5. Lebesgue Integration

In the following, $(\Omega, \mathcal{F}, \mu)$ is a measure space.

Definition 39 *Let $A \subseteq \Omega$. We call* **characteristic function** *of A, the map $1_A : \Omega \to \mathbf{R}$, defined by:*

$$\forall \omega \in \Omega \ , \ 1_A(\omega) \stackrel{\triangle}{=} \left\{ \begin{array}{lll} 1 & \textit{if} & \omega \in A \\ 0 & \textit{if} & \omega \notin A \end{array} \right.$$

Exercise 1. Given $A \subseteq \Omega$, show that $1_A : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable if and only if $A \in \mathcal{F}$.

Definition 40 *Let $(\Omega, \mathcal{F})$ be a measurable space. We say that a map $s : \Omega \to \mathbf{R}^+$ is a* **simple function** *on $(\Omega, \mathcal{F})$, if and only if s is of the form :*

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$

where $n \geq 1$, $\alpha_i \in \mathbf{R}^+$ and $A_i \in \mathcal{F}$, for all $i = 1, \ldots, n$.

Exercise 2. Show that $s : (\Omega, \mathcal{F}) \to (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$ is measurable, whenever s is a simple function on $(\Omega, \mathcal{F})$.

Exercise 3. Let s be a simple function on $(\Omega, \mathcal{F})$ with representation $s = \sum_{i=1}^n \alpha_i 1_{A_i}$. Consider the map $\phi : \Omega \to \{0,1\}^n$ defined by $\phi(\omega) = (1_{A_1}(\omega), \ldots, 1_{A_n}(\omega))$. For each $y \in s(\Omega)$, pick one $\omega_y \in \Omega$ such that $y = s(\omega_y)$. Consider the map $\psi : s(\Omega) \to \{0,1\}^n$ defined by $\psi(y) = \phi(\omega_y)$.

1. Show that ψ is injective, and that $s(\Omega)$ is a finite subset of $\mathbf{R}^+$.

2. Show that $s = \sum_{\alpha \in s(\Omega)} \alpha 1_{\{s=\alpha\}}$

3. Show that any simple function s can be represented as:

$$s = \sum_{i=1}^n \alpha_i 1_{A_i}$$

where $n \geq 1, \alpha_i \in \mathbf{R}^+, A_i \in \mathcal{F}$ and $\Omega = A_1 \uplus \ldots \uplus A_n$.

Definition 41 *Let $(\Omega, \mathcal{F})$ be a measurable space, and s be a simple function on $(\Omega, \mathcal{F})$. We call* **partition** *of the simple function s, any representation of the form:*

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$

where $n \geq 1$, $\alpha_i \in \mathbf{R}^+$, $A_i \in \mathcal{F}$ and $\Omega = A_1 \uplus \ldots \uplus A_n$.

Exercise 4. Let s be a simple function on $(\Omega, \mathcal{F})$ with two partitions:

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i} = \sum_{j=1}^{m} \beta_j 1_{B_j}$$

1. Show that $s = \sum_{i,j} \alpha_i 1_{A_i \cap B_j}$ is a partition of s.
2. Recall the convention $0 \times (+\infty) = 0$ and $\alpha \times (+\infty) = +\infty$ if $\alpha > 0$. For all $a_1, \ldots, a_p$ in $[0, +\infty], p \geq 1$ and $x \in [0, +\infty]$, prove the distributive property: $x(a_1 + \ldots + a_p) = xa_1 + \ldots + xa_p$.

3. Show that $\sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{j=1}^{m} \beta_j \mu(B_j)$.

4. Explain why the following definition is legitimate.

Definition 42 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and s be a simple function on $(\Omega, \mathcal{F})$. We define the* **integral** *of s with respect to μ, as the sum, denoted $I^{\mu}(s)$, defined by:*

$$I^{\mu}(s) \stackrel{\triangle}{=} \sum_{i=1}^{n} \alpha_i \mu(A_i) \in [0, +\infty]$$

where $s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$ is any partition of s.

Exercise 5. Let s, t be two simple functions on $(\Omega, \mathcal{F})$ with partitions $s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$ and $t = \sum_{j=1}^{m} \beta_j 1_{B_j}$. Let $\alpha \in \mathbf{R}^+$.

1. Show that $s + t$ is a simple function on $(\Omega, \mathcal{F})$ with partition:

$$s + t = \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i + \beta_j) 1_{A_i \cap B_j}$$

2. Show that $I^\mu(s + t) = I^\mu(s) + I^\mu(t)$.

3. Show that αs is a simple function on $(\Omega, \mathcal{F})$.

4. Show that $I^\mu(\alpha s) = \alpha I^\mu(s)$.

5. Why is the notation $I^\mu(\alpha s)$ meaningless if $\alpha = +\infty$ or $\alpha < 0$.

6. Show that if $s \leq t$ then $I^\mu(s) \leq I^\mu(t)$.

Exercise 6. Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be a non-negative and measurable map. For all $n \geq 1$, we define:

$$s_n \triangleq \sum_{k=0}^{n2^n-1} \frac{k}{2^n} 1_{\{\frac{k}{2^n} \leq f < \frac{k+1}{2^n}\}} + n 1_{\{n \leq f\}} \tag{1}$$

1. Show that s_n is a simple function on $(\Omega, \mathcal{F})$, for all $n \geq 1$.
2. Show that equation (1) is a partition s_n, for all $n \geq 1$.
3. Show that $s_n \leq s_{n+1} \leq f$, for all $n \geq 1$.
4. Show that $s_n \uparrow f$ as $n \to +\infty$[1].

[1] i.e. for all $\omega \in \Omega$, the sequence $(s_n(\omega))_{n\geq 1}$ is non-decreasing and converges to $f(\omega) \in [0, +\infty]$.

Theorem 18 *Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be a non-negative and measurable map, where $(\Omega, \mathcal{F})$ is a measurable space. There exists a sequence $(s_n)_{n\geq 1}$ of simple functions on $(\Omega, \mathcal{F})$ such that $s_n \uparrow f$.*

Definition 43 *Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be a non-negative and measurable map, where $(\Omega, \mathcal{F}, \mu)$ is a measure space. We define the* **Lebesgue integral** *of f with respect to μ, denoted $\int f d\mu$, as:*

$$\int f d\mu \stackrel{\triangle}{=} \sup\{I^\mu(s) \ : \ s \text{ simple function on } (\Omega, \mathcal{F}) \ , \ s \leq f\}$$

where, given any simple function s on $(\Omega, \mathcal{F})$, $I^\mu(s)$ denotes its integral with respect to μ.

Exercise 7. Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be a non-negative and measurable map.

1. Show that $\int f d\mu \in [0, +\infty]$.

2. Show that $\int f d\mu = I^\mu(f)$, whenever f is a simple function.

3. Show that $\int g d\mu \leq \int f d\mu$, whenever $g : (\Omega, \mathcal{F}) \to [0, +\infty]$ is non-negative and measurable map with $g \leq f$.

4. Show that $\int (cf) d\mu = c \int f d\mu$, if $0 < c < +\infty$. Explain why both integrals are well defined. Is the equality still true for $c = 0$.

5. For $n \geq 1$, put $A_n = \{f > 1/n\}$, and $s_n = (1/n) 1_{A_n}$. Show that s_n is a simple function on $(\Omega, \mathcal{F})$ with $s_n \leq f$. Show that $A_n \uparrow \{f > 0\}$.

6. Show that $\int f d\mu = 0 \ \Rightarrow \mu(\{f > 0\}) = 0$.

7. Show that if s is a simple function on $(\Omega, \mathcal{F})$ with $s \leq f$, then $\mu(\{f > 0\}) = 0$ implies $I^{\mu}(s) = 0$.

8. Show that $\int f d\mu = 0 \ \Leftrightarrow \ \mu(\{f > 0\}) = 0$.

9. Show that $\int (+\infty) f d\mu = (+\infty) \int f d\mu$. Explain why both integrals are well defined.

10. Show that $(+\infty)1_{\{f=+\infty\}} \leq f$ and:

$$\int (+\infty)1_{\{f=+\infty\}}d\mu = (+\infty)\mu(\{f = +\infty\})$$

11. Show that $\int f d\mu < +\infty \Rightarrow \mu(\{f = +\infty\}) = 0$.

12. Suppose that $\mu(\Omega) = +\infty$ and take $f = 1$. Show that the converse of the previous implication is not true.

Exercise 8. Let s be a simple function on $(\Omega, \mathcal{F})$. Let $A \in \mathcal{F}$.

1. Show that $s1_A$ is a simple function on $(\Omega, \mathcal{F})$.

2. Show that for any partition $s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$ of s, we have:

$$I^{\mu}(s1_A) = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap A)$$

3. Let $\nu : \mathcal{F} \to [0, +\infty]$ be defined by $\nu(A) = I^\mu(s1_A)$. Show that ν is a measure on $\mathcal{F}$.

4. Suppose $A_n \in \mathcal{F}, A_n \uparrow A$. Show that $I^\mu(s1_{A_n}) \uparrow I^\mu(s1_A)$.

Exercise 9. Let $(f_n)_{n\geq 1}$ be a sequence of non-negative and measurable maps $f_n : (\Omega, \mathcal{F}) \to [0, +\infty]$, such that $f_n \uparrow f$.

1. Recall what the notation $f_n \uparrow f$ means.

2. Explain why $f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

3. Let $\alpha = \sup_{n\geq 1} \int f_n d\mu$. Show that $\int f_n d\mu \uparrow \alpha$.

4. Show that $\alpha \leq \int f d\mu$.

5. Let s be any simple function on $(\Omega, \mathcal{F})$ such that $s \leq f$. Let $c \in]0, 1[$. For $n \geq 1$, define $A_n = \{cs \leq f_n\}$. Show that $A_n \in \mathcal{F}$ and $A_n \uparrow \Omega$.

6. Show that $cI^{\mu}(s1_{A_n}) \leq \int f_n d\mu$, for all $n \geq 1$.

7. Show that $cI^{\mu}(s) \leq \alpha$.

8. Show that $I^{\mu}(s) \leq \alpha$.

9. Show that $\int f d\mu \leq \alpha$.

10. Conclude that $\int f_n d\mu \uparrow \int f d\mu$.

Theorem 19 (Monotone Convergence) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $(f_n)_{n\geq 1}$ be a sequence of non-negative and measurable maps $f_n : (\Omega, \mathcal{F}) \to [0, +\infty]$ such that $f_n \uparrow f$. Then $\int f_n d\mu \uparrow \int f d\mu$.*

Exercise 10. Let $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Let $a, b \in [0, +\infty]$.

1. Show that if $(f_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ are two sequences of non-negative and measurable maps such that $f_n \uparrow f$ and $g_n \uparrow g$, then $f_n + g_n \uparrow f + g$.

2. Show that $\int(f+g)d\mu = \int f d\mu + \int g d\mu$.

3. Show that $\int(af+bg)d\mu = a\int f d\mu + b\int g d\mu$.

Exercise 11. Let $(f_n)_{n\geq 1}$ be a sequence of non-negative and measurable maps $f_n : (\Omega, \mathcal{F}) \to [0, +\infty]$. Define $f = \sum_{n=1}^{+\infty} f_n$.

1. Explain why $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ is well defined, non-negative and measurable.

2. Show that $\int f d\mu = \sum_{n=1}^{+\infty} \int f_n d\mu$.

Definition 44 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\mathcal{P}(\omega)$ be a property depending on $\omega \in \Omega$. We say that the property $\mathcal{P}(\omega)$ holds* **μ-almost surely**, *and we write $\mathcal{P}(\omega)$ μ-a.s., if and only if:*

$$\exists N \in \mathcal{F} \ , \ \mu(N) = 0 \ , \ \forall \omega \in N^c, \mathcal{P}(\omega) \ \textit{holds}$$

Exercise 12. Let $\mathcal{P}(\omega)$ be a property depending on $\omega \in \Omega$, such that $\{\omega \in \Omega : \mathcal{P}(\omega) \text{ holds}\}$ is an element of the σ-algebra $\mathcal{F}$.

1. Show that $\mathcal{P}(\omega)$, μ-a.s. $\Leftrightarrow$ $\mu(\{\omega \in \Omega : \mathcal{P}(\omega) \text{ holds}\}^c) = 0$.
2. Explain why in general, the right-hand side of this equivalence cannot be used to defined μ-almost sure properties.

Exercise 13. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $(A_n)_{n\geq 1}$ be a sequence of elements of $\mathcal{F}$. Show that $\mu(\cup_{n=1}^{+\infty} A_n) \leq \sum_{n=1}^{+\infty} \mu(A_n)$.

Exercise 14. Let $(f_n)_{n\geq 1}$ be a sequence of maps $f_n : \Omega \to [0, +\infty]$.

1. Translate formally the statement $f_n \uparrow f$ μ-a.s.
2. Translate formally $f_n \to f$ μ-a.s. and $\forall n, (f_n \leq f_{n+1}$ μ-a.s.)
3. Show that the statements 1. and 2. are equivalent.

Exercise 15. Suppose that $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ are non-negative and measurable with $f = g$ μ-a.s.. Let $N \in \mathcal{F}$, $\mu(N) = 0$ such that $f = g$ on N^c. Explain why $\int f d\mu = \int (f 1_N) d\mu + \int (f 1_{N^c}) d\mu$, all integrals being well defined. Show that $\int f d\mu = \int g d\mu$.

Exercise 16. Suppose $(f_n)_{n\geq 1}$ is a sequence of non-negative and measurable maps and f is a non-negative and measurable map, such that $f_n \uparrow f$ μ-a.s.. Let $N \in \mathcal{F}$, $\mu(N) = 0$, such that $f_n \uparrow f$ on N^c. Define $\bar{f}_n = f_n 1_{N^c}$ and $\bar{f} = f 1_{N^c}$.

1. Explain why $\bar{f}$ and the $\bar{f}_n$'s are non-negative and measurable.

2. Show that $\bar{f}_n \uparrow \bar{f}$.

3. Show that $\int f_n d\mu \uparrow \int f d\mu$.

EXERCISE 17. Let $(f_n)_{n\geq 1}$ be a sequence of non-negative and measurable maps $f_n : (\Omega, \mathcal{F}) \to [0, +\infty]$. For $n \geq 1$, we define $g_n = \inf_{k\geq n} f_k$.

1. Explain why the g_n's are non-negative and measurable.

2. Show that $g_n \uparrow \liminf f_n$.

3. Show that $\int g_n d\mu \leq \int f_n d\mu$, for all $n \geq 1$.

4. Show that if $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$ are two sequences in $\bar{\mathbf{R}}$ with $u_n \leq v_n$ for all $n \geq 1$, then $\liminf u_n \leq \liminf v_n$.

5. Show that $\int(\liminf f_n)d\mu \leq \liminf \int f_n d\mu$, and recall why all integrals are well defined.

Theorem 20 (Fatou Lemma) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $(f_n)_{n\geq 1}$ be a sequence of non-negative and measurable maps $f_n : (\Omega, \mathcal{F}) \to [0, +\infty]$. Then:*

$$\int (\liminf_{n\to+\infty} f_n) d\mu \leq \liminf_{n\to+\infty} \int f_n d\mu$$

Exercise 18. Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be a non-negative and measurable map. Let $A \in \mathcal{F}$.

1. Recall what is meant by the induced measure space $(A, \mathcal{F}_{|A}, \mu_{|A})$. Why is it important to have $A \in \mathcal{F}$. Show that the restriction of f to A, $f_{|A} : (A, \mathcal{F}_{|A}) \to [0, +\infty]$ is measurable.

2. We define the map $\mu^A : \mathcal{F} \to [0, +\infty]$ by $\mu^A(E) = \mu(A \cap E)$, for all $E \in \mathcal{F}$. Show that $(\Omega, \mathcal{F}, \mu^A)$ is a measure space.

3. Consider the equalities:

$$\int (f 1_A) d\mu = \int f d\mu^A = \int (f_{|A}) d\mu_{|A} \qquad (2)$$

For each of the above integrals, what is the underlying measure space on which the integral is considered. What is the map being integrated. Explain why each integral is well defined.

4. Show that in order to prove (2), it is sufficient to consider the case when f is a simple function on $(\Omega, \mathcal{F})$.

5. Show that in order to prove (2), it is sufficient to consider the case when f is of the form $f = 1_B$, for some $B \in \mathcal{F}$.

6. Show that (2) is indeed true.

Definition 45 *Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be a non-negative and measurable map, where $(\Omega, \mathcal{F}, \mu)$ is a measure space. let $A \in \mathcal{F}$. We call* **partial Lebesgue integral** *of f with respect to μ over A, the integral denoted $\int_A f d\mu$, defined as:*

$$\int_A f d\mu \stackrel{\triangle}{=} \int (f 1_A) d\mu = \int f d\mu^A = \int (f_{|A}) d\mu_{|A}$$

where μ^A is the measure on $(\Omega, \mathcal{F})$, $\mu^A = \mu(A \cap \bullet)$, $f_{|A}$ is the restriction of f to A and $\mu_{|A}$ is the restriction of μ to $\mathcal{F}_{|A}$, the trace of $\mathcal{F}$ on A.

Exercise 19. Let $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Let $\nu : \mathcal{F} \to [0, +\infty]$ be defined by $\nu(A) = \int_A f d\mu$, for all $A \in \mathcal{F}$.

1. Show that ν is a measure on $\mathcal{F}$.

2. Show that:

$$\int g d\nu = \int g f d\mu$$

Theorem 21 *Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be a non-negative and measurable map, where $(\Omega, \mathcal{F}, \mu)$ is a measure space. Let $\nu : \mathcal{F} \to [0, +\infty]$ be defined by $\nu(A) = \int_A f d\mu$, for all $A \in \mathcal{F}$. Then, ν is a measure on $\mathcal{F}$, and for all $g : (\Omega, \mathcal{F}) \to [0, +\infty]$ non-negative and measurable, we have:*

$$\int g d\nu = \int g f d\mu$$

Definition 46 *The* **L^1-spaces** *on a measure space $(\Omega, \mathcal{F}, \mu)$, are:*

$$L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} \left\{ f : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R})) \text{ measurable}, \int |f| d\mu < +\infty \right\}$$

$$L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} \left\{ f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C})) \text{ measurable}, \int |f| d\mu < +\infty \right\}$$

EXERCISE 20. Let $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ be a measurable map.

1. Explain why the integral $\int |f| d\mu$ makes sense.
2. Show that $f : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is measurable, if $f(\Omega) \subseteq \mathbf{R}$.
3. Show that $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \subseteq L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.
4. Show that $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) = \{f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)\ ,\ f(\Omega) \subseteq \mathbf{R}\}$
5. Show that $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbf{R}$-linear combinations.
6. Show that $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbf{C}$-linear combinations.

Definition 47 *Let $u : \Omega \to \mathbf{R}$ be a real-valued function defined on a set Ω. We call* **positive part** *and* **negative part** *of u the maps u^+ and u^- respectively, defined as $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$.*

Exercise 21. Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Let $u = Re(f)$ and $v = Im(f)$.

1. Show that $u = u^+ - u^-$, $v = v^+ - v^-$, $f = u^+ - u^- + i(v^+ - v^-)$.

2. Show that $|u| = u^+ + u^-, |v| = v^+ + v^-$

3. Show that $u^+, u^-, v^+, v^-, |f|, u, v, |u|, |v|$ all lie in $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

4. Explain why the integrals $\int u^+ d\mu, \int u^- d\mu, \int v^+ d\mu, \int v^- d\mu$ are all well defined.

5. We define the integral of f with respect to μ, denoted $\int f d\mu$, as $\int f d\mu = \int u^+ d\mu - \int u^- d\mu + i \left(\int v^+ d\mu - \int v^- d\mu\right)$. Explain why $\int f d\mu$ is a well defined complex number.

6. In the case when $f(\Omega) \subseteq \mathbf{C} \cap [0, +\infty] = \mathbf{R}^+$, explain why this new definition of the integral of f with respect to μ is consistent with the one already known (43) for non-negative and measurable maps.

7. Show that $\int f d\mu = \int u d\mu + i \int v d\mu$ and explain why all integrals involved are well defined.

Definition 48 *Let* $f = u + iv \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ *where* $(\Omega, \mathcal{F}, \mu)$ *is a measure space. We define the* **Lebesgue integral** *of* f *with respect to* μ*, denoted* $\int f d\mu$*, as:*

$$\int f d\mu \stackrel{\triangle}{=} \int u^+ d\mu - \int u^- d\mu + i\left(\int v^+ d\mu - \int v^- d\mu\right)$$

EXERCISE 22. Let $f = u + iv \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $A \in \mathcal{F}$.

1. Show that $f1_A \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.
2. Show that $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu^A)$.
3. Show that $f_{|A} \in L^1_{\mathbf{C}}(A, \mathcal{F}_{|A}, \mu_{|A})$
4. Show that $\int (f1_A) d\mu = \int f d\mu^A = \int f_{|A} d\mu_{|A}$.
5. Show that 4. is:$\int_A u^+ d\mu - \int_A u^- d\mu + i\left(\int_A v^+ d\mu - \int_A v^- d\mu\right)$.

Definition 49 *Let* $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ *, where* $(\Omega, \mathcal{F}, \mu)$ *is a measure space. let* $A \in \mathcal{F}$*. We call* **partial Lebesgue integral** *of* f *with respect to* μ *over* A*, the integral denoted* $\int_A f d\mu$*, defined as:*

$$\int_A f d\mu \stackrel{\triangle}{=} \int (f 1_A) d\mu = \int f d\mu^A = \int (f_{|A}) d\mu_{|A}$$

where μ^A *is the measure on* $(\Omega, \mathcal{F})$*,* $\mu^A = \mu(A \cap \bullet)$*,* $f_{|A}$ *is the restriction of* f *to* A *and* $\mu_{|A}$ *is the restriction of* μ *to* $\mathcal{F}_{|A}$*, the trace of* $\mathcal{F}$ *on* A*.*

Exercise 23. Let $f, g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ and let $h = f + g$

1. Show that:

$$\int h^+ d\mu + \int f^- d\mu + \int g^- d\mu = \int h^- d\mu + \int f^+ d\mu + \int g^+ d\mu$$

2. Show that $\int h d\mu = \int f d\mu + \int g d\mu$.

3. Show that $\int (-f) d\mu = - \int f d\mu$

4. Show that if $\alpha \in \mathbf{R}$ then $\int (\alpha f) d\mu = \alpha \int f d\mu$.

5. Show that if $f \leq g$ then $\int f d\mu \leq \int g d\mu$

6. Show the following theorem.

Theorem 22 *For all $f, g \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $\alpha \in \mathbf{C}$, we have:*

$$\int (\alpha f + g) d\mu = \alpha \int f d\mu + \int g d\mu$$

Exercise 24. Let f, g be two maps, and $(f_n)_{n \geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$, such that:

$$\begin{array}{rl} (i) & \forall \omega \in \Omega \ , \ \lim_{n \to +\infty} f_n(\omega) = f(\omega) \text{ in } \mathbf{C} \\ (ii) & \forall n \geq 1 \ , \ |f_n| \leq g \\ (iii) & g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \end{array}$$

Let $(u_n)_{n \geq 1}$ be an arbitrary sequence in $\bar{\mathbf{R}}$.

1. Show that $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $f_n \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ for all $n \geq 1$.
2. For $n \geq 1$, define $h_n = 2g - |f_n - f|$. Explain why Fatou lemma (20) can be applied to the sequence $(h_n)_{n\geq 1}$.
3. Show that $\liminf(-u_n) = -\limsup u_n$.
4. Show that if $\alpha \in \mathbf{R}$, then $\liminf(\alpha + u_n) = \alpha + \liminf u_n$.
5. Show that $u_n \to 0$ as $n \to +\infty$ if and only if $\limsup |u_n| = 0$.
6. Show that $\int (2g)d\mu \leq \int (2g)d\mu - \limsup \int |f_n - f|d\mu$
7. Show that $\limsup \int |f_n - f|d\mu = 0$.
8. Conclude that $\int |f_n - f|d\mu \to 0$ as $n \to +\infty$.

Theorem 23 (Dominated Convergence) *Let $(f_n)_{n\geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ such that $f_n \to f$ in $\mathbf{C}$*[2] *. Suppose that there exists some $g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ such that $|f_n| \leq g$ for all $n \geq 1$. Then $f, f_n \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ for all $n \geq 1$, and:*

$$\lim_{n\to+\infty} \int |f_n - f| d\mu = 0$$

Exercise 25. Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and put $z = \int f d\mu$. Let $\alpha \in \mathbf{C}$, be such that $|\alpha| = 1$ and $\alpha z = |z|$. Put $u = Re(\alpha f)$.

1. Show that $u \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$
2. Show that $u \leq |f|$
3. Show that $|\int f d\mu| = \int (\alpha f) d\mu$.
4. Show that $\int (\alpha f) d\mu = \int u d\mu$.

[2] i.e. for all $\omega \in \Omega$, the sequence $(f_n(\omega))_{n\geq 1}$ converges to $f(\omega) \in \mathbf{C}$

5. Prove the following theorem.

Theorem 24 *Let* $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ *where* $(\Omega, \mathcal{F}, \mu)$ *is a measure space. We have:*

$$\left|\int f d\mu\right| \leq \int |f| d\mu$$

Solutions to Exercises

Exercise 1. Let $A \subseteq \Omega$. Suppose 1_A is measurable. Then in particular $A = (1_A)^{-1}(\{1\}) \in \mathcal{F}$. Conversely, suppose $A \in \mathcal{F}$. Let $B \in \mathcal{B}(\bar{\mathbf{R}})$. If $\{0,1\} \subseteq B$, then $(1_A)^{-1}(B) = \Omega$. If $\{0,1\} \cap B = \{1\}$, then $(1_A)^{-1}(B) = A$. If $\{0,1\} \cap B = \{0\}$, then $(1_A)^{-1}(B) = A^c$. Finally, if $\{0,1\} \cap B = \emptyset$, then $(1_A)^{-1}(B) = \emptyset$. In any case, $(1_A)^{-1}(B) \in \mathcal{F}$. We have proved that $1_A : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, if and only if $A \in \mathcal{F}$.

Exercise 1

Exercise 2. Let $s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$ be a simple function on $(\Omega, \mathcal{F})$. For all $i = 1, \ldots, n$, $A_i \in \mathcal{F}$. From exercise (1), each characteristic function 1_{A_i} is measurable. Using exercise (19) of the previous tutorial, each $\alpha_i 1_{A_i}$ is measurable. In fact, since $\alpha_i \in \mathbf{R}^+$, $\alpha_i 1_{A_i}$ is a measurable map with values in $\mathbf{R}$, (it is also a non-negative and measurable map). It follows from exercise (19), that $s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$ is measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\bar{\mathbf{R}})$. However, s has values in $\mathbf{R}^+$, and $\mathcal{B}(\mathbf{R}^+) \subseteq \mathcal{B}(\bar{\mathbf{R}})$. So s is also measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\mathbf{R}^+)$.

Exercise 2

Exercise 3.

1. Suppose $x, y \in s(\Omega)$ and $\psi(x) = \psi(y)$. Then $\phi(\omega_x) = \phi(\omega_y)$. So for all $i = 1, \ldots, n$, $1_{A_i}(\omega_x) = 1_{A_i}(\omega_y)$. Hence, $s(\omega_x) = s(\omega_y)$. However, ω_x and ω_y have been chosen to be such that $x = s(\omega_x)$ and $y = s(\omega_y)$. It follows that $x = y$, and $\psi : s(\Omega) \to \{0,1\}^n$ is an injective map. Since $\{0,1\}^n$ is a finite set, we conclude that $s(\Omega)$ is itself a finite set. By definition (40), it is also a subset of $\mathbf{R}^+$.

2. Let $t = \sum_{\alpha \in s(\Omega)} \alpha 1_{\{s=\alpha\}}$. From 1., $s(\Omega)$ is a finite set, and t is therefore well defined as a finite sum of weighted characteristic functions. Let $\omega \in \Omega$. Let $\alpha' = s(\omega)$. Then, $1_{\{s=\alpha'\}}(\omega) = 1$, and $1_{\{s=\alpha\}}(\omega) = 0$ for all $\alpha \in s(\Omega)$ such that $\alpha \neq \alpha'$. It follows that $t(\omega) = \alpha'$. Hence, $t(\omega) = s(\omega)$. This being true for all $\omega \in \Omega$, we have proved that $t = s$.

3. From 2., s can be represented as $s = \sum_{\alpha \in s(\Omega)} \alpha 1_{\{s=\alpha\}}$. $s(\Omega)$ being a finite set, there exists a bijection $\gamma : \{1, \ldots, n\} \to s(\Omega)$,

for some $n \geq 1$ [3]. For all $i = 1, \ldots, n$, we define $\alpha_i = \gamma(i)$ and $A_i = \{s = \gamma(i)\}$. Then, it is clear that $s = \sum_{i=1}^n \alpha_i 1_{A_i}$. Moreover, each α_i is an element of $\mathbf{R}^+$. From exercise (2), s is a measurable map, and $A_i \in \mathcal{F}$ for all $i = 1, \ldots, n$. Let $\omega \in \Omega$ and $\alpha = s(\omega)$. γ being onto, there exists $i \in \{1, \ldots, n\}$ such that $\gamma(i) = \alpha$. So $\omega \in \{s = \gamma(i)\} = A_i$ and we have proved that $\Omega \subseteq A_1 \cup \ldots \cup A_n$. Each A_i being a subset of Ω, we have $\Omega = A_1 \cup \ldots \cup A_n$. Finally, suppose there exists $\omega \in A_i \cap A_j$. Then, $s(\omega) = \gamma(i)$ and $s(\omega) = \gamma(j)$. γ being injective, $i = j$. It follows that the A_i's are pairwise disjoint, and therefore $\Omega = A_1 \uplus \ldots \uplus A_n$. We have proved that any simple function s on $(\Omega, \mathcal{F})$, can be expressed as $s = \sum_{i=1}^n \alpha_i 1_{A_i}$, where $n \geq 1$, $\alpha_i \in \mathbf{R}^+$, $A_i \in \mathcal{F}$ and $\Omega = A_1 \uplus \ldots \uplus A_n$.

Exercise 3

[3] If $\Omega = \emptyset$ and $s(\Omega) = \emptyset$, write $s = 1_\emptyset$ and there is nothing else to prove.

Exercise 4.

1. Let $t = \sum_{i,j} \alpha_i 1_{A_i \cap B_j}$. For each (i, j), $\alpha_i \in \mathbf{R}^+$ and $A_i \cap B_j \in \mathcal{F}$. If $(i, j) \neq (i', j')$, then $i \neq i'$ or $j \neq j'$. In the first case, the A_i's being pairwise disjoint, $A_i \cap A_{i'} = \emptyset$. In the second case, $B_j \cap B_{j'} = \emptyset$. In any case, $(A_i \cap B_j) \cap (A_{i'} \cap B_{j'}) = \emptyset$. It follows that the $A_i \cap B_j$'s are pairwise disjoint, and $\Omega = \uplus_{i,j} A_i \cap B_j$. Let $\omega \in \Omega$. There exists a unique (i, j) such that $\omega \in A_i \cap B_j$. We have $t(\omega) = \alpha_i = s(\omega)$. It follows that $s = t$. We have proved that $t = \sum_{i,j} \alpha_i 1_{A_i \cap B_j}$ is a partition of the simple function s.

2. Let $\mathcal{P}$ be the property $x(a_1 + \ldots + a_p) = xa_1 + \ldots + xa_p$. Suppose $x = 0$. Then $x(a_1 + \ldots + a_p) = 0$. Moreover, for all $i = 1, \ldots, p$, we have $xa_i = 0$. It follows that property $\mathcal{P}$ is true. Suppose $x = +\infty$ and $a_i = 0$ for all $i = 1, \ldots, p$. Then $a_1 + \ldots + a_p = 0$, and $x(a_1 + \ldots + a_p) = 0$. Moreover, $xa_i = 0$ for all i and property $\mathcal{P}$ is true. Suppose $x = +\infty$ and $a_i > 0$ for some $i = 1, \ldots, p$. Then $xa_i = +\infty$, and therefore $xa_1 + \ldots + xa_p = +\infty$. However, $a_1 + \ldots + a_p$ is also strictly

positive with $x = +\infty$. Hence, $x(a_1 + \ldots + a_p) = +\infty$ and property $\mathcal{P}$ is true. Suppose $0 < x < +\infty$. If $a_i < +\infty$ for all i, then property $\mathcal{P}$ is true by virtue of the distributive law in $\mathbf{R}$. Suppose $a_i = +\infty$ for some i. Then $xa_i = +\infty$ and $xa_1 + \ldots + xa_p = +\infty$. However, $a_1 + \ldots + a_p$ is also equal to $+\infty$, with $x > 0$. So $x(a_1 + \ldots + a_p) = +\infty$ and property $\mathcal{P}$ is true. We have proved that property $\mathcal{P}$ is true in all cases.

3. Since $\Omega = B_1 \uplus \ldots \uplus B_m$, we have $A_i = \uplus_{j=1}^m (A_i \cap B_j)$, for all $i = 1, \ldots, n$. μ being a measure on $(\Omega, \mathcal{F})$, it follows that $\mu(A_i) = \sum_{j=1}^m \mu(A_i \cap B_j)$. Hence:

$$\sum_{i=1}^n \alpha_i \mu(A_i) = \sum_{i=1}^n \alpha_i \left(\sum_{j=1}^m \mu(A_i \cap B_j) \right)$$

From the distributive property proved in 2., we obtain:

$$\sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \mu(A_i \cap B_j) \tag{3}$$

Similarly, we have:

$$\sum_{j=1}^{m} \beta_j \mu(B_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_j \mu(A_i \cap B_j) \tag{4}$$

Suppose $A_i \cap B_j = \emptyset$. Then in particular, $\mu(A_i \cap B_j) = 0$ and $\alpha_i \mu(A_i \cap B_j) = \beta_j \mu(A_i \cap B_j)$. If $A_i \cap B_j \neq \emptyset$, there exists $\omega \in A_i \cap B_j$ in which case, $\alpha_i = s(\omega) = \beta_j$. In any case, $\alpha_i \mu(A_i \cap B_j) = \beta_j \mu(A_i \cap B_j)$, and we conclude from (3) and (4) that:

$$\sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{j=1}^{m} \beta_j \mu(B_j)$$

4. Given a simple function s on $(\Omega, \mathcal{F})$, the integral of s with respect to μ is defined from (42) as $I^{\mu}(s) = \sum_{i=1}^{n} \alpha_i \mu(A_i)$, where $\sum_{i=1}^{n} \alpha_i 1_{A_i}$ is an arbitrary partition of s. We know from exercise (3) that such partition exists, but it may not be unique. However, since we proved in 3. that the sum $\sum_{i=1}^{n} \alpha_i \mu(A_i)$ is invariant across all partitions of s, there is no ambiguity as to what $I^{\mu}(s)$ actually refers to, and definition (42) is therefore legitimate.

Exercise 4

Exercise 5.

1. From definition (40), $s+t = \sum_{i=1}^n \alpha_i 1_{A_i} + \sum_{j=1}^m \beta_j 1_{B_j}$ is clearly a simple function on $(\Omega, \mathcal{F})$. Since $\Omega = \uplus_{i=1}^n A_i$ and $\Omega = \uplus_{j=1}^m B_j$, we have $\Omega = \uplus_{i,j} A_i \cap B_j$. Furthermore:

$$s = \sum_{i=1}^n \sum_{j=1}^m \alpha_i 1_{A_i \cap B_j} \tag{5}$$

and:

$$t = \sum_{i=1}^n \sum_{j=1}^m \beta_j 1_{A_i \cap B_j} \tag{6}$$

It follows that:

$$s + t = \sum_{i=1}^n \sum_{j=1}^m (\alpha_i + \beta_j) 1_{A_i \cap B_j} \tag{7}$$

As a finite sum involving $\alpha_i + \beta_j \in \mathbf{R}^+$ and $A_i \cap B_j \in \mathcal{F}$, with $\Omega = \uplus_{i,j} A_i \cap B_j$, equation (7) defines a partition of s+t.

2. Since $\Omega = \uplus_{i,j} A_i \cap B_j$, equations (5), (6) and (7) are partitions of s, t and $s+t$ respectively. From definition (42), we obtain:

$$I^{\mu}(s+t) = \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i + \beta_j)\mu(A_i \cap B_j) = I^{\mu}(s) + I^{\mu}(t)$$

3. $\alpha s = \sum_{i=1}^{n} \alpha\alpha_i 1_{A_i}$. Since $\alpha \in \mathbf{R}^+$, each $\alpha\alpha_i \in \mathbf{R}^+$. It follows from definition (40) that αs is a simple function on $(\Omega, \mathcal{F})$.

4. $\sum_{i=1}^{n} \alpha\alpha_i 1_{A_i}$ being a partition of αs, From definition (42) and the distributive property of exercise (4), we have:

$$I^{\mu}(\alpha s) = \sum_{i=1}^{n} \alpha\alpha_i \mu(A_i) = \alpha \left(\sum_{i=1}^{n} \alpha_i \mu(A_i) \right) = \alpha I^{\mu}(s)$$

5. If $\alpha = +\infty$ or $\alpha < 0$, the map αs may not have values in $\mathbf{R}^+$. In particular, αs may not be a simple function. As definition (42) only defines the integral of simple functions, $I^{\mu}(\alpha s)$ may not be meaningful.

6. Suppose $s \leq t$. Equations (5) and (6) being partitions of s and t respectively, from definition (42), we have:

$$I^{\mu}(s) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \mu(A_i \cap B_j)$$

and:

$$I^{\mu}(t) = \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_j \mu(A_i \cap B_j)$$

If $A_i \cap B_j = \emptyset$, then in particular $\mu(A_i \cap B_j) = 0$, and we have $\alpha_i \mu(A_i \cap B_j) \leq \beta_j \mu(A_i \cap B_j)$. If $A_i \cap B_j \neq \emptyset$, then there exists $\omega \in A_i \cap B_j$, in which case, $\alpha_i = s(\omega) \leq t(\omega) = \beta_j$. In any case, we have $\alpha_i \mu(A_i \cap B_j) \leq \beta_j \mu(A_i \cap B_j)$. This being true for all (i, j), it follows that $I^{\mu}(s) \leq I^{\mu}(t)$.

Exercise 5

Exercise 6.

1. Since f is measurable, each set $\{k/2^n \leq f < (k+1)/2^n\}$ belongs to $\mathcal{F}$, for $n \geq 1$ and $k = 0, \ldots, n2^n - 1$. $\{n \leq f\}$ is also an element of $\mathcal{F}$. Moreover, $k/2^n \in \mathbf{R}^+$ and $n \in \mathbf{R}^+$. It follows from definition (40) that each s_n as defined by (1), is indeed a simple function on $(\Omega, \mathcal{F})$.

2. $[0, +\infty] = \left(\uplus_{k=0}^{n2^n-1}[k/2^n, (k+1)/2^n[\right) \uplus [n, +\infty]$. Hence:

$$\Omega = f^{-1}([0, +\infty]) = \left(\biguplus_{k=0}^{n2^n-1} \left\{\frac{k}{2^n} \leq f < \frac{k+1}{2^n}\right\}\right) \uplus \{n \leq f\}$$

   It follows that equation (1) is indeed a partition of s_n.

3. Let $n \geq 1$ and $\omega \in \Omega$. Suppose $f(\omega) \in [0, n[$. Then, there exists $k \in \{0, \ldots, n2^n - 1\}$, such that $f(\omega) \in [k/2^n, (k+1)/2^n[$. In particular, $s_n(\omega) = k/2^n \leq f(\omega)$. If $f(\omega) \in [n, +\infty]$, then $s_n(\omega) = n \leq f(\omega)$. In any case, $s_n(\omega) \leq f(\omega)$. This being

true for all $\omega \in \Omega$, $s_n \leq f$. Suppose $f(\omega) \in [k/2^n, (k+1)/2^n[$. Then, $f(\omega) \in [(2k)/2^{n+1}, (2k+1)/2^{n+1}[$ or alternatively, we have $f(\omega) \in [(2k+1)/2^{n+1}, (2k+2)/2^{n+1}[$. In the first case, $s_n(\omega) = k/2^n = (2k)/2^{n+1} = s_{n+1}(\omega)$. In the second case, $s_n(\omega) = k/2^n \leq (2k+1)/2^{n+1} = s_{n+1}(\omega)$. In both cases, we have $s_n(\omega) \leq s_{n+1}(\omega)$. Suppose that $f(\omega) \in [n, +\infty]$. Then, either $f(\omega) \in [n, n+1[$ or $f(\omega) \in [n+1, +\infty]$. In the first case, $s_{n+1}(\omega) = k/2^{n+1}$ for some $k \in \{n2^{n+1}, \ldots, (n+1)2^{n+1} - 1\}$, and in particular, $s_n(\omega) = n \leq k/2^{n+1} = s_{n+1}(\omega)$. In the second case, $s_n(\omega) = n \leq n+1 = s_{n+1}(\omega)$. In both cases, we have $s_n(\omega) \leq s_{n+1}(\omega)$. We have proved that $s_n \leq s_{n+1} \leq f$.

4. Let $\omega \in \Omega$. If $f(\omega) = +\infty$, then $\omega \in \{n \leq f\}$, for all $n \geq 1$. It follows that $s_n(\omega) = n$ for all $n \geq 1$, and $s_n(\omega) \to +\infty = f(\omega)$. If $f(\omega) < +\infty$, then $f(\omega) \in [0, N[$ for some integer $N \geq 1$. For all $n \geq N$, $f(\omega) \in [0, n[$, and therefore $s_n(\omega) = k/2^n$, for some $k \in \{0, \ldots, n2^n - 1\}$, such that $k/2^n \leq f(\omega) < (k+1)/2^n$. In particular, $0 \leq f(\omega) - s_n(\omega) < 1/2^n$. This being true for all

$n \geq N$, we see that $s_n(\omega) \to f(\omega)$. We have proved that for all $\omega \in \Omega$, the sequence $(s_n(\omega))_{n\geq 1}$ converges to $f(\omega)$. From 3., this sequence is non-decreasing. Finally, we have $s_n \uparrow f$. The purpose of this exercise is to prove theorem (18).

Exercise 6

Exercise 7.

1. $0 = 0.1_\Omega$ is a simple function on $(\Omega, \mathcal{F})$. Since f is non-negative, $0 \leq f$. From definition (43), it follows that $I^\mu(0) \leq \int f d\mu$. Since $I^\mu(0) = 0$, we conclude that $\int f d\mu \in [0, +\infty]$.

2. Suppose f is a simple function on $(\Omega, \mathcal{F})$. Let s be another simple function on $(\Omega, \mathcal{F})$, such that $s \leq f$. From exercise (5), we have $I^\mu(s) \leq I^\mu(f)$. It follows that $I^\mu(f)$ is an upper-bound of all $I^\mu(s)$ for s simple function on $(\Omega, \mathcal{F})$ with $s \leq f$. The Lebesgue integral $\int f d\mu$ being the smallest of such upper-bound, we have $\int f d\mu \leq I^\mu(f)$. However, since $f \leq f$ and f is a simple function on $(\Omega, \mathcal{F})$, from definition (43), $I^\mu(f) \leq \int f d\mu$. We conclude that $\int f d\mu = I^\mu(f)$.

3. Let $g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be non-negative and measurable such that $g \leq f$. Let s be a simple function on $(\Omega, \mathcal{F})$ such that $s \leq g$. Then in particular, $s \leq f$, and it follows from definition (43) that $I^\mu(s) \leq \int f d\mu$. Hence, $\int f d\mu$ is an upper-bound of all

$I^\mu(s)$, for s simple function on $(\Omega, \mathcal{F})$ with $s \leq g$. The Lebesgue integral $\int g d\mu$ being the smallest of such upper-bound, we have $\int g d\mu \leq \int f d\mu$.

4. Let $0 < c < +\infty$. Since f is non-negative and measurable, $\int f d\mu$ is well-defined by virtue of definition (43). However, cf is also non-negative and measurable[4]. So $\int (cf) d\mu$ is also well-defined. Let s be a simple function on $(\Omega, \mathcal{F})$ such that $s \leq f$. Since $c \in \mathbf{R}^+$, from exercise (5), cs is also a simple function on $(\Omega, \mathcal{F})$. We have $cs \leq cf$. From definition (43), it follows that $I^\mu(cs) \leq \int (cf) d\mu$. However, from exercise (5), $I^\mu(cs) = cI^\mu(s)$. Since $c > 0$, we have $I^\mu(s) \leq c^{-1} \int (cf) d\mu$. Hence, $c^{-1} \int (cf) d\mu$ is an upper-bound of all $I^\mu(s)$, for s simple function on $(\Omega, \mathcal{F})$ with $s \leq f$. The Lebesgue integral $\int f d\mu$ being the smallest of such upper-bound, we have $\int f d\mu \leq c^{-1} \int (cf) d\mu$. Multiplying both sides by c, we obtain that $c \int f d\mu \leq \int (cf) d\mu$. Similarly, since $0 < 1/c < +\infty$, we have $c^{-1} \int (cf) d\mu \leq \int c^{-1}(cf) d\mu$, i.e.

[4]See exercise (19) of the previous tutorial. (Beware of external links !)

$\int(cf)d\mu \leq c\int fd\mu$. We conclude that $\int(cf)d\mu = c\int fd\mu$. If $c = 0$, whether or not $\int fd\mu = +\infty$, we have $c\int fd\mu = 0$. Since 0 is a simple function on $(\Omega, \mathcal{F})$, we have $\int 0d\mu = I^{\mu}(0) = 0$. It follows that the equality $\int(cf)d\mu = c\int fd\mu$ is still true in the case when $c = 0$.

5. f being measurable, $A_n = \{f > 1/n\}$ is an element of the σ-algebra $\mathcal{F}$. Since $1/n \in \mathbf{R}^+$, from definition (40) it follows that $s_n = (1/n)1_{A_n}$ is a simple function on $(\Omega, \mathcal{F})$. Suppose that $\omega \in \Omega$. If $\omega \notin A_n$, then $s_n(\omega) = 0 \leq f(\omega)$. If $\omega \in A_n$, then $s_n(\omega) = 1/n < f(\omega)$. In any case, $s_n(\omega) \leq f(\omega)$. It follows that $s_n \leq f$. Let $n \geq 1$, if $\omega \in A_n$, then $f(\omega) > 1/n$ and in particular $f(\omega) > 1/(n+1)$. So $\omega \in A_{n+1}$ and we see that $A_n \subseteq A_{n+1}$. For all $n \geq 1$, $A_n \subseteq \{f > 0\}$. It follows that $\cup_{n=1}^{+\infty} A_n \subseteq \{f > 0\}$. Conversely, if $f(\omega) > 0$, then there exists $n \geq 1$ such that $f(\omega) > 1/n$. So $\{f > 0\} \subseteq \cup_{n=1}^{+\infty} A_n$. We have proved that $A_n \subseteq A_{n+1}$ with $\cup_{n=1}^{+\infty} A_n = \{f > 0\}$, i.e. $A_n \uparrow \{f > 0\}$.

6. Suppose that $\int f d\mu = 0$. Given $n \geq 1$, let s_n and A_n be defined as in 5. s_n being a simple function on $(\Omega, \mathcal{F})$ with $s_n \leq f$, from definition (43) we have $I^{\mu}(s_n) \leq \int f d\mu = 0$. Hence, we have $I^{\mu}(s_n) = 0$. From definition (42), $I^{\mu}(s_n) = (1/n)\mu(A_n)$. It follows that $\mu(A_n) = 0$ for all $n \geq 1$. However, from 5., we have $A_n \uparrow \{f > 0\}$. Using theorem (7), $\mu(A_n) \uparrow \mu(\{f > 0\})$. It follows that $\mu(\{f > 0\}) = \lim_{n \to +\infty} \mu(A_n) = 0$. We have proved that $\int f d\mu = 0 \Rightarrow \mu(\{f > 0\}) = 0$.

7. Let s be a simple function on $(\Omega, \mathcal{F})$ with $s \leq f$. Suppose that $\mu(\{f > 0\}) = 0$. Let $s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$ be a partition of the simple function s. From definition (42), $I^{\mu}(s) = \sum_{i=1}^{n} \alpha_i \mu(A_i)$. Let $i \in \{1, \ldots, n\}$. If $\alpha_i > 0$ and $\omega \in A_i$, $A_1, \ldots, A_n$ being pairwise disjoint, $\alpha_i = s(\omega) \leq f(\omega)$. In particular, $0 < f(\omega)$. Hence, $A_i \subseteq \{f > 0\}$. μ being a measure on $\mathcal{F}$, we have[5] $\mu(A_i) \leq \mu(\{f > 0\})$. It follows that $\mu(A_i) = 0$. In particular, $\alpha_i \mu(A_i) = 0$. If $\alpha_i = 0$, whether or not $\mu(A_i) = +\infty$, we still

[5]See exercise (9) of Tutorial 2. (Beware of external links !)

have $\alpha_i\mu(A_i) = 0$. We conclude that $I^\mu(s) = \sum_{i=1}^n \alpha_i\mu(A_i) = 0$.

8. $\int f d\mu = 0 \Rightarrow \mu(\{f > 0\}) = 0$ was proved in 6. Suppose conversely that $\mu(\{f > 0\}) = 0$. Let s be a simple function on $(\Omega, \mathcal{F})$ such that $s \leq f$. From 7., $I^\mu(s) = 0$. It follows that 0 is an upper-bound of all $I^\mu(s)$ for s simple function on $(\Omega, \mathcal{F})$ with $s \leq f$. The Lebesgue integral $\int f d\mu$ being the smallest of such upper-bound, we have $\int f d\mu \leq 0$. However, from 1., $\int f d\mu \geq 0$. We have proved that $\int f d\mu = 0$, if and only if $\mu(\{f > 0\} = 0$.

9. f being non-negative and measurable, $\int f d\mu$ is well-defined, by virtue of definition (43). However, $(+\infty)f$ is also non-negative and measurable[6]. So $\int (+\infty) f d\mu$ is also well-defined. Suppose that $\int f d\mu = 0$. Then, $(+\infty)\int f d\mu = 0$. From 8. (or 6.), we have $\mu(\{f > 0\}) = 0$. However, $\{f > 0\} = \{(+\infty)f > 0\}$. So $\mu(\{(+\infty)f > 0\}) = 0$. Hence, from 8., $\int (+\infty) f d\mu = 0$. It follows that $\int (+\infty) f d\mu = (+\infty)\int f d\mu$. Suppose $\int f d\mu > 0$.

[6]See exercise (19) of the previous tutorial. (Beware of external links !)

Then, $(+\infty)\int f d\mu = +\infty$. However, from 8., $\mu(\{f > 0\}) > 0$. Let $A = \{f > 0\} = \{(+\infty)f = +\infty\}$. For all $n \geq 1$, we have $n1_A \leq (+\infty)f$. Using 3., 2., and the fact that $n1_A$ is a simple function on $(\Omega, \mathcal{F})$, we see that $n\mu(A) \leq \int(+\infty)f d\mu$, for all $n \geq 1$. Since $\mu(A) > 0$, we have $\int(+\infty)f d\mu = +\infty$. We conclude that $\int(+\infty)f d\mu = (+\infty)\int f d\mu$ is true in all possible cases. Looking back at 4., $\int(cf)d\mu = c\int f d\mu$ is therefore true for all $c \in [0, +\infty]$.

10. If $\omega \in \{f = +\infty\}$, then $(+\infty)1_{\{f=+\infty\}}(\omega) = +\infty = f(\omega)$. If $\omega \notin \{f = +\infty\}$, then $(+\infty)1_{\{f=+\infty\}}(\omega) = 0 \leq f(\omega)$. In any case, $(+\infty)1_{\{f=+\infty\}}(\omega) \leq f(\omega)$. Using 9. and 2., we have:

$$\int(+\infty)1_{\{f=+\infty\}}d\mu = (+\infty)\int 1_{\{f=+\infty\}}d\mu = (+\infty)\mu(\{f = +\infty\})$$

11. Suppose $\int f d\mu < +\infty$. From 10., $(+\infty)1_{\{f=+\infty\}} \leq f$. Using 3. and 10., we have $(+\infty)\mu(\{f = +\infty\}) \leq \int f d\mu$. It follows that $(+\infty)\mu(\{f = +\infty\}) < +\infty$. Hence, $\mu(\{f = +\infty\}) = 0$.

12. If $f = 1$, then $f = 1.1_\Omega$ and $\int f d\mu = I^\mu(f) = \mu(\Omega) = +\infty$. However, $\mu(\{f = +\infty\}) = \mu(\emptyset) = 0$. Hence, the converse of 11. is not true in general.

Exercise 7

Exercise 8.

1. If $s = \sum_{i=1}^n \alpha_i 1_{A_i}$ is a simple function on $(\Omega, \mathcal{F})$, then we have $s1_A = \sum_{i=1}^n \alpha_i 1_{A\cap A_i}$ with $\alpha_i \in \mathbf{R}^+$ and $A \cap A_i \in \mathcal{F}$. From definition (40), $s1_A$ is indeed a simple function on $(\Omega, \mathcal{F})$.

2. If $s = \sum_{i=1}^n \alpha_i 1_{A_i}$ is a partition of s, from definition (41), we have $\Omega = \uplus_{i=1}^n A_i$. It follows that $\Omega = (\uplus_{i=1}^n (A \cap A_i)) \uplus A^c$. Hence, $s1_A = \sum_{i=1}^n \alpha_i 1_{A\cap A_i} + 0.1_{A^c}$ is a partition of $s1_A$. From definition (42), we have:

$$I^\mu(s1_A) = \sum_{i=1}^n \alpha_i \mu(A \cap A_i) + 0.\mu(A^c) = \sum_{i=1}^n \alpha_i \mu(A \cap A_i)$$

3. $\nu(\emptyset) = I^\mu(0) = 0$. Let $(B_k)_{k\geq 1}$ be a sequence of pairwise disjoint elements of $\mathcal{F}$. Let $A = \uplus_{k=1}^{+\infty} B_k$. Let $s = \sum_{i=1}^n \alpha_i 1_{A_i}$ be a partition of s. For all $i = 1, \ldots, n$, $A \cap A_i = \uplus_{k=1}^{+\infty} (B_k \cap A_i)$. μ being a measure on $\mathcal{F}$, we have $\mu(A \cap A_i) = \sum_{k=1}^{+\infty} \mu(B_k \cap A_i)$.

Hence, using 2.:

$$I^{\mu}(s1_A) = \sum_{i=1}^{n} \alpha_i \mu(A \cap A_i) = \sum_{k=1}^{+\infty} \sum_{i=1}^{n} \alpha_i \mu(B_k \cap A_i) = \sum_{k=1}^{+\infty} I^{\mu}(s1_{B_k})$$

It follows that $\nu(A) = \sum_{k=1}^{+\infty} \nu(B_k)$. We have proved that ν is indeed a measure on $\mathcal{F}$[7].

4. From 3., ν is a measure on $\mathcal{F}$. If $(A_n)_{n \geq 1}$ is a sequence of elements of $\mathcal{F}$, such that $A_n \uparrow A$, using theorem (7), we have $\nu(A_n) \uparrow \nu(A)$. In other words, $I^{\mu}(s1_{A_n}) \uparrow I^{\mu}(s1_A)$.

Exercise 8

[7]See definition (9). (Beware of external links !)

Exercise 9.

1. $f_n \uparrow f$ means that for all $\omega \in \Omega$, $f_n(\omega) \uparrow f(\omega)$. In other words, the sequence $(f_n(\omega))_{n\geq 1}$ is non-decreasing and converges to $f(\omega)$ in $\bar{\mathbf{R}}$.

2. The fact that $f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, is a consequence of exercise (15), and the fact that $f = \sup_{n\geq 1} f_n$. One can also apply theorem (17), and argue that as a limit of measurable maps with values in the metrizable space $\bar{\mathbf{R}}$, f is itself a measurable map.

3. Let $\alpha = \sup_{n\geq 1} \int f_n d\mu$. Since $f_n \leq f_{n+1}$ for all $n \geq 1$, from exercise (7), $\int f_n d\mu \leq \int f_{n+1} d\mu$. Being a non-decreasing sequence in $\bar{\mathbf{R}}$, $(\int f_n d\mu)_{n\geq 1}$ converges to its supremum. So $\int f_n d\mu \uparrow \alpha$.

4. Since $f = \sup_{n\geq 1} f_n$, for all $n \geq 1$, $f_n \leq f$. From exercise (7), $\int f_n d\mu \leq \int f d\mu$. It follows that $\int f d\mu$ is an upper-bound of all $\int f_n d\mu$ for $n \geq 1$. Since α is the smallest of such upper-bound, we have $\alpha \leq \int f d\mu$.

5. From exercise (5), cs is itself a simple function on $(\Omega, \mathcal{F})$. From exercise (2), it is therefore measurable. Hence, given $n \geq 1$, both cs and f_n are measurable. It follows that[8] $A_n = \{cs \leq f_n\} \in \mathcal{F}$. Let $n \geq 1$. Suppose $\omega \in A_n$. Then, $cs(\omega) \leq f_n(\omega) \leq f_{n+1}(\omega)$. So $\omega \in A_{n+1}$ and $A_n \subseteq A_{n+1}$. Let $\omega \in \Omega$. If $s(\omega) = 0$, then $\omega \in A_n$ for all $n \geq 1$. Suppose $s(\omega) > 0$. Then, we have $0 < s(\omega) < +\infty$. Since $c \in]0,1[$, we have $cs(\omega) < s(\omega)$. It follows that $cs(\omega) < f(\omega) = \sup_{n\geq 1} f_n(\omega)$. Since $f(\omega)$ is the smallest upper-bound of all $f_n(\omega)$ for $n \geq 1$, we see that $cs(\omega)$ cannot be such upper-bound. There exists $n \geq 1$ such that $cs(\omega) < f_n(\omega)$. In particular, there exists $n \geq 1$, such that $\omega \in A_n$. Hence, $\Omega = \cup_{n=1}^{+\infty} A_n$, with $A_n \subseteq A_{n+1}$, i.e. $A_n \uparrow \Omega$.

6. For all $n \geq 1$, we have $cs1_{A_n} \leq f_n$. Hence, using exercise (7), $\int cs1_{A_n} d\mu \leq \int f_n d\mu$. But $\int cs1_{A_n} d\mu = c \int s1_{A_n} d\mu$. From exercise (8), $s1_{A_n}$ is a simple function on $(\Omega, \mathcal{F})$. Using exercise (7) once more, $\int s1_{A_n} d\mu = I^{\mu}(s1_{A_n})$. We conclude that

[8]See exercise (17) of the previous tutorial. (Beware of external links !)

$cI^\mu(s1_{A_n}) \leq \int f_n d\mu$ for all $n \geq 1$.

7. From exercise (8), since $A_n \uparrow \Omega$, $I^\mu(s1_{A_n}) \uparrow I^\mu(s)$. In particular, $cI^\mu(s1_{A_n}) \uparrow cI^\mu(s)$[9]. From 3., $\int f_n d\mu \uparrow \alpha$. From 6., $cI^\mu(s1_{A_n}) \leq \int f_n d\mu$ for all $n \geq 1$. Taking the limit as $n \to +\infty$, we conclude that $cI^\mu(s) \leq \alpha$.

8. Since $cI^\mu(s) \leq \alpha$ for all $c \in]0,1[$, we have $I^\mu(s) \leq \alpha$.

9. From 8., α is an upper-bound of all $I^\mu(s)$ for s simple function on $(\Omega, \mathcal{F})$, such that $s \leq f$. The Lebesgue integral $\int f d\mu$ being the smallest of such upper-bound, we have $\int f d\mu \leq \alpha$.

10. From 4. and 9., we have $\alpha = \int f d\mu$. Using 3., we conclude that $\int f_n d\mu \uparrow \int f d\mu$. In other words, $(\int f_n d\mu)_{n\geq 1}$ is a non-decreasing sequence in $[0, +\infty]$, converging to $\int f d\mu$. The purpose of this exercise is to prove theorem (19).

Exercise 9

[9] If we had $c = +\infty$ and $\alpha_n = 1/n$, then $\alpha_n \downarrow 0$, but $c\alpha_n \downarrow 0$ fails to be true.

Exercise 10.

1. Given two sequences $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ in $\mathbf{R}$ converging to $\alpha \in \mathbf{R}$ and $\beta \in \mathbf{R}$ respectively, the fact that $\alpha_n + \beta_n \to \alpha + \beta$ is known and easy to prove. However, when we allow $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ to be sequences in $\bar{\mathbf{R}}$, with limits α, β in $\bar{\mathbf{R}}$, problems may occur. For a start, the sum $\alpha_n + \beta_n$ may not be meaningful. Or indeed, even if $\alpha_n + \beta_n$ does make sense, it is possible that the sum $\alpha + \beta$ doesn't. In the case when $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ are sequences in $[0, +\infty]$, then all $\alpha_n + \beta_n$'s and $\alpha + \beta$ are meaningful. If both α and β are finite, then $\alpha_n + \beta_n \to \alpha + \beta$ stems from the known real case[10]. If $\alpha = +\infty$ or $\beta = +\infty$, then $\alpha + \beta = +\infty$, and it is easy to prove that $\alpha_n + \beta_n \to +\infty$. Now, if $f_n \uparrow f$ and $g_n \uparrow g$, then for all $\omega \in \Omega$, $(f_n(\omega))_{n\geq 1}$ and $(g_n(\omega))_{n\geq 1}$ are non-decreasing sequences in $[0, +\infty]$ converging to $f(\omega)$ and $g(\omega)$ respectively. So $(f_n(\omega) + g_n(\omega))_{n\geq 1}$ is non-decreasing, and converges to $f(\omega) + g(\omega)$, i.e. $f_n + g_n \uparrow f + g$.

[10] Both sequences are eventually with values in $\mathbf{R}$.

2. Let $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. From theorem (18), there exist two sequences $(s_n)_{n\geq 1}$ and $(t_n)_{n\geq 1}$ of simple functions on $(\Omega, \mathcal{F})$, such that $s_n \uparrow f$ and $t_n \uparrow g$. Hence, $s_n + t_n \uparrow f + g$. From the monotone convergence theorem (19), we have $\int (s_n + t_n) d\mu \uparrow \int (f + g) d\mu$. From exercise (5), $s_n + t_n$ is a simple function on $(\Omega, \mathcal{F})$. It follows from exercise (7) that $\int (s_n + t_n) d\mu = I^\mu(s_n + t_n)$. Hence, $I^\mu(s_n + t_n) \uparrow \int (f + g) d\mu$. Similarly, $I^\mu(s_n) \uparrow \int f d\mu$ and $I^\mu(t_n) \uparrow \int g d\mu$. However from exercise (5), we have:

$$I^\mu(s_n + t_n) = I^\mu(s_n) + I^\mu(t_n)$$

Taking the limit as $n \to +\infty$, we obtain:

$$\int (f + g) d\mu = \int f d\mu + \int g d\mu$$

3. This is an immediate application of 2. and exercise (7).

Exercise 10

Exercise 11.

1. Given $\omega \in \Omega$, $f(\omega) = \sum_{k=1}^{+\infty} f_k(\omega)$ is a series of non-negative terms. It is therefore well-defined and non-negative. Given $n \geq 1$, all f_k's being measurable, the partial sum $g_n = \sum_{k=1}^{n} f_k$ is itself measurable[11]. So $f = \sup_{n\geq 1} g_n$ is measurable[12]. We conclude that $f = \sum_{k=1}^{+\infty} f_k$ is well-defined, non-negative and measurable.

2. Given $n \geq 1$, let $g_n = \sum_{k=1}^{n} f_k$. Since $g_n \uparrow f$, from the monotone convergence theorem (19), we have $\int g_n d\mu \uparrow \int f d\mu$. However, from exercise (10), $\int g_n d\mu = \sum_{k=1}^{n} \int f_k d\mu$. Hence, we see that the sequence $(\sum_{k=1}^{n} \int f_k d\mu)_{n\geq 1}$ converges to $\int f d\mu$. In other words, we have $\int f d\mu = \sum_{k=1}^{+\infty} \int f_k d\mu$.

Exercise 11

[11]See exercise (19) of the previous tutorial. (Beware of external links !)
[12]See exercise (15) of the previous tutorial.

Exercise 12.

1. Let $M = \{\omega \in \Omega : \mathcal{P}(\omega) \text{ holds}\}^c$. By assumption, $M \in \mathcal{F}$. Suppose that $\mathcal{P}(\omega)$ holds μ-almost surely. From definition (44), there exists $N \in \mathcal{F}$ such that $\mu(N) = 0$ and $\mathcal{P}(\omega)$ holds for all $\omega \in N^c$. In particular, $N^c \subseteq M^c$. So $M \subseteq N$, and therefore $\mu(M) \leq \mu(N)$[13]. Since $\mu(N) = 0$, we see that $\mu(M) = 0$. Conversely, suppose that $\mu(M) = 0$. From the very definition of M, for all $\omega \in M^c$, $\mathcal{P}(\omega)$ holds. From definition (44), it follows that $\mathcal{P}(\omega)$ holds μ-almost surely. We have proved that $\mathcal{P}(\omega)$ holds μ-almost surely, if and only if $\mu(M) = 0$.

2. In all generality, the set $\{\omega \in \Omega : \mathcal{P}(\omega) \text{ holds}\}$ may not be an element of $\mathcal{F}$. Hence, a notation such as $\mu(\{\omega \in \Omega : \mathcal{P}(\omega) \text{ holds}\}^c)$ may not be meaningful. It follows that such notation cannot be used in any criterion defining μ-almost sure properties.

Exercise 12

[13]See exercise (9) of Tutorial 2. (Beware of external links !)

Exercise 13. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $(A_n)_{n\geq 1}$ be a sequence of elements of $\mathcal{F}$. Define $B_1 = A_1$ and for all $n \geq 1$, $B_{n+1} = A_{n+1} \setminus (B_1 \cup \ldots \cup B_n)$. Then $(B_n)_{n\geq 1}$ is a sequence of elements of $\mathcal{F}$, and we claim that $\cup_{n\geq 1} A_n = \uplus_{n\geq 1} B_n$. Indeed, it is clear that $B_n \subseteq A_n$ for all $n \geq 1$ and consequently $\cup_{n\geq 1} B_n \subseteq \cup_{n\geq 1} A_n$. Furthermore, if $x \in \cup_{n\geq 1} A_n$ there exists $n \geq 1$ such that $x \in A_n$. The set $\{n \in \mathbf{N} : x \in A_n\}$ is therefore a non-empty subset of $\mathbf{N}$ and has a smallest element, say $p \geq 1$. Then $x \in A_p$ and for all $k < p$ we have $x \notin A_k$. In particular for all $k < p$, $x \notin B_k$. Hence, it is clear that $x \in B_p$. We have proved that $\cup_{n\geq 1} A_n \subseteq \cup_{n\geq 1} B_n$ and finally $\cup_{n\geq 1} A_n = \cup_{n\geq 1} B_n$. It remains to show that the B_n's are pairwise disjoint. Suppose $n \neq m$ and $x \in B_n \cap B_m$. Without loss of generality, we may assume that $n < m$. But $x \in B_m$ implies $x \notin B_n$ which is a contradiction. So the B_n's are indeed pairwise disjoint. Having proved that $\cup_{n\geq 1} A_n = \uplus_{n\geq 1} B_n$, we conclude from the fact that $B_n \subseteq A_n$ implies $\mu(B_n) \leq \mu(A_n)$ [14] and:

$$\mu\left(\bigcup_{n=1}^{+\infty} A_n\right) = \mu\left(\biguplus_{n=1}^{+\infty} B_n\right) = \sum_{n=1}^{+\infty} \mu(B_n) \leq \sum_{n=1}^{+\infty} \mu(A_n)$$

Exercise 13

[14] See exercise (9) of Tutorial 2.

Exercise 14.

1. From definition (44), the statement $f_n \uparrow f$ μ-a.s. is formally translated as follows: there exists $N \in \mathcal{F}$ such that $\mu(N) = 0$, and for all $\omega \in N^c$, we have $f_n(\omega) \uparrow f(\omega)$, i.e. the sequence $(f_n(\omega))_{n\geq 1}$ is non-decreasing and converges to $f(\omega)$.

2. From definition (44), $f_n \to f$ μ-a.s. and $f_n \leq f_{n+1}$ μ-a.s. for all $n \geq 1$, is formally translated as follows: there exist $N \in \mathcal{F}$ and a sequence $(N_n)_{n\geq 1}$ of elements of $\mathcal{F}$, such that $\mu(N) = 0$ and $\mu(N_n) = 0$ for all $n \geq 1$, and for all $\omega \in N^c$, $f_n(\omega) \to f(\omega)$, and given $n \geq 1$ and $\omega \in N_n^c$, $f_n(\omega) \leq f_{n+1}(\omega)$.

3. Suppose that $f_n \uparrow f$ μ-a.s., i.e. that statement 1. is satisfied. Taking $N_n = N$ for all $n \geq 1$, it is clear that statement 2. is also satisfied. Conversely, suppose that statement 2. is satisfied. Define $M = N \cup (\cup_{n=1}^{+\infty} N_n)$. Then $M \in \mathcal{F}$, and from exercise (13), we have $\mu(M) \leq \mu(N) + \sum_{n=1}^{+\infty} \mu(N_n)$. So $\mu(M) = 0$. Moreover, for all $\omega \in M^c$, it is clear that $f_n(\omega) \uparrow f(\omega)$. It follows that

$f_n \uparrow f$ μ-a.s. is true. We have proved that both statements 1. and 2. are equivalent. This exercise is pretty important. More generally, if a condition $\mathcal{P}(\omega)$ is true μ-a.s and another condition $\mathcal{Q}(\omega)$ is true μ-a.s., then $(\mathcal{P}(\omega)$ and $\mathcal{Q}(\omega))$ is also true μ-a.s.. In fact, we have just seen that this *factoring* of 'μ-a.s.' is valid for a countable number of conditions, which is a straightforward application of the fact that a countable union of measurable sets (belonging to $\mathcal{F}$) of μ-measure 0, is itself measurable (belonging to $\mathcal{F}$) of μ-measure 0.

Exercise 14

Exercise 15. Given $B \in \mathcal{B}(\bar{\mathbf{R}})$, $\{f1_N \in B\}$ is equal to $\{f \in B\} \cap N$ if $0 \notin B$, or equal to $(\{f \in B\} \cap N) \cup N^c$ if $0 \in B$. In any case, $\{f1_N \in B\} \in \mathcal{F}$ and $f1_N$ is therefore non-negative and measurable. Similarly $f1_{N^c}$ is non-negative and measurable. So both integrals $\int f1_N d\mu$ and $\int f1_{N^c} d\mu$ are well-defined by virtue of definition (43). Since $f = f1_N + f1_{N^c}$, we have $\int f d\mu = \int f1_N d\mu + \int f1_{N^c} d\mu$, from exercise (10). Similarly, $\int g d\mu = \int g1_N d\mu + \int g1_{N^c} d\mu$. However, for all $\omega \in N^c$, $f(\omega) = g(\omega)$. It follows that $f1_{N^c} = g1_{N^c}$. Moreover, $\mu(N) = 0$. Since $\{f1_N > 0\} \subseteq N$, we see that $\mu(\{f1_N > 0\}) = 0$. Hence, from exercise (7), $\int f1_N d\mu = 0$. Similarly, $\int g1_N d\mu = 0$. We conclude that:

$$\int f d\mu = \int f1_{N^c} d\mu = \int g1_{N^c} d\mu = \int g d\mu$$

Exercise 15

Exercise 16.

1. Given $B \in \mathcal{B}(\bar{\mathbf{R}})$, $\{f1_{N^c} \in B\}$ is either equal to $\{f \in B\} \cap N^c$ or $(\{f \in B\} \cap N^c) \cup N$, depending on whether $0 \in B$ or not. In any case $\{f1_{N^c} \in B\} \in \mathcal{F}$, and $\bar{f} = f1_{N^c}$ is therefore non-negative and measurable. Similarly, for all $n \geq 1$, $\bar{f}_n = f_n 1_{N^c}$ is non-negative and measurable.

2. If $\omega \in N^c$, then $\bar{f}_n(\omega) = f_n(\omega) \uparrow f(\omega) = \bar{f}(\omega)$. If $\omega \in N$, then $\bar{f}_n(\omega) = 0$ for all $n \geq 1$, and $\bar{f}(\omega) = 0$. In any case, $\bar{f}_n(\omega) \uparrow \bar{f}(\omega)$. We have proved that $\bar{f}_n \uparrow \bar{f}$.

3. From 2., we have $\bar{f}_n \uparrow \bar{f}$. Hence, from the monotone convergence theorem (19), $\int \bar{f}_n d\mu \uparrow \int \bar{f} d\mu$. However, from the very definition of $\bar{f}$ and $\bar{f}_n$, there exists $N \in \mathcal{F}$ with $\mu(N) = 0$, such that for all $\omega \in N^c$, $\bar{f}(\omega) = f(\omega)$ and $\bar{f}_n(\omega) = f_n(\omega)$. In other words, from definition (44), $\bar{f} = f$ μ-a.s. and $\bar{f}_n = f_n$ μ-a.s.. From exercise (15), it follows that $\int \bar{f} d\mu = \int f d\mu$ and $\int \bar{f}_n d\mu = \int f_n d\mu$ for all $n \geq 1$. We conclude that $\int f_n d\mu \uparrow \int f d\mu$. Although it

may not appear to be the case, this exercise is very important. The monotone convergence theorem (19) states that whenever $f_n \uparrow f$, we have $\int f_n d\mu \uparrow \int f d\mu$. In this exercise, we proved that in fact, a weaker condition of $f_n \uparrow f$ μ-a.s. is sufficient to ensure that $\int f_n d\mu \uparrow \int f d\mu$. We obtained that result with a standard technique of *cleaning up* our functions f and f_n's, to ensure that $f_n \uparrow f$ everywhere, as opposed to μ-a.s.. It is important to be familiar with this technique. In my experience, theorems with *almost sure* conditions are confusing to students, and are an encouragement to poor rigor and sloppy reasoning[15]. Hence, most theorems in these tutorials, at least in the early stages, will be stated with *everywhere* conditions. So you may need to *clean up* your assumptions again in the future...

Exercise 16

[15]Particularly when dealing with questions of measurability in a non-complete measure space.

Exercise 17.

1. Since $g_n = \inf_{k\geq n} f_k$, g_n is a countable infimum of measurable maps. It is therefore measurable[16], and is obviously non-negative.

2. Let $\omega \in \Omega$ and $n \geq 1$. For all $k \geq n$, we have $g_n(\omega) \leq f_k(\omega)$. In particular, $g_n(\omega)$ is a lower-bound of all $f_k(\omega)$ for $k \geq n+1$. Since $g_{n+1}(\omega)$ is the greatest of such lower-bound, we have $g_n(\omega) \leq g_{n+1}(\omega)$. It follows that $(g_n(\omega))_{n\geq 1}$ is a non-decreasing sequence in $\bar{\mathbf{R}}$, which therefore converges to its supremum. Hence, $g_n \uparrow \sup_{n\geq 1} g_n = \liminf f_n$[17].

3. For all $n \geq 1$, we have $g_n \leq f_n$. From exercise (7), it follows that $\int g_n d\mu \leq \int f_n d\mu$.

4. Let $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$ be two sequences in $\bar{\mathbf{R}}$ with $u_n \leq v_n$ for all $n \geq 1$. For all $k \geq n$, we have $\inf_{k\geq n} u_k \leq u_k \leq v_k$.

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[16] See exercise (15) of the previous tutorial. (Beware of external links !)

[17] See definition (36) of the previous tutorial.

Hence, $\inf_{k\geq n} u_k$ is a lower-bound of all v_k's for $k \geq n$. It follows that $\inf_{k\geq n} u_k \leq \inf_{k\geq n} v_k$. Hence, for all $n \geq 1$, we have $\inf_{k\geq n} u_k \leq \sup_{n\geq 1} \inf_{k\geq n} v_k = \liminf v_n$. In other words, $\liminf v_n$ is an upper-bound of all $\inf_{k\geq n} u_k$ for $n \geq 1$. It follows that $\sup_{n\geq 1} \inf_{k\geq 1} u_k \leq \liminf v_n$, i.e. $\liminf u_n \leq \liminf v_n$.

5. $\liminf f_n$ is measurable[18], and is obviously non-negative. The integral $\int(\liminf f_n)d\mu$ is therefore well-defined by virtue of definition (43). The same can be said of $\int f_n d\mu$ for all $n \geq 1$. From 3., we have $\int g_n d\mu \leq \int f_n d\mu$, for all $n \geq 1$. It follows from 4. that:

$$\liminf_{n\to+\infty} \int g_n d\mu \leq \liminf_{n\to+\infty} \int f_n d\mu \tag{8}$$

However, from 2., $g_n \uparrow \liminf f_n$. From the monotone convergence theorem (19), $\int g_n d\mu \uparrow \int(\liminf f_n)d\mu$. In particular, the sequence $(\int g_n d\mu)_{n\geq 1}$ converges to $\int(\liminf f_n)d\mu$. It follows

[18]See exercise (18) of the previous tutorial. (Beware of external links !)

from theorem (16), that:

$$\liminf_{n\to+\infty}\int g_n d\mu = \int(\liminf_{n\to+\infty} f_n)d\mu \tag{9}$$

Comparing (8) with (9), we conclude that:

$$\int(\liminf_{n\to+\infty} f_n)d\mu \leq \liminf_{n\to+\infty}\int f_n d\mu$$

The purpose of this exercise is to prove Fatou lemma (20).

Exercise 17

Exercise 18.

1. $\mathcal{F}_{|A} = \{A \cap B : B \in \mathcal{F}\}$ is the trace on A of the σ-algebra $\mathcal{F}$[19], which is a σ-algebra on A[20]. Since $A \in \mathcal{F}$, $\mathcal{F}_{|A} \subseteq \mathcal{F}$. It is therefore meaningful to define $\mu_{|A}$ as the restriction of μ to $\mathcal{F}_{|A}$, which is a measure[21] on $\mathcal{F}_{|A}$. It is important that we have $A \in \mathcal{F}$, since otherwise, $\mu_{|A}$ would not be meaningful. Let $B \in \mathcal{B}(\bar{\mathbf{R}})$. $f_{|A}$ being the restriction of f to A, we have $(f_{|A})^{-1}(B) = \{x \in A : f(x) \in B\} = A \cap f^{-1}(B)$. Since f is measurable, $f^{-1}(B) \in \mathcal{F}$. It follows that $(f_{|A})^{-1}(B) \in \mathcal{F}_{|A}$. We have proved that $f_{|A} : (A, \mathcal{F}_{|A}) \to [0, +\infty]$ is measurable.

2. Let $(E_n)_{n\geq 1}$ be a sequence of pairwise disjoint elements of $\mathcal{F}$. Let $E = \uplus_{n=1}^{+\infty} E_n$. Then, $A \cap E = \uplus_{n=1}^{+\infty}(A \cap E_n)$. μ being a measure on $\mathcal{F}$, $\mu(A \cap E) = \sum_{n=1}^{+\infty} \mu(A \cap E_n)$. It follows that $\mu^A(E) = \sum_{n=1}^{+\infty} \mu^A(E_n)$. It is clear that $\mu^A(\emptyset) = 0$. We have

[19] See definition (22). (Beware of external links !)
[20] See exercise (15) of Tutorial 3.
[21] See definition (9).

proved that μ^A is a measure on $\mathcal{F}$. $(\Omega, \mathcal{F}, \mu^A)$ is therefore a measure space[22].

3. Consider the following equality:

$$\int (f1_A)d\mu = \int f d\mu^A = \int (f_{|A})d\mu_{|A} \tag{10}$$

$\int (f1_A)d\mu$ is an integral defined on $(\Omega, \mathcal{F}, \mu)$. The map being integrated is $f1_A$ which is non-negative and measurable. The integral is therefore well-defined. $\int f d\mu^A$ is an integral defined on $(\Omega, \mathcal{F}, \mu^A)$. The map being integrated is f which is non-negative and measurable. The integral is therefore well-defined. $\int (f_{|A})d\mu_{|A}$ is an integral defined on $(A, \mathcal{F}_{|A}, \mu_{|A})$. The map being integrated is the restriction $f_{|A}$ which is non-negative and measurable with respect to $\mathcal{F}_{|A}$. The integral is therefore well-defined. At this stage, we do not know whether equation (10) is true, but at least, all its terms are meaningful...

[22]See definition (19). (Beware of external links !)

4. Suppose that equation (10) is true, whenever f is a simple function on $(\Omega, \mathcal{F})$. Suppose that f is an arbitrary non-negative and measurable map. From theorem (18), f can be approximated by a non-decreasing sequence of simple functions on $(\Omega, \mathcal{F})$. In other words, there exists a sequence $(s_n)_{n\geq 1}$ of simple functions on $(\Omega, \mathcal{F})$, such that $s_n \uparrow f$. In particular, $s_n 1_A \uparrow f 1_A$ and $(s_n)_{|A} \uparrow f_{|A}$. Having assumed that equation (10) is true for all simple functions on $(\Omega, \mathcal{F})$, for all $n \geq 1$, we have:

$$\int (s_n 1_A) d\mu = \int s_n d\mu^A = \int (s_n)_{|A} d\mu_{|A} \tag{11}$$

From the monotone convergence theorem (19), taking the limit as $n \to +\infty$ in (11) , we obtain equation (10). We conclude that in order to prove equation (10), it is sufficient to consider the case when f is a simple function on $(\Omega, \mathcal{F})$.

5. Suppose that equation (10) is true whenever f is of the form $f = 1_B$, for $B \in \mathcal{F}$. Let $s = \sum_{i=1}^n \alpha_i 1_{A_i}$ be a simple function on

$(\Omega, \mathcal{F})$. Then, $s1_A = \sum_{i=1}^n \alpha_i(1_{A_i}1_A)$ and $s_{|A} = \sum_{i=1}^n \alpha_i(1_{A_i})_{|A}$. Using the linearity of the integral proved in exercise (10):

$$\int s1_A d\mu = \sum_{i=1}^n \alpha_i \int 1_{A_i}1_A d\mu \tag{12}$$

$$\int s d\mu^A = \sum_{i=1}^n \alpha_i \int 1_{A_i} d\mu^A \tag{13}$$

$$\int s_{|A} d\mu_{|A} = \sum_{i=1}^n \alpha_i \int (1_{A_i})_{|A} d\mu_{|A} \tag{14}$$

Having assumed that equation (10) is true for all measurable characteristic functions, for all $i = 1, \ldots, n$, we have:

$$\int 1_{A_i}1_A d\mu = \int 1_{A_i} d\mu^A = \int (1_{A_i})_{|A} d\mu_{|A} \tag{15}$$

We conclude from (12), (13), (14) and (15) that equation (10) is true for all simple functions s on $(\Omega, \mathcal{F})$. Using, 4., equation (10) is therefore true for any non-negative and measurable map f. Hence, in order to prove equation (10), it is sufficient to consider the case when f is of the form $f = 1_B$ for $B \in \mathcal{F}$.

6. Suppose f is of the form $f = 1_B$ with $B \in \mathcal{F}$. Then, we have $f1_A = 1_{A\cap B}$, and $\int f1_A d\mu = \mu(A \cap B)$. Moreover, we have $\int f d\mu^A = \mu^A(B) = \mu(A \cap B)$. Finally, since[23] $(1_B)_{|A} = 1^*_{A\cap B}$, we have $\int (1_B)_{|A} d\mu_{|A} = \mu_{|A}(A \cap B) = \mu(A \cap B)$. We conclude that equation (10) is true for f. From 5., it follows that equation (10) is true for all non-negative and measurable maps. The purpose of this exercise is to justify definition (45). The techniques used in this exercise will be used over and over again in the future. Very often, when an equality between integrals has to be proved, one starts by verifying such equality for characteristic functions. By linearity, the equality can be extended

[23] We write $1^*_{A\cap B}$ as opposed to $1_{A\cap B}$ to emphasize the fact that it is the characteristic function of $A \cap B$, viewed as a subset of A. In other words, it is a map defined on A, not Ω...

to all simple functions. Using theorem (18) and the monotone convergence theorem (19), it can then be proved to be true for all non-negative and measurable maps.

Exercise 18

Exercise 19.

1. Let $(A_n)_{n\geq 1}$ be a sequence of pairwise disjoint elements of $\mathcal{F}$. Let $A = \uplus_{n=1}^{+\infty} A_n$. Then, $1_A = \sum_{n=1}^{+\infty} 1_{A_n}$, and consequently $f1_A = \sum_{n=1}^{+\infty} f1_{A_n}$. Hence, $\int f1_A d\mu = \sum_{n=1}^{+\infty} \int f1_{A_n} d\mu$, as proved in exercise (11). It follows that $\nu(A) = \sum_{n=1}^{+\infty} \nu(A_n)$. It is clear that $\nu(\emptyset) = \int f1_{\emptyset} d\mu = 0$. We conclude that ν is indeed a measure on $\mathcal{F}$.

2. Suppose g is of the form $g = 1_B$ with $B \in \mathcal{F}$. Then, we have $\int g d\nu = \nu(B) = \int_B f d\mu = \int f1_B d\mu = \int f g d\mu$. By linearity, it follows that $\int g d\nu = \int g f d\mu$ is true whenever g is a simple function on $(\Omega, \mathcal{F})$. If g is an arbitrary non-negative and measurable map, from theorem (18), there exists a sequence $(s_n)_{n\geq 1}$ of simple functions in $(\Omega, \mathcal{F})$, such that $s_n \uparrow g$. From $\int s_n d\nu = \int s_n f d\mu$ and the monotone convergence theorem (19), taking the limit as $n \to +\infty$, we conclude that $\int g d\nu = \int g f d\mu$.

Exercise 19

Exercise 20.

1. $|f|$ is non-negative and measurable. The integral $\int |f|d\mu$ is therefore well-defined.

2. if f is real-valued, and measurable with respect to $\mathcal{B}(\mathbf{C})$, then it is also measurable with respect to $\mathcal{B}(\mathbf{R})$, since $\mathcal{B}(\mathbf{R}) \subseteq \mathcal{B}(\mathbf{C})$. We have not proved this inclusion before. Here is one way of doing it: the usual metric on $\mathbf{R}$ is the metric induced by the usual metric on $\mathbf{C}$. From theorem (12), $\mathcal{T}_{\mathbf{R}} = (\mathcal{T}_{\mathbf{C}})_{|\mathbf{R}}$, i.e. the usual topology on $\mathbf{R}$ is induced from the usual topology on $\mathbf{C}$. From the trace theorem (10), it follows that $\mathcal{B}(\mathbf{R}) = \mathcal{B}(\mathbf{C})_{|\mathbf{R}}$, i.e. that the Borel σ-algebra on $\mathbf{R}$ is the trace on $\mathbf{R}$, of the Borel σ-algebra on $\mathbf{C}$. In particular, since $\mathbf{R} \in \mathcal{B}(\mathbf{C})$ (it is closed in $\mathbf{C}$), we have $\mathcal{B}(\mathbf{R}) \subseteq \mathcal{B}(\mathbf{C})$.

3. If f is measurable with respect to $\mathcal{B}(\mathbf{R})$, then it is also measurable with respect to $\mathcal{B}(\mathbf{C})$. Indeed, given $B \in \mathcal{B}(\mathbf{C})$, we have $B \cap \mathbf{R} \in \mathcal{B}(\mathbf{R})$ and therefore, $f^{-1}(B) = f^{-1}(B \cap \mathbf{R}) \in \mathcal{F}$. It

follows that $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \subseteq L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

4. If $f \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, then it is real-valued, and from 3., it is also an element of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Conversely, if f is real-valued and belongs to $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, then from 2., it is also measurable with respect to $\mathcal{B}(\mathbf{R})$, and therefore lies in $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. We have proved that $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) = \{f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \, : \, f(\Omega) \subseteq \mathbf{R}\}$.

5. Let $f, g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ and $\alpha, \beta \in \mathbf{R}$. Then $\alpha f + \beta g$ is measurable[24]. Moreover, since $|\alpha f + \beta g| \leq |\alpha||f| + |\beta||g|$, from exercise (7), and by linearity, we have:

$$\int |\alpha f + \beta g| d\mu \leq |\alpha| \int |f| d\mu + |\beta| \int |g| d\mu < +\infty$$

We conclude that $\alpha f + \beta g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

6. Let $f, g \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $\alpha, \beta \in \mathbf{C}$. Then, $\alpha f + \beta g$ is mea-

[24]See exercise (19) of the previous tutorials. (Beware of external links !)

surable[25]. Moreover, since $|\alpha f + \beta g| \leq |\alpha||f| + |\beta||g|$, from exercise (7), and by linearity, we have:

$$\int |\alpha f + \beta g| d\mu \leq |\alpha| \int |f| d\mu + |\beta| \int |g| d\mu < +\infty$$

We conclude that $\alpha f + \beta g \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

Exercise 20

[25] Both the real and imaginary parts of $\alpha f + \beta g$ are measurable. Conclude with exercise (25) of the previous tutorial. (Beware of external links !)

Exercise 21.

1. $u^+ - u^- = \max(u,0) - \max(-u,0) = \max(u,0) + \min(u,0)$. Hence, $u^+ - u^- = u + 0 = u$, and similarly, $v^+ - v^- = v$. Finally, we have $f = u + iv = u^+ - u^- + i(v^+ - v^-)$.

2. Let $\omega \in \Omega$. If $u(\omega) \geq 0$, then $u^+(\omega) = u(\omega)$ and $u^-(\omega) = 0$. If $u(\omega) \leq 0$, then $u^+(\omega) = 0$ and $u^-(\omega) = -u(\omega)$. In any case, $u^+(\omega) + u^-(\omega) = |u|(\omega)$. So $|u| = u^+ + u^-$, and similarly $|v| = v^+ + v^-$.

3. f being measurable, $|f|$, u and v are also measurable[26]. It follows that $|u|$ and $|v|$ are also measurable. From 1. and 2., we have $u^+ = (|u| + u)/2$ and $u^- = (|u| - u)/2$. So u^+, u^- and similarly v^+, v^- are measurable. Moreover, u^+, u^-, v^+, v^-, $|f|$, u, v, $|u|$ and $|v|$ are all maps with values in $\mathbf{R}$. Finally, we have $u^-, u^+ \leq |u| \leq |f|$, and consequently, using exercise (7), $\int u^- d\mu \leq \int |u| d\mu \leq \int |f| d\mu < +\infty$. It follows that u^- (and

[26]See exercise (24) of the previous tutorial. (Beware of external links !)

u^+ since $\int u^+ d\mu < +\infty$), u, $|u|$ and $|f|$ are all elements of $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. Similarly, v^-, v^+, v, $|v|$ also lie in $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

4. u^+, u^-, v^+ and v^- are all non-negative and measurable. Their integrals $\int u^+ d\mu$, $\int u^- d\mu$, $\int v^+ d\mu$ and $\int v^- d\mu$ are therefore well-defined.

5. $\int f d\mu = \int u^+ d\mu - \int u^- d\mu + i(\int v^+ d\mu - \int v^- d\mu)$. Each integral $\int u^+ d\mu$, $\int u^- d\mu$, $\int v^+ d\mu$ and $\int v^- d\mu$, is not only well-defined, but is also finite, i.e. lie in $\mathbf{R}^+$. It follows that $\int f d\mu$ is a well-defined complex number.

6. In the case when $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is such that $f(\Omega) \subseteq \mathbf{R}^+$, then $\int f d\mu$ is potentially ambiguous. On the one hand, f being non-negative and measurable, $\int f d\mu$ is defined by virtue of definition (43). On the other hand, f being an element of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, $\int f d\mu = \int u^+ d\mu - \int u^- d\mu + i(\int v^+ d\mu - \int v^- d\mu)$. However, since f has value in $\mathbf{R}^+$, $f = u^+$ and $u^- = v^+ = v^- = 0$. it follows that the two definitions of $\int f d\mu$ coincide.

7. From 3., $u, v \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \subseteq L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. It follows that $\int u d\mu$ and $\int v d\mu$ are well-defined, as $\int u d\mu = \int u^+ d\mu - \int u^- d\mu$ and $\int v d\mu = \int v^+ d\mu - \int v^- d\mu$. So $\int f d\mu = \int u d\mu + i \int v d\mu$.

Exercise 21

Exercise 22.

1. Let $B \in \mathcal{B}(\mathbf{C})$. If $0 \in B$, then $(f1_A)^{-1}(B) = (A \cap f^{-1}(B)) \uplus A^c$. If $0 \notin B$, then $(f1_A)^{-1}(B) = A \cap f^{-1}(B)$. In any case, since f is measurable and $A \in \mathcal{F}$, we have $(f1_A)^{-1}(B) \in \mathcal{F}$. It follows that $f1_A$ is measurable. From $|f1_A| = |f|1_A \leq |f|$, we have $\int |f1_A| d\mu \leq \int |f| d\mu < +\infty$. We conclude that $f1_A$ is an element of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

2. From definition (45), $\int |f| d\mu^A = \int_A |f| d\mu = \int |f| 1_A d\mu < +\infty$. f being complex valued and measurable, $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu^A)$.

3. Let $B \in \mathcal{B}(\mathbf{C})$. Then, $(f_{|A})^{-1}(B) = A \cap f^{-1}(B) \in \mathcal{F}_{|A}$. It follows that $f_{|A} : (A, \mathcal{F}_{|A}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is measurable. Moreover, using definition (45):

$$\int |f_{|A}| d\mu_{|A} = \int |f|_{|A} d\mu_{|A} = \int_A |f| d\mu = \int |f| 1_A d\mu < +\infty$$

We conclude that $f_{|A} \in L^1_{\mathbf{C}}(A, \mathcal{F}_{|A}, \mu_{|A})$.

4. Since $f1_A \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, $\int f1_A d\mu$ is well-defined by virtue of definition (48). We have:

$$\int f1_A d\mu = \int u^+ 1_A d\mu - \int u^- 1_A d\mu + i\left(\int v^+ 1_A d\mu - \int v^- 1_A d\mu\right)$$

Since $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu^A)$, $\int f d\mu^A$ is well-defined, and:

$$\int f d\mu^A = \int u^+ d\mu^A - \int u^- d\mu^A + i\left(\int v^+ d\mu^A - \int v^- d\mu^A\right)$$

Since $f_{|A} \in L^1_{\mathbf{C}}(A, \mathcal{F}_{|A}, \mu_{|A})$, $\int f_{|A} d\mu_{|A}$ is well-defined, and:

$$\int f_{|A} d\mu_{|A} = \int u^+_{|A} d\mu_{|A} - \int u^-_{|A} d\mu_{|A} + i\left(\int v^+_{|A} d\mu_{|A} - \int v^-_{|A} d\mu_{|A}\right)$$

Using definition (45), $\int u^+ 1_A d\mu = \int u^+ d\mu^A = \int u^+_{|A} d\mu_{|A}$, with similar expressions involving u^-, v^+ and v^-. We conclude that $\int f1_A d\mu = \int f d\mu^A = \int f_{|A} d\mu_{|A}$.

5. From:

$$\int f1_A d\mu = \int u^+ 1_A d\mu - \int u^- 1_A d\mu + i\left(\int v^+ 1_A d\mu - \int v^- 1_A d\mu\right)$$

and definition (45), we have:

$$\int f1_A d\mu = \int_A u^+ d\mu - \int_A u^- d\mu + i\left(\int_A v^+ d\mu - \int_A v^- d\mu\right)$$

Exercise 22

Exercise 23.

1. From $h = h^+ - h^-$, $f = f^+ - f^-$ and $g = g^+ - g^-$, we obtain that $h^+ + f^- + g^- = h^- + f^+ + g^+$. By linearity, proved in exercise (10), we conclude that:

$$\int h^+ d\mu + \int f^- d\mu + \int g^- d\mu = \int h^- d\mu + \int f^+ d\mu + \int g^+ d\mu \quad (16)$$

2. Since f, g and h belong to $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, all six integrals in equation (16) are finite. It follows that equation (16) can be re-arranged as:

$$\int h^+ d\mu - \int h^- d\mu = \int f^+ d\mu - \int f^- d\mu + \int g^+ d\mu - \int g^- d\mu$$

From definition (48), we conclude that:

$$\int h d\mu = \int f d\mu + \int g d\mu \quad (17)$$

3. From definition (47), $(-f)^+ = f^-$ and $(-f)^- = f^+$. It follows from definition (48) that:

$$\int (-f) d\mu = \int f^- d\mu - \int f^+ d\mu = -\int f d\mu \tag{18}$$

4. Suppose $\alpha \in \mathbf{R}^+$. Then, $(\alpha f)^+ = \alpha f^+$ and $(\alpha f)^- = \alpha f^-$. From definition (48), and by linearity proved in exercise (10) for non-negative maps and $\alpha \geq 0$, we have:

$$\int (\alpha f) d\mu = \int \alpha f^+ d\mu - \int \alpha f^- d\mu = \alpha \int f d\mu \tag{19}$$

If $\alpha \leq 0$, applying equation (19) to $(-\alpha) f$ and then using equation (18), we see that:

$$\int (\alpha f) d\mu = \alpha \int f d\mu \tag{20}$$

We conclude that equation (20) is satisfied for all $\alpha \in \mathbf{R}$.

5. If $f \leq g$, then $f^+ + g^- \leq f^- + g^+$. From exercise (7) and by linearity for non-negative maps, we obtain:

$$\int f^+ d\mu + \int g^- d\mu \leq \int f^- d\mu + \int g^+ d\mu$$

All integrals being finite, this can be re-arranged as:

$$\int f^+ d\mu - \int f^- d\mu \leq \int g^+ d\mu - \int g^- d\mu$$

We conclude that $\int f d\mu \leq \int g d\mu$. This is an extension of exercise (7) (3.) to the case when $f, g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

6. Proving theorem (22) may be seen as an immediate consequence of equations (17) and (20). In fact, these equalities have only been established for $\alpha \in \mathbf{R}$, and $f, g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. Hence, a little more work is required. Suppose that $f, g \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Let us write $f = u + iv$, and $g = u' + iv'$. From exercise (21), all maps u, v, u' and v' are elements of $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. It follows from equation (17) that $\int (u + u') d\mu = \int u d\mu + \int u' d\mu$

and $\int(v+v')d\mu = \int v d\mu + \int v' d\mu$. However, also from exercise (21), $\int f d\mu = \int u d\mu + i\int v d\mu$, with similar equalities, $\int g d\mu = \int u' d\mu + i\int v' d\mu$ and:

$$\int(f+g)d\mu = \int(u+u')d\mu + i\int(v+v')d\mu$$

We conclude that $\int(f+g)d\mu = \int f d\mu + \int g d\mu$, and equation (17) is therefore satisfied for $f, g \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Furthermore, if $\alpha \in \mathbf{R}$, Then $\alpha f = (\alpha u) + i(\alpha v)$, with αu and αv in $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. It follows from equation (20) that we have $\int(\alpha u)d\mu = \alpha\int u d\mu$ and $\int(\alpha v)d\mu = \alpha\int v d\mu$. However, again from exercise (21), $\int(\alpha f)d\mu = \int(\alpha u)d\mu + i\int(\alpha v)d\mu$. Hence, $\int(\alpha f)d\mu = \alpha\int f d\mu$, and equation (20) is true for $\alpha \in \mathbf{R}$, and $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. If $\alpha = i$, then $\alpha f = -v + iu$ and therefore:

$$\int(\alpha f)d\mu = -\int v d\mu + i\int u d\mu = \alpha\int f d\mu$$

Finally, if $\alpha = x + iy \in \mathbf{C}$, with $x, y \in \mathbf{R}$, we have:

$$\int (\alpha f) d\mu = \int (xf) d\mu + \int (iyf) d\mu$$

with $\int (xf) d\mu = x \int f d\mu$, and furthermore:

$$\int (iyf) d\mu = i \int (yf) d\mu = iy \int f d\mu$$

We conclude that $\int (\alpha f) d\mu = \alpha \int f d\mu$, and equation (20) is therefore satisfied for all $\alpha \in \mathbf{C}$, and $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. This completes the proof of theorem (22).

Exercise 23

Exercise 24.

1. Let $n \geq 1$. By assumption, f_n is $\mathbf{C}$-valued and measurable. Moreover, since $0 \leq |f_n| \leq g$ and $g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$:

$$\int |f_n| d\mu \leq \int g d\mu < +\infty$$

It follows that $f_n \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Given $\omega \in \Omega$, the sequence $(f_n(\omega))_{n \geq 1}$ converges to $f(\omega)$ in $\mathbf{C}$. This excludes possible limits like $+\infty$ or $-\infty$. So f is $\mathbf{C}$-valued. As a limit of measurable maps with values in a metrizable space, f is itself a measurable map[27]. Finally, since $|f_n(\omega)| \leq g(\omega)$ for all $n \geq 1$ and $\omega \in \Omega$, taking the limit as $n \to +\infty$, we see that $|f(\omega)| \leq g(\omega)$, and consequently:

$$\int |f| d\mu \leq \int g d\mu < +\infty$$

We conclude that $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

[27] See theorem (17). (Beware of external links !)

2. Given $n \geq 1$, since $f, f_n \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, $f_n - f$ is also an element of $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. So $|f_n - f| \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, and since $g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, we have $h_n = 2g - |f_n - f| \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. In particular, h_n is a measurable map. Moreover, we have $|f_n - f| \leq |f_n| + |f| \leq 2g$, and consequently $h_n \geq 0$. It follows that $(h_n)_{n\geq 1}$ is a sequence of non-negative and measurable maps. We conclude that Fatou lemma (20) can legitimately be applied to it.

3. Let $(u_n)_{n\geq 1}$ be a sequence in $\bar{\mathbf{R}}$. Given $n \geq 1$ and $k \geq n$, we have $\inf_{k\geq n}(-u_k) \leq -u_k$, and consequently $u_k \leq -\inf_{k\geq n}(-u_k)$. It follows that $\sup_{k\geq n} u_k \leq -\inf_{k\geq n}(-u_k)$. In particular:

$$\limsup_{n\to+\infty} u_n = \inf_{n\geq 1}\left(\sup_{k\geq n} u_k\right) \leq \sup_{k\geq n} u_k \leq -\inf_{k\geq n}(-u_k)$$

or equivalently, $\inf_{k\geq n}(-u_k) \leq -\limsup u_n$. It follows that $-\limsup u_n$ is an upper-bound of all $\inf_{k\geq n}(-u_k)$, for $n \geq 1$. $\liminf(-u_n)$ being the smallest of such upper-bound, we conclude that $\liminf(-u_n) \leq -\limsup u_n$. Given $n \geq 1$ and $k \geq n$, we have $u_k \leq \sup_{k\geq n} u_k$, and consequently $-\sup_{k\geq n} u_k \leq -u_k$. It follows that $-\sup_{k\geq n} u_k \leq \inf_{k\geq n}(-u_k)$. In particular:

$$-\sup_{k\geq n} u_k \leq \inf_{k\geq n}(-u_k) \leq \sup_{n\geq 1}\left(\inf_{k\geq n}(-u_k)\right) = \liminf_{n\to+\infty}(-u_n)$$

or equivalently $-\liminf(-u_n) \leq \sup_{k\geq n} u_k$. It follows that $-\liminf(-u_n)$ is a lower-bound of all $\sup_{k\geq n} u_k$, for $n \geq 1$. $\limsup u_n$ being the greatest of such lower-bound, we conclude that $-\liminf(-u_n) \leq \limsup u_n$. We have proved that:

$$\liminf_{n\to+\infty}(-u_n) = -\limsup_{n\to+\infty} u_n$$

4. Since $\alpha \in \mathbf{R}$, for all $n \geq 1$, the sum '$\alpha + u_n$' is always meaningful in $\bar{\mathbf{R}}$. The sum '$\alpha + \liminf u_n$' is also meaningful in $\bar{\mathbf{R}}$. Let $n \geq 1$ and $k \geq n$. We have $\inf_{k\geq n}(\alpha + u_k) \leq \alpha + u_k$. Since $\alpha \in \mathbf{R}$, this inequality can be re-arranged as $-\alpha + \inf_{k\geq n}(\alpha + u_k) \leq u_k$.

It follows that:

$$-\alpha + \inf_{k\geq n}(\alpha + u_k) \leq \inf_{k\geq n} u_k \leq \sup_{n\geq 1}\left(\inf_{k\geq n} u_k\right) = \liminf_{n\to+\infty} u_n$$

Re-arranging this inequality, we see that $\alpha + \liminf u_n$ is an upper-bound of all $\inf_{k\geq n}(\alpha+u_k)$ for $n \geq 1$. Since $\liminf(\alpha+u_n)$ is the smallest of such upper-bound, we conclude that we have $\liminf(\alpha + u_n) \leq \alpha + \liminf u_n$. Similarly:

$$\liminf_{n\to+\infty} u_n = \liminf_{n\to+\infty}(-\alpha + \alpha + u_n) \leq -\alpha + \liminf_{n\to+\infty}(\alpha + u_n)$$

We have proved that for all $\alpha \in \mathbf{R}$:

$$\liminf_{n\to+\infty}(\alpha + u_n) = \alpha + \liminf_{n\to+\infty} u_n$$

5. Suppose that $u_n \to 0$ as $n \to +\infty$. Then $|u_n| \to 0$ and consequently, using theorem (16), $\liminf |u_n| = \limsup |u_n| = 0$. Conversely, if $\limsup |u_n| = 0$, then:

$$0 \leq \liminf_{n\to+\infty} |u_n| \leq \limsup_{n\to+\infty} |u_n| = 0$$

Hence, we see that $\liminf |u_n| = \limsup |u_n| = 0$. From theorem (16), we conclude that $(|u_n|)_{n\geq 1}$ converges to 0. We have proved that $u_n \to 0$, if and only if $\limsup |u_n| = 0$.

6. Let h_n be defined as in 2. Since $f_n \to f$, we have $h_n \to 2g$. In particular, $\liminf h_n = 2g$. Applying Fatou lemma (20) to the sequence $(h_n)_{n\geq 1}$, we obtain:

$$\int (2g)d\mu \leq \liminf_{n\to+\infty} \int (2g - |f_n - f|)d\mu$$

By linearity proved in theorem (22):

$$\int (2g)d\mu \leq \liminf_{n\to+\infty} \left(\int (2g)d\mu - \int |f_n - f|d\mu \right)$$

Since $g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, $\int (2g)d\mu \in \mathbf{R}$. From 4.:

$$\int (2g)d\mu \leq \int (2g)d\mu + \liminf_{n\to+\infty} \left(-\int |f_n - f|d\mu \right)$$

Finally, using 3., we obtain:

$$\int (2g)d\mu \leq \int (2g)d\mu - \limsup_{n\to+\infty} \int |f_n - f|d\mu \qquad (21)$$

7. Since $\int (2g)d\mu \in \mathbf{R}$, inequality (21) can be simplified as:

$$0 \leq - \limsup_{n\to+\infty} \int |f_n - f|d\mu$$

from which we conclude that $\limsup \int |f_n - f|d\mu = 0$.

8. It follows from 5. and 7. that $\int |f_n - f|d\mu \to 0$, as $n \to +\infty$. The purpose of this exercise is to prove theorem (23). Called the *Dominated Convergence Theorem*, this theorem is one of the corner stones of the Lebesgue integration theory, together with the *Monotone Convergence Theorem* (19), and *Fatou Lemma* (20).

Exercise 24

Exercise 25.

1. Since $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $\alpha \in \mathbf{C}$, $\alpha f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. From exercise (21), it follows that $u = Re(\alpha f) \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

2. We have $u = Re(\alpha f) \leq |Re(\alpha f)| \leq |\alpha f| = |f|$.

3. We have $|\int f d\mu| = |z| = \alpha z = \alpha \int f d\mu = \int (\alpha f) d\mu$.

4. From 3., $\int (\alpha f) d\mu \in \mathbf{R}$. However, from exercise (21), we have:

$$\int (\alpha f) d\mu = \int Re(\alpha f) d\mu + i \int Im(\alpha f) d\mu$$

   It follows that $\int (\alpha f) d\mu = \int Re(\alpha f) d\mu = \int u d\mu$.

5. From 3. and 4., we have $|\int f d\mu| = \int u d\mu$. However, from 2., we have $u \leq |f|$. From exercise (23) (5.), $\int u d\mu \leq \int |f| d\mu$. Finally, we conclude that $|\int f d\mu| \leq \int |f| d\mu$. This proves theorem (24).

Exercise 25