). Let $t_0 > 0$. Since $b : \mathbf{R}^+ \to E$ is cadlag, the limit $b(t_0-)$ exists in $E$. Let $U$ be open in $E$ such that $b(t_0-) \in U$. There exists $t_1 \in \mathbf{R}^+$, $t_1 < t_0$, such that:

$$t \in ]t_1, t_0[ \ \Rightarrow \ b(t) \in U$$

In particular, since $b(t) \in E'$ for all $t \in \mathbf{R}^+$, we have $U \cap E' \neq \emptyset$. Hence, for all $U$ open subsets of $E$ with $b(t_0-) \in U$, we have proved that $U \cap E' \neq \emptyset$. This shows that $b(t_0-) \in \bar{E}'$. Hence, for all $t_0 > 0$, we have shown that $b(t_0-)$ exists in $\bar{E}'$. We conclude that $b : \mathbf{R}^+ \to \bar{E}'$ is cadlag. In other words, $b$ is cadlag with respect to $\bar{E}'$.

5. Let $b : \mathbf{R}^+ \to \mathbf{R}$ be a map. The right-continuity of $b$ is independent of whether $b$ is viewed as a map with values in $\mathbf{R}$ or values $\mathbf{C}$. If $b : \mathbf{R}^+ \to \mathbf{R}$ is cadlag, then for all $t > 0$, $b(t-)$ exists in $\mathbf{R}$. In particular, $b(t-)$ exists in $\mathbf{C}$. So $b : \mathbf{R}^+ \to \mathbf{C}$ is cadlag. Conversely, if $b$ is cadlag with respect to $\mathbf{C}$ with values in $\mathbf{R}$, from 4. it is cadlag with respect to the closure of $\mathbf{R}$ in $\mathbf{C}$. However, $\mathbf{R}$ is a closed subset of $\mathbf{C}$, hence equal to its own closure. So $b$ is cadlag with respect to $\mathbf{R}$. We have proved that $b : \mathbf{R}^+ \to \mathbf{R}$ is cadlag, if and only if $b : \mathbf{R}^+ \to \mathbf{C}$ is cadlag.

Exercise 28

**Exercise 29.**

1. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be cadlag. Suppose $b$ is continuous with $b(0) = 0$. From definition (111), $b(0-)$ is defined as $b(0-) = 0$. Hence, $\Delta b(0) = b(0) - b(0-) = 0$. Suppose $t > 0$. Since $b$ is continuous at $t$, in particular it is left-continuous at $t$. Hence:

$$b(t) = \lim_{s \uparrow\uparrow t} b(s) \stackrel{\triangle}{=} b(t-)$$

It follows that $\Delta b(t) = b(t) - b(t-) = 0$. Conversely, suppose $\Delta b(t) = 0$ for all $t \in \mathbf{R}^+$. In particular $\Delta b(0) = 0$ and consequently $b(0) = b(0-) = 0$. Furthermore, for all $t > 0$, $\Delta b(t) = 0$. So $b$ is left-continuous at $t$. Being cadlag, $b$ is also right-continuous at $t$. Being right-continuous at 0, $b$ is in fact continuous at every point of $\mathbf{R}^+$. We have proved that $b$ is continuous with $b(0) = 0$, if and only if $\Delta b(t) = 0$ for all $t \in \mathbf{R}^+$.

2. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. We claim that $a$ is cadlag. From exercise (28), since

$\mathbf{R}^+$ is a closed subset of $\mathbf{R}$, being cadlag with respect to $\mathbf{R}$ or $\mathbf{R}^+$ is equivalent. To show that $a$ is cadlag, we only need to show that for all $t > 0$, the left-limit $a(t-)$ exists in $\mathbf{R}$. Given $t > 0$, define:

$$l \stackrel{\triangle}{=} \sup_{s\in]0,t[} a(s)$$

Since $a$ is non-decreasing, we have $l \leq a(t) < +\infty$. In particular, $\epsilon > 0$ being given, we have $l-\epsilon < l$. So $l-\epsilon$ cannot be an upper-bound of all $a(s)$'s as $s \in ]0,t[$. There exists $u \in ]0,t[$ such that $l - \epsilon < a(u)$. Since $a$ is non-decreasing, we obtain:

$$s \in ]u,t[ \Rightarrow \; l - \epsilon < a(s) \leq l$$

which shows that $a(t-)$ exists and is equal to $l$. We have proved that $a$ is cadlag. Since $a(0) \geq 0$ and by convention $a(0-) = 0$, it is clear that $\Delta a(0) \geq 0$. Let $t > 0$. We have seen that $a(t-) = l \leq a(t)$. So $\Delta a(t) \geq 0$. Having proved that $\Delta a(t) \geq 0$ for all $t \in \mathbf{R}^+$, we conclude that $\Delta a \geq 0$.

3. Let $b_1, b_2 : \mathbf{R}^+ \to \mathbf{C}$ be two cadlag maps. Let $\alpha \in \mathbf{C}$. Then, $b_1 + \alpha b_2$ is right-continuous, and for all $t > 0$:

$$\lim_{s \uparrow\uparrow t} (b_1 + \alpha b_2)(s) = b_1(t-) + \alpha b_2(t-)$$

So the left-limit $(b_1 + \alpha b_2)(t-)$ exists in $\mathbf{C}$. This shows that $b_1 + \alpha b_2$ is cadlag.

4. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. From exercise (17), $b$ can be expressed as $b = b_1 - b_2 + i(b_3 - b_4)$ where each $b_i$ is right-continuous, non-decreasing with $b_i(0) \geq 0$. From 2. each $b_i$ is cadlag. From 3. a linear combination of cadlag maps is cadlag. We conclude that $b$ is cadlag. We have proved that any right-continuous map of finite variation is cadlag.

5. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $da$ be its associated Stieltjes measure, as per definition (24). We have:

$$\Delta a(0) = a(0) - a(0-) = a(0) = da(\{0\})$$

Furthermore, for all $t > 0$, given an arbitrary sequence $(t_n)_{n\geq 1}$ in $]0, t[$ such that $t_n \uparrow\uparrow t$, we have $]t_n, t] \downarrow \{t\}$. Moreover:

$$da(]t_1, t]) = a(t) - a(t_1) \leq a(t) < +\infty$$

Applying theorem (8), we obtain:

$$\begin{aligned} da(\{t\}) &= \lim_{n\to+\infty} da(]t_n, t]) \\ &= \lim_{n\to+\infty} (a(t) - a(t_n)) \\ &= a(t) - a(t-) = \Delta a(t) \end{aligned}$$

We have proved that $da(\{t\}) = \Delta a(t)$ for all $t \in \mathbf{R}^+$.

6. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of bounded variation. Let $db$ be its associated complex Stieltjes measure, as per definition (110). We have:

$$\Delta b(0) = b(0) - b(0-) = b(0) = db(\{0\})$$

Furthermore, for all $t > 0$, given an arbitrary sequence $(t_n)_{n\geq 1}$ in $]0, t[$ such that $t_n \uparrow\uparrow t$, we have $]t_n, t] \downarrow \{t\}$ and in particular

$1_{]t_n,t]} \to 1_{\{t\}}$. Using exercise (13) of Tutorial 12, we obtain:

$$\begin{aligned} db(\{t\}) &= \lim_{n\to+\infty} db(]t_n,t]) \\ &= \lim_{n\to+\infty} (b(t) - b(t_n)) \\ &= b(t) - b(t-) = \Delta b(t) \end{aligned}$$

We have proved that $db(\{t\}) = \Delta b(t)$ for all $t \in \mathbf{R}^+$.

7. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. Let $T \in \mathbf{R}^+$. Let $t \in \mathbf{R}^+$. Suppose that $t \leq T$. Then $b^T(s)$ and $b(s)$ coincide for $s < t$. So $b^T(t-) = b(t-)$. Suppose that $T < t$. Then $b(s) = b(T)$ on $]T,t[$ and consequently $b^T(t-) = b(T)$. We have proved that:

$$\forall t \in \mathbf{R}^+ \ , \ b^T(t-) = \begin{cases} b(t-) & \text{if} \quad t \leq T \\ b(T) & \text{if} \quad T < t \end{cases}$$

Furthermore, we have:

$$\Delta b^T(0) = b^T(0) = b(0) = \Delta b(0) 1_{[0,T]}(0)$$

Moreover, if $t \in ]0,T]$:

$$\Delta b^T(t) = b^T(t) - b^T(t-) = b(t) - b(t-) = \Delta b(t) 1_{[0,T]}(t)$$

and if $t \in ]T, +\infty[$:

$$\Delta b^T(t) = b^T(t) - b^T(t-) = b(T) - b(T) = 0 = \Delta b(t) 1_{[0,T]}(t)$$

We have proved that $\Delta b^T(t) = \Delta b(t) 1_{[0,T]}(t)$ for all $t \in \mathbf{R}^+$. Finally, since $b^T$ is right-continuous of bounded variation, from 6. we have $db^T(\{t\}) = \Delta b^T(t) = \Delta b(t) 1_{[0,T]}(t)$.

Exercise 29

**Exercise 30.**

1. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be cadlag and $T \in \mathbf{R}^+$. Suppose that $b(t-)$ is not bounded on $[0, T]$. For all $n \geq 1$, there exists $t \in [0, T]$ such that $|b(t-)| > n$. Define $U_n = \{z \in \mathbf{C} : |z| > n\}$. Then $U_n$ is an open subset of $\mathbf{C}$ and $b(t-) \in U_n$. There exists $u_n \in [0, t[$, such that:
$$s \in ]u_n, t[ \ \Rightarrow \ b(s) \in U_n$$
Choosing an arbitrary $t_n \in ]u_n, t[$, we have $b(t_n) \in U_n$. The sequence $(t_n)_{n\geq 1}$ is a sequence of elements of $[0, T]$ such that $|b(t_n)| > n$ and in particular $|b(t_n)| \to +\infty$.

2. Suppose $b$ is not bounded on $[0, T]$. For all $n \geq 1$, there exists some $s_n \in [0, T]$ such that $|b(s_n)| > n$. Since $[0, T]$ is metrizable and compact, from the sequence $(s_n)_{n\geq 1}$ we can extract a converging sub-sequence, say $(s_{\phi(n)})_{n\geq 1}$ (see theorem (47)). Let $t \in [0, T]$ be its limit. Defining $t_n = s_{\phi(n)}$, we have found a sequence $(t_n)_{n\geq 1}$ on $[0, T]$ such that $t_n \to t$ for some $t \in [0, T]$, and $|b(t_n)| \to +\infty$.

3. Let $R = \{n \geq 1 : t \leq t_n\}$ and $L = \{n \geq 1 : t_n < t\}$. Since $\mathbf{N} = R \cup L$, $R$ and $L$ cannot be both finite.

4. Suppose $R$ is infinite. Since $t_n \to t$, there exists $N_1 \geq 1$ such that:
$$n \geq N_1 \;\Rightarrow\; t_n \in ]t-1, t+1[\cap[0,T]$$
Let $A_1 = \{n \geq 1 : t_n \in [t, t+1[\cap[0,T]\}$. Since $n \in R$ implies $n < N_1$ or $n \in A_1$, the fact that $R$ is infinite implies that $A_1$ is infinite. In particular, $A_1$ is not empty, and there exists $n_1 \geq 1$ such that:
$$t_{n_1} \in [t, t+1[\cap[0,T]$$

5. $R$ being assumed infinite, suppose we have $n_1 < \ldots < n_k$, $k \geq 1$, such that $t_{n_j} \in [t, t+1/j[\cap[0,T]$ for all $j \in \{1,\ldots,k\}$. Since $t_n \to t$, there exists $N_{k+1} \geq 1$ such that:
$$n \geq N_{k+1} \;\Rightarrow\; t_n \in \left]t - \frac{1}{k+1}, t + \frac{1}{k+1}\right[ \cap [0,T]$$

Let $A_{k+1} = \{n > n_k : t_n \in [t, t+1/(k+1)[\cap[0,T]\}$. Then $n \in R$ implies $n < N_{k+1}$ or $n \leq n_k$ or $n \in A_{k+1}$. So the fact that $R$ is infinite implies that $A_{k+1}$ is itself infinite. In particular, $A_{k+1}$ is not empty, and there exists $n_{k+1} > n_k$ such that:

$$t_{n_{k+1}} \in \left[t, t + \frac{1}{k+1}\right[ \cap [0,T]$$

This induction argument shows that we can construct a sequence $n_1 < n_2 < \ldots$ such that

$$t_{n_k} \in \left[t, t + \frac{1}{k}\right[ \cap [0,T] \ , \ \forall k \geq 1$$

6. By construction we have $t_{n_k} \to t$ while $t \leq t_{n_k}$. Since $b$ is cadlag, in particular $b$ is right-continuous. So $b(t_{n_k}) \to b(t)$ and $|b(t_{n_k})|$ cannot converge to $+\infty$. This contradicts the fact that $|b(t_n)| \to +\infty$.

7. Suppose $L$ is infinite. In particular $L$ is not empty. There exists $n \geq 1$ such that $t_n < t$. So $t > 0$. Since $t_n \to t$, there exists

$N_1 \geq 1$ such that:

$$n \geq N_1 \;\Rightarrow\; t_n \in ]t-1, t+1[\cap[0,T]$$

Let $B_1 = \{n \geq 1 : t \in ]t-1, t[\cap[0,T]\}$. Then $n \in L$ implies that $n < N_1$ or $n \in B_1$. So the fact that $L$ is infinite implies that $B_1$ is infinite. In particular, $B_1$ is not empty, and there exists $n_1 \geq 1$ such that $t_{n_1} \in ]t-1, t[\cap[0,T]$. Following an induction argument identical to that of 5. we can construct a sequence $n_1 < n_2 < \ldots$ such that:

$$t_{n_k} \in \left]t - \frac{1}{k}, t\right[ \cap [0,T] \;,\; \forall k \geq 1$$

8. Since $b$ is cadlag, the left-limit $b(t-)$ exists in $\mathbf{C}$. By construction, we have $t_{n_k} \to t$, while $t_{n_k} < t$. It follows that $b(t_{n_k}) \to b(t-)$ and consequently $|b(t_{n_k})|$ cannot converge to $+\infty$. This contradicts the fact that $|b(t_n)| \to +\infty$.

9. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a cadlag map. Let $T \in \mathbf{R}^+$. Suppose

$b(t)$ or $b(t-)$ is not bounded on $[0,T]$. If $b(t-)$ is not bounded on $[0,T]$, then from 1. there exists $(t_n)_{n\geq 1}$ in $[0,T]$ such that $|b(t_n)| \to +\infty$. Hence, without loss of generality, we may assume that $b(t)$ is not bounded on $[0,T]$. From 2. we can construct a sequence $(t_n)_{n\geq 1}$ in $[0,T]$ such that $t_n \to t$ for some $t \in [0,T]$ and $|b(t_n)| \to +\infty$. However, assuming $R$ infinite leads to a contradiction in 6. while assuming $L$ infinite leads to a contradiction in 8.. Since $R$ and $L$ cannot be both finite, we conclude that our initial assumption is absurd. This shows that $b(t)$ and $b(t-)$ are both bounded on $[0,T]$, which completes the proof of theorem (85).

Exercise 30