4. Measurability

Definition 25 *Let A and B be two sets, and $f : A \to B$ be a map. Given $A' \subseteq A$, we call* **direct image** *of A' by f the set denoted $f(A')$, and defined by $f(A') = \{f(x) \ : \ x \in A'\}$.*

Definition 26 *Let A and B be two sets, and $f : A \to B$ be a map. Given $B' \subseteq B$, we call* **inverse image** *of B' by f the set denoted $f^{-1}(B')$, and defined by $f^{-1}(B') = \{x \ : \ x \in A \ , \ f(x) \in B'\}$.*

Exercise 1. Let A and B be two sets, and $f : A \to B$ be a bijection from A to B. Let $A' \subseteq A$ and $B' \subseteq B$.

1. Explain why the notation $f^{-1}(B')$ is potentially ambiguous.
2. Show that the inverse image of B' by f is in fact equal to the direct image of B' by f^{-1}.
3. Show that the direct image of A' by f is in fact equal to the inverse image of A' by f^{-1}.

Definition 27 *Let* $(\Omega, \mathcal{T})$ *and* $(S, \mathcal{T}_S)$ *be two topological spaces. A map* $f : \Omega \to S$ *is said to be* **continuous** *if and only if:*

$$\forall B \in \mathcal{T}_S \ , \ f^{-1}(B) \in \mathcal{T}$$

In other words, if and only if the inverse image of any open set in S *is an open set in* Ω*.*

We Write $f : (\Omega, \mathcal{T}) \to (S, \mathcal{T}_S)$ *is continuous*, as a way of emphasizing the two topologies $\mathcal{T}$ and $\mathcal{T}_S$ with respect to which f is continuous.

Definition 28 *Let* E *be a set. A map* $d : E \times E \to [0, +\infty[$ *is said to be a* **metric** *on* E*, if and only if:*

$$\begin{array}{ll} (i) & \forall x, y \in E \ , \ d(x,y) = 0 \ \Leftrightarrow \ x = y \\ (ii) & \forall x, y \in E \ , \ d(x,y) = d(y,x) \\ (iii) & \forall x, y, z \in E \ , \ d(x,y) \le d(x,z) + d(z,y) \end{array}$$

Definition 29 *A* **metric space** *is an ordered pair* (E,d) *where* E *is a set, and* d *is a metric on* E.

Definition 30 *Let* (E,d) *be a metric space. For all* $x \in E$ *and* $\epsilon > 0$, *we define the so-called* **open ball** *in* E:

$$B(x,\epsilon) \stackrel{\triangle}{=} \{y \ : \ y \in E \ , \ d(x,y) < \epsilon\}$$

We call **metric topology** *on* E, *associated with* d, *the topology* $\mathcal{T}_E^d$ *defined by:*

$$\mathcal{T}_E^d \stackrel{\triangle}{=} \{U \subseteq E \ , \forall x \in U, \exists \epsilon > 0, B(x,\epsilon) \subseteq U\}$$

EXERCISE 2. Let $\mathcal{T}_E^d$ be the metric topology associated with d, where (E,d) is a metric space.

1. Show that $\mathcal{T}_E^d$ is indeed a topology on E.

2. Given $x \in E$ and $\epsilon > 0$, show that $B(x,\epsilon)$ is an open set in E.

Exercise 3. Show that the usual topology on $\mathbf{R}$ is nothing but the metric topology associated with $d(x,y) = |x - y|$.

Exercise 4. Let (E,d) and (F,δ) be two metric spaces. Show that a map $f : E \to F$ is continuous, if and only if for all $x \in E$ and $\epsilon > 0$, there exists $\eta > 0$ such that for all $y \in E$:

$$d(x,y) < \eta \quad \Rightarrow \quad \delta(f(x), f(y)) < \epsilon$$

Definition 31 *Let $(\Omega, \mathcal{T})$ and $(S, \mathcal{T}_S)$ be two topological spaces. A map $f : \Omega \to S$ is said to be a* **homeomorphism**, *if and only if f is a continuous bijection, such that f^{-1} is also continuous.*

Definition 32 *A topological space $(\Omega, \mathcal{T})$ is said to be* **metrizable**, *if and only if there exists a metric d on Ω, such that the associated metric topology coincides with $\mathcal{T}$, i.e. $\mathcal{T}^d_\Omega = \mathcal{T}$.*

Definition 33 *Let (E,d) be a metric space and $F \subseteq E$. We call* **induced metric** *on F, denoted $d_{|F}$, the restriction of the metric d to $F \times F$, i.e. $d_{|F} = d_{|F\times F}$.*

Exercise 5. Let (E,d) be a metric space and $F \subseteq E$. We define $\mathcal{T}_F = (\mathcal{T}_E^d)_{|F}$ as the topology on F induced by the metric topology on E. Let $\mathcal{T}'_F = \mathcal{T}_F^{d_{|F}}$ be the metric topology on F associated with the induced metric $d_{|F}$ on F.

1. Show that $\mathcal{T}_F \subseteq \mathcal{T}'_F$.
2. Given $A \in \mathcal{T}'_F$, show that $A = (\cup_{x\in A} B(x,\epsilon_x)) \cap F$ for some $\epsilon_x > 0$, $x \in A$, where $B(x,\epsilon_x)$ denotes the open ball in E.
3. Show that $\mathcal{T}'_F \subseteq \mathcal{T}_F$.

Theorem 12 *Let (E, d) be a metric space and $F \subseteq E$. Then, the topology on F induced by the metric topology, is equal to the metric topology on F associated with the induced metric, i.e. $(\mathcal{T}_E^d)_{|F} = \mathcal{T}_F^{d_{|F}}$.*

Exercise 6. Let $\phi : \mathbf{R} \to]-1,1[$ be the map defined by:

$$\forall x \in \mathbf{R} \quad , \quad \phi(x) \stackrel{\triangle}{=} \frac{x}{|x|+1}$$

1. Show that $[-1,0[$ is not open in $\mathbf{R}$.
2. Show that $[-1,0[$ is open in $[-1,1]$.
3. Show that ϕ is a homeomorphism between $\mathbf{R}$ and $]-1,1[$.
4. Show that $\lim_{x\to+\infty}\phi(x) = 1$ and $\lim_{x\to-\infty}\phi(x) = -1$.

Exercise 7. Let $\bar{\mathbf{R}} = [-\infty, +\infty] = \mathbf{R} \cup \{-\infty, +\infty\}$. Let ϕ be defined as in exercise (6), and $\bar{\phi} : \bar{\mathbf{R}} \to [-1,1]$ be the map defined by:

$$\bar{\phi}(x) = \left\{ \begin{array}{rcl} \phi(x) & \text{if} & x \in \mathbf{R} \\ 1 & \text{if} & x = +\infty \\ -1 & \text{if} & x = -\infty \end{array} \right.$$

Define:

$$\mathcal{T}_{\bar{\mathbf{R}}} \stackrel{\triangle}{=} \{U \subseteq \bar{\mathbf{R}} \ , \ \bar{\phi}(U) \text{ is open in } [-1,1]\}$$

1. Show that $\bar{\phi}$ is a bijection from $\bar{\mathbf{R}}$ to $[-1,1]$, and let $\bar{\psi} = \bar{\phi}^{-1}$.

2. Show that $\mathcal{T}_{\bar{\mathbf{R}}}$ is a topology on $\bar{\mathbf{R}}$.

3. Show that $\bar{\phi}$ is a homeomorphism between $\bar{\mathbf{R}}$ and $[-1,1]$.

4. Show that $[-\infty, 2[,\ ]3,+\infty],\ ]3,+\infty[$ are open in $\bar{\mathbf{R}}$.

5. Show that if $\phi' : \bar{\mathbf{R}} \to [-1,1]$ is an arbitrary homeomorphism, then $U \subseteq \bar{\mathbf{R}}$ is open, if and only if $\phi'(U)$ is open in $[-1,1]$.

Definition 34 *The* **usual topology** *on* $\bar{\mathbf{R}}$ *is defined as:*

$$\mathcal{T}_{\bar{\mathbf{R}}} \stackrel{\triangle}{=} \{U \subseteq \bar{\mathbf{R}} \ , \ \bar{\phi}(U) \ \textit{is open in} \ [-1,1]\}$$

where $\bar{\phi} : \bar{\mathbf{R}} \to [-1,1]$ *is defined by* $\bar{\phi}(-\infty) = -1$, $\bar{\phi}(+\infty) = 1$ *and:*

$$\forall x \in \mathbf{R} \quad , \quad \bar{\phi}(x) \stackrel{\triangle}{=} \frac{x}{|x|+1}$$

Exercise 8. Let ϕ and $\bar{\phi}$ be as in exercise (7). Define:

$$\mathcal{T}' \stackrel{\triangle}{=} (\mathcal{T}_{\bar{\mathbf{R}}})_{|\mathbf{R}} \stackrel{\triangle}{=} \{U \cap \mathbf{R} \ , \ U \in \mathcal{T}_{\bar{\mathbf{R}}}\}$$

1. Recall why $\mathcal{T}'$ is a topology on $\mathbf{R}$.
2. Show that for all $U \subseteq \bar{\mathbf{R}}$, $\phi(U \cap \mathbf{R}) = \bar{\phi}(U) \cap]-1,1[$.
3. Explain why if $U \in \mathcal{T}_{\bar{\mathbf{R}}}$, $\phi(U \cap \mathbf{R})$ is open in $]-1,1[$.
4. Show that $\mathcal{T}' \subseteq \mathcal{T}_{\mathbf{R}}$, (the usual topology on $\mathbf{R}$).

5. Let $U \in \mathcal{T}_{\mathbf{R}}$. Show that $\bar{\phi}(U)$ is open in $]-1,1[$ and $[-1,1]$.

6. Show that $\mathcal{T}_{\mathbf{R}} \subseteq \mathcal{T}_{\bar{\mathbf{R}}}$

7. Show that $\mathcal{T}_{\mathbf{R}} = \mathcal{T}'$, i.e. that the usual topology on $\bar{\mathbf{R}}$ induces the usual topology on $\mathbf{R}$.

8. Show that $\mathcal{B}(\mathbf{R}) = \mathcal{B}(\bar{\mathbf{R}})_{|\mathbf{R}} = \{B \cap \mathbf{R}\ ,\ B \in \mathcal{B}(\bar{\mathbf{R}})\}$

Exercise 9. Let $d : \bar{\mathbf{R}} \times \bar{\mathbf{R}} \to [0, +\infty[$ be defined by:

$$\forall (x,y) \in \bar{\mathbf{R}} \times \bar{\mathbf{R}} \quad , \quad d(x,y) = |\phi(x) - \phi(y)|$$

where ϕ is an arbitrary homeomorphism from $\bar{\mathbf{R}}$ to $[-1,1]$.

1. Show that d is a metric on $\bar{\mathbf{R}}$.

2. Show that if $U \in \mathcal{T}_{\bar{\mathbf{R}}}$, then $\phi(U)$ is open in $[-1,1]$

3. Show that for all $U \in \mathcal{T}_{\bar{\mathbf{R}}}$ and $y \in \phi(U)$, there exists $\epsilon > 0$ such that:
$$\forall z \in [-1,1] \ , \ |z-y| < \epsilon \ \Rightarrow \ z \in \phi(U)$$

4. Show that $\mathcal{T}_{\bar{\mathbf{R}}} \subseteq \mathcal{T}^d_{\bar{\mathbf{R}}}$.

5. Show that for all $U \in \mathcal{T}^d_{\bar{\mathbf{R}}}$ and $x \in U$, there is $\epsilon > 0$ such that:
$$\forall y \in \bar{\mathbf{R}} \ , \ |\phi(x) - \phi(y)| < \epsilon \ \Rightarrow \ y \in U$$

6. Show that for all $U \in \mathcal{T}^d_{\bar{\mathbf{R}}}$, $\phi(U)$ is open in $[-1,1]$.

7. Show that $\mathcal{T}^d_{\bar{\mathbf{R}}} \subseteq \mathcal{T}_{\bar{\mathbf{R}}}$

8. Prove the following theorem.

Theorem 13 *The topological space $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is metrizable.*

Definition 35 *Let $(\Omega, \mathcal{F})$ and (S, Σ) be two measurable spaces. A map $f : \Omega \to S$ is said to be* **measurable** *with respect to $\mathcal{F}$ and Σ, if and only if:*

$$\forall B \in \Sigma \ , \ f^{-1}(B) \in \mathcal{F}$$

We Write $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ *is measurable*, as a way of emphasizing the two σ-algebras $\mathcal{F}$ and Σ with respect to which f is measurable.

Exercise 10. Let $(\Omega, \mathcal{F})$ and (S, Σ) be two measurable spaces. Let S' be a set and $f : \Omega \to S$ be a map such that $f(\Omega) \subseteq S' \subseteq S$. We define Σ' as the trace of Σ on S', i.e. $\Sigma' = \Sigma_{|S'}$.

1. Show that for all $B \in \Sigma$, we have $f^{-1}(B) = f^{-1}(B \cap S')$
2. Show that $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ is measurable, if and only if $f : (\Omega, \mathcal{F}) \to (S', \Sigma')$ is itself measurable.
3. Let $f : \Omega \to \mathbf{R}^+$. Show that the following are equivalent:

$$(i) \qquad f : (\Omega, \mathcal{F}) \to (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \text{ is measurable}$$

$$(ii) \qquad f : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R})) \text{ is measurable}$$
$$(iii) \qquad f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}})) \text{ is measurable}$$

Exercise 11. Let $(\Omega, \mathcal{F})$, (S, Σ), (S_1, Σ_1) be three measurable spaces. let $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ and $g : (S, \Sigma) \to (S_1, \Sigma_1)$ be two measurable maps.

1. For all $B \subseteq S_1$, show that $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$
2. Show that $g \circ f : (\Omega, \mathcal{F}) \to (S_1, \Sigma_1)$ is measurable.

Exercise 12. Let $(\Omega, \mathcal{F})$ and (S, Σ) be two measurable spaces. Let $f : \Omega \to S$ be a map. We define:

$$\Gamma \stackrel{\triangle}{=} \{B \in \Sigma\ ,\ f^{-1}(B) \in \mathcal{F}\}$$

1. Show that $f^{-1}(S) = \Omega$.

2. Show that for all $B \subseteq S$, $f^{-1}(B^c) = (f^{-1}(B))^c$.

3. Show that if $B_n \subseteq S, n \geq 1$, then $f^{-1}(\cup_{n=1}^{+\infty} B_n) = \cup_{n=1}^{+\infty} f^{-1}(B_n)$

4. Show that Γ is a σ-algebra on S.

5. Prove the following theorem.

Theorem 14 *Let $(\Omega, \mathcal{F})$ and (S, Σ) be two measurable spaces, and $\mathcal{A}$ be a set of subsets of S generating Σ, i.e. such that $\Sigma = \sigma(\mathcal{A})$. Then $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ is measurable, if and only if:*

$$\forall B \in \mathcal{A} \quad , \quad f^{-1}(B) \in \mathcal{F}$$

EXERCISE 13. Let $(\Omega, \mathcal{T})$ and $(S, \mathcal{T}_S)$ be two topological spaces. Let $f : \Omega \to S$ be a map. Show that if $f : (\Omega, \mathcal{T}) \to (S, \mathcal{T}_S)$ is continuous, then $f : (\Omega, \mathcal{B}(\Omega)) \to (S, \mathcal{B}(S))$ is measurable.

EXERCISE 14. We define the following subsets of the power set $\mathcal{P}(\bar{\mathbf{R}})$:

$$\begin{aligned}
\mathcal{C}_1 &\stackrel{\triangle}{=} \{[-\infty, c]\ ,\ c \in \mathbf{R}\} \\
\mathcal{C}_2 &\stackrel{\triangle}{=} \{[-\infty, c[\ ,\ c \in \mathbf{R}\} \\
\mathcal{C}_3 &\stackrel{\triangle}{=} \{[c, +\infty]\ ,\ c \in \mathbf{R}\} \\
\mathcal{C}_4 &\stackrel{\triangle}{=} \{]c, +\infty]\ ,\ c \in \mathbf{R}\}
\end{aligned}$$

1. Show that $\mathcal{C}_2$ and $\mathcal{C}_4$ are subsets of $\mathcal{T}_{\bar{\mathbf{R}}}$.
2. Show that the elements of $\mathcal{C}_1$ and $\mathcal{C}_3$ are closed in $\bar{\mathbf{R}}$.
3. Show that for all $i = 1, 2, 3, 4$, $\sigma(\mathcal{C}_i) \subseteq \mathcal{B}(\bar{\mathbf{R}})$.
4. Let U be open in $\bar{\mathbf{R}}$. Explain why $U \cap \mathbf{R}$ is open in $\mathbf{R}$.

5. Show that any open subset of $\mathbf{R}$ is a countable union of open bounded intervals in $\mathbf{R}$.

6. Let $a < b$, $a, b \in \mathbf{R}$. Show that we have:

$$]a,b[=\bigcup_{n=1}^{+\infty}]a,b-1/n]=\bigcup_{n=1}^{+\infty}[a+1/n,b[$$

7. Show that for all $i = 1, 2, 3, 4$, $]a, b[\in \sigma(\mathcal{C}_i)$.

8. Show that for all $i = 1, 2, 3, 4$, $\{\{-\infty\}, \{+\infty\}\} \subseteq \sigma(\mathcal{C}_i)$.

9. Show that for all $i = 1, 2, 3, 4$, $\sigma(\mathcal{C}_i) = \mathcal{B}(\bar{\mathbf{R}})$

10. Prove the following theorem.

Theorem 15 *Let $(\Omega, \mathcal{F})$ be a measurable space, and $f : \Omega \to \bar{\mathbf{R}}$ be a map. The following are equivalent:*

$$
\begin{array}{rl}
(i) & f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}})) \text{ is measurable} \\
(ii) & \forall B \in \mathcal{B}(\bar{\mathbf{R}})\ ,\ \{f \in B\} \in \mathcal{F} \\
(iii) & \forall c \in \mathbf{R}\ ,\ \{f \leq c\} \in \mathcal{F} \\
(iv) & \forall c \in \mathbf{R}\ ,\ \{f < c\} \in \mathcal{F} \\
(v) & \forall c \in \mathbf{R}\ ,\ \{c \leq f\} \in \mathcal{F} \\
(vi) & \forall c \in \mathbf{R}\ ,\ \{c < f\} \in \mathcal{F}
\end{array}
$$

Exercise 15. Let $(\Omega, \mathcal{F})$ be a measurable space. Let $(f_n)_{n\geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$. Let g and h be the maps defined by $g(\omega) = \inf_{n\geq 1} f_n(\omega)$ and $h(\omega) = \sup_{n\geq 1} f_n(\omega)$, for all $\omega \in \Omega$.

1. Let $c \in \mathbf{R}$. Show that $\{c \leq g\} = \cap_{n=1}^{+\infty}\{c \leq f_n\}$.
2. Let $c \in \mathbf{R}$. Show that $\{h \leq c\} = \cap_{n=1}^{+\infty}\{f_n \leq c\}$.

3. Show that $g, h : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are measurable.

Definition 36 *Let $(v_n)_{n\geq 1}$ be a sequence in $\bar{\mathbf{R}}$. We define:*

$$u \stackrel{\triangle}{=} \liminf_{n\to+\infty} v_n \stackrel{\triangle}{=} \sup_{n\geq 1}\left(\inf_{k\geq n} v_k\right)$$

and:

$$w \stackrel{\triangle}{=} \limsup_{n\to+\infty} v_n \stackrel{\triangle}{=} \inf_{n\geq 1}\left(\sup_{k\geq n} v_k\right)$$

Then, $u, w \in \bar{\mathbf{R}}$ are respectively called **lower limit** *and* **upper limit** *of the sequence $(v_n)_{n\geq 1}$.*

EXERCISE 16. Let $(v_n)_{n\geq 1}$ be a sequence in $\bar{\mathbf{R}}$. for $n \geq 1$ we define $u_n = \inf_{k\geq n} v_k$ and $w_n = \sup_{k\geq n} v_k$. Let u and w be the lower limit and upper limit of $(v_n)_{n\geq 1}$, respectively.

1. Show that $u_n \leq u_{n+1} \leq u$, for all $n \geq 1$.

2. Show that $w \leq w_{n+1} \leq w_n$, for all $n \geq 1$.

3. Show that $u_n \to u$ and $w_n \to w$ as $n \to +\infty$.

4. Show that $u_n \leq v_n \leq w_n$, for all $n \geq 1$.

5. Show that $u \leq w$.

6. Show that if $u = w$ then $(v_n)_{n\geq 1}$ converges to a limit $v \in \bar{\mathbf{R}}$, with $u = v = w$.

7. Show that if $a, b \in \mathbf{R}$ are such that $u < a < b < w$ then for all $n \geq 1$, there exist $N_1, N_2 \geq n$ such that $v_{N_1} < a < b < v_{N_2}$.

8. Show that if $a, b \in \mathbf{R}$ are such that $u < a < b < w$ then there exist two strictly increasing sequences of integers $(n_k)_{k\geq 1}$ and $(m_k)_{k\geq 1}$ such that for all $k \geq 1$, we have $v_{n_k} < a < b < v_{m_k}$.

9. Show that if $(v_n)_{n\geq 1}$ converges to some $v \in \bar{\mathbf{R}}$, then $u = w$.

Theorem 16 *Let $(v_n)_{n\geq 1}$ be a sequence in $\bar{\mathbf{R}}$. Then, the following are equivalent:*

$$(i) \qquad \liminf_{n\to+\infty} v_n = \limsup_{n\to+\infty} v_n$$

$$(ii) \qquad \lim_{n\to+\infty} v_n \quad \textit{exists in } \bar{\mathbf{R}}.$$

in which case:

$$\lim_{n\to+\infty} v_n = \liminf_{n\to+\infty} v_n = \limsup_{n\to+\infty} v_n$$

Exercise 17. Let $f, g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ be two measurable maps, where $(\Omega, \mathcal{F})$ is a measurable space.

1. Show that $\{f < g\} = \cup_{r\in\mathbf{Q}}(\{f < r\} \cap \{r < g\})$.
2. Show that the sets $\{f < g\}, \{f > g\}, \{f = g\}, \{f \leq g\}$, $\{f \geq g\}$ belong to the σ-algebra $\mathcal{F}$.

EXERCISE 18. Let $(\Omega, \mathcal{F})$ be a measurable space. Let $(f_n)_{n\geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$. We define $g = \liminf f_n$ and $h = \limsup f_n$ in the obvious way:

$$\forall \omega \in \Omega \ , \ g(\omega) \stackrel{\triangle}{=} \liminf_{n \to +\infty} f_n(\omega)$$

$$\forall \omega \in \Omega \ , \ h(\omega) \stackrel{\triangle}{=} \limsup_{n \to +\infty} f_n(\omega)$$

1. Show that $g, h : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are measurable.
2. Show that $g \leq h$, i.e. $\forall \omega \in \Omega \ , \ g(\omega) \leq h(\omega)$.
3. Show that $\{g = h\} \in \mathcal{F}$.
4. Show that $\{\omega : \omega \in \Omega \ , \ \lim_{n \to +\infty} f_n(\omega) \text{ exists in } \bar{\mathbf{R}}\} \in \mathcal{F}$.
5. Suppose $\Omega = \{g = h\}$, and let $f(\omega) = \lim_{n \to +\infty} f_n(\omega)$, for all $\omega \in \Omega$. Show that $f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

Exercise 19. Let $f, g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ be two measurable maps, where $(\Omega, \mathcal{F})$ is a measurable space.

1. Show that $-f, |f|, f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ are measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\bar{\mathbf{R}})$.

2. Let $a \in \bar{\mathbf{R}}$. Explain why the map $a + f$ may not be well defined.

3. Show that $(a+f) : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, whenever $a \in \mathbf{R}$.

4. Show that $(a.f) : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, for all $a \in \bar{\mathbf{R}}$. (Recall the convention $0.\infty = 0$).

5. Explain why the map $f + g$ may not be well defined.

6. Suppose that $f \geq 0$ and $g \geq 0$, i.e. $f(\Omega) \subseteq [0, +\infty]$ and also $g(\Omega) \subseteq [0, +\infty]$. Show that $\{f + g < c\} = \{f < c - g\}$, for all $c \in \mathbf{R}$. Show that $f + g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

7. Show that $f+g:(\Omega,\mathcal{F})\to(\bar{\mathbf{R}},\mathcal{B}(\bar{\mathbf{R}}))$ is measurable whenever $f+g$ is well-defined, i.e. when the following condition holds:

$$(\{f=+\infty\}\cap\{g=-\infty\})\cup(\{f=-\infty\}\cap\{g=+\infty\})=\emptyset$$

8. Show that $1/f:(\Omega,\mathcal{F})\to(\bar{\mathbf{R}},\mathcal{B}(\bar{\mathbf{R}}))$ is measurable, in the case when $f(\Omega)\subseteq\mathbf{R}\setminus\{0\}$.

9. Suppose that f is $\mathbf{R}$-valued. Show that $\bar{f}$ defined by $\bar{f}(\omega)=f(\omega)$ if $f(\omega)\neq 0$ and $\bar{f}(\omega)=1$ if $f(\omega)=0$, is measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\bar{\mathbf{R}})$.

10. Suppose f and g take values in $\mathbf{R}$. Let $\bar{f}$ be defined as in 9. Show that for all $c\in\mathbf{R}$, the set $\{fg<c\}$ can be expressed as:

$$(\{f>0\}\cap\{g<c/\bar{f}\})\uplus(\{f<0\}\cap\{g>c/\bar{f}\})\uplus(\{f=0\}\cap\{f<c\})$$

11. Show that $fg:(\Omega,\mathcal{F})\to(\bar{\mathbf{R}},\mathcal{B}(\bar{\mathbf{R}}))$ is measurable, in the case when f and g take values in $\mathbf{R}$.

Exercise 20. Let $f, g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ be two measurable maps, where $(\Omega, \mathcal{F})$ is a measurable space. Let $\bar{f}, \bar{g}$, be defined by:

$$\bar{f}(\omega) \stackrel{\triangle}{=} \left\{ \begin{array}{rcl} f(\omega) & \text{if} & f(\omega) \not\in \{-\infty, +\infty\} \\ 1 & \text{if} & f(\omega) \in \{-\infty, +\infty\} \end{array} \right.$$

$\bar{g}(\omega)$ being defined in a similar way. Consider the partitions of Ω, $\Omega = A_1 \uplus A_2 \uplus A_3 \uplus A_4 \uplus A_5$ and $\Omega = B_1 \uplus B_2 \uplus B_3 \uplus B_4 \uplus B_5$, where $A_1 = \{f \in]0, +\infty[\}$, $A_2 = \{f \in] - \infty, 0[\}$, $A_3 = \{f = 0\}$, $A_4 = \{f = -\infty\}$, $A_5 = \{f = +\infty\}$ and B_1, B_2, B_3, B_4, B_5 being defined in a similar way with g. Recall the conventions $0 \times (+\infty) = 0$, $(-\infty) \times (+\infty) = (-\infty)$, etc...

1. Show that $\bar{f}$ and $\bar{g}$ are measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\bar{\mathbf{R}})$.

2. Show that all A_i's and B_j's are elements of $\mathcal{F}$.

3. Show that for all $B \in \mathcal{B}(\bar{\mathbf{R}})$:

$$\{fg \in B\} = \biguplus_{i,j=1}^{5} (A_i \cap B_j \cap \{fg \in B\})$$

4. Show that $A_i \cap B_j \cap \{fg \in B\} = A_i \cap B_j \cap \{\bar{f}\bar{g} \in B\}$, in the case when $1 \leq i \leq 3$ and $1 \leq j \leq 3$.

5. Show that $A_i \cap B_j \cap \{fg \in B\}$ is either equal to $\emptyset$ or $A_i \cap B_j$, in the case when $i \geq 4$ or $j \geq 4$.

6. Show that $fg : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

Definition 37 *Let $(\Omega, \mathcal{T})$ be a topological space, and $A \subseteq \Omega$. We call* **closure** *of A in Ω, denoted $\bar{A}$, the set defined by:*

$$\bar{A} \stackrel{\triangle}{=} \{x \in \Omega \ : \ x \in U \in \mathcal{T} \Rightarrow U \cap A \neq \emptyset\}$$

Exercise 21. Let $(E, \mathcal{T})$ be a topological space, and $A \subseteq E$. Let $\bar{A}$ be the closure of A.

1. Show that $A \subseteq \bar{A}$ and that $\bar{A}$ is closed.

2. Show that if B is closed and $A \subseteq B$, then $\bar{A} \subseteq B$.

3. Show that $\bar{A}$ is the smallest closed set in E containing A.

4. Show that A is closed if and only if $A = \bar{A}$.

5. Show that if $(E, \mathcal{T})$ is metrizable, then:

$$\bar{A} = \{x \in E \ : \ \forall \epsilon > 0 \ , \ B(x, \epsilon) \cap A \neq \emptyset\}$$

where $B(x, \epsilon)$ is relative to any metric d such that $\mathcal{T}_E^d = \mathcal{T}$.

Exercise 22. Let (E, d) be a metric space. Let $A \subseteq E$. For all $x \in E$, we define:

$$d(x, A) \stackrel{\triangle}{=} \inf\{d(x, y) \ : \ y \in A\} \stackrel{\triangle}{=} \Phi_A(x)$$

where it is understood that $\inf \emptyset = +\infty$.

1. Show that for all $x \in E$, $d(x, A) = d(x, \bar{A})$.
2. Show that $d(x, A) = 0$, if and only if $x \in \bar{A}$.
3. Show that for all $x, y \in E$, $d(x, A) \leq d(x, y) + d(y, A)$.
4. Show that if $A \neq \emptyset$, $|d(x, A) - d(y, A)| \leq d(x, y)$.
5. Show that $\Phi_A : (E, \mathcal{T}_E^d) \to (\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is continuous.
6. Show that if A is closed, then $A = \Phi_A^{-1}(\{0\})$

EXERCISE 23. Let $(\Omega, \mathcal{F})$ be a measurable space. Let $(f_n)_{n\geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$, where (E, d) is a metric space. We assume that for all $\omega \in \Omega$, the sequence $(f_n(\omega))_{n\geq 1}$ converges to some $f(\omega) \in E$.

1. Explain why $\liminf f_n$ and $\limsup f_n$ may not be defined in an arbitrary metric space E.

2. Show that $f : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is measurable, if and only if $f^{-1}(A) \in \mathcal{F}$ for all closed subsets A of E.

3. Show that for all A closed in E, $f^{-1}(A) = (\Phi_A \circ f)^{-1}(\{0\})$, where the map $\Phi_A : E \to \bar{\mathbf{R}}$ is defined as in exercise (22).

4. Show that $\Phi_A \circ f_n : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

5. Show that $f : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is measurable.

Theorem 17 *Let $(\Omega, \mathcal{F})$ be a measurable space. Let $(f_n)_{n\geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$, where (E, d) is a metric space. Then, if the limit $f = \lim f_n$ exists on Ω, the map $f : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is itself measurable.*

Definition 38 *The* **usual topology** *on* $\mathbf{C}$*, the set of complex numbers, is defined as the metric topology associated with* $d(z, z') = |z - z'|$.

Exercise 24. Let $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ be a measurable map, where $(\Omega, \mathcal{F})$ is a measurable space. Let $u = Re(f)$ and $v = Im(f)$. Show that $u, v, |f| : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are all measurable.

Exercise 25. Define the subset of the power set $\mathcal{P}(\mathbf{C})$:

$$\mathcal{C} \stackrel{\triangle}{=} \{]a, b[\times]c, d[\ ,\ a, b, c, d \in \mathbf{R}\}$$

where it is understood that:

$$]a, b[\times]c, d[\stackrel{\triangle}{=} \{z = x + iy \in \mathbf{C}\ ,\ (x, y) \in]a, b[\times]c, d[\}$$

1. Show that any element of $\mathcal{C}$ is open in $\mathbf{C}$.
2. Show that $\sigma(\mathcal{C}) \subseteq \mathcal{B}(\mathbf{C})$.
3. Let $z = x + iy \in \mathbf{C}$. Show that if $|x| < \eta$ and $|y| < \eta$ then we have $|z| < \sqrt{2}\eta$.

4. Let U be open in $\mathbf{C}$. Show that for all $z \in U$, there are rational numbers a_z, b_z, c_z, d_z such that $z \in]a_z, b_z[\times]c_z, d_z[\subseteq U$.

5. Show that U can be written as $U = \cup_{n=1}^{+\infty} A_n$ where $A_n \in \mathcal{C}$.

6. Show that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbf{C})$.

7. Let $(\Omega, \mathcal{F})$ be a measurable space, and $u, v : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ be two measurable maps. Show that $u+iv : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is measurable.

Solutions to Exercises

Exercise 1.

1. $f : A \to B$ being a bijection, the notation f^{-1} by itself is meaningful. From definition (26), $f^{-1}(B')$ denotes the inverse image of B' by f. However, from definition (25), $f^{-1}(B')$ also denotes the direct image of B' by f^{-1}. So $f^{-1}(B')$ is ambiguous.

2. Let $f^{-1}(B')$ denote the inverse image of B' by f. Let $g = f^{-1}$ and $g(B')$ be the direct image of B' by g. Let $x \in f^{-1}(B')$. Then $x \in A$ and $f(x) \in B'$. Let $y = f(x)$. Then $x = g(y)$ with $y \in B'$. It follows that $x \in g(B')$, and $f^{-1}(B') \subseteq g(B')$. Conversely, let $x \in g(B')$. There exists $y \in B'$ such that $x = g(y) \in A$. Hence, $f(x) = y \in B'$, and we see that $x \in f^{-1}(B')$. It follows that $g(B') \subseteq f^{-1}(B')$. We have proved that $f^{-1}(B') = g(B')$.

3. Let $g = f^{-1}$. Then $f = g^{-1}$, and applying 2. to g, we have $g^{-1}(A') = f(A')$, where $g^{-1}(A')$ denotes an inverse image.

Exercise 1

Exercise 2.

1. Any statement of the form $\forall x \in \emptyset, \ldots,$ is true. Hence, $\emptyset \in \mathcal{T}_E^d$. It is clear that $E \in \mathcal{T}_E^d$, and (i) of definition (13) is satisfied for $\mathcal{T}_E^d$. Let $A, B \in \mathcal{T}_E^d$, and $x \in A \cap B$. Since $x \in A \in \mathcal{T}_E^d$, there exists $\epsilon_1 > 0$ such that $B(x, \epsilon_1) \subseteq A$. Similarly, there exist $\epsilon_2 > 0$ such that $B(x, \epsilon_2) \subseteq B$. Let $\epsilon = \min(\epsilon_1, \epsilon_2)$. Then $\epsilon > 0$ and $B(x, \epsilon) \subseteq A \cap B$. It follows that $A \cap B \in \mathcal{T}_E^d$ and (ii) of definition (13) is satisfied for $\mathcal{T}_E^d$. Let $(A_i)_{i \in I}$ be a family of elements of $\mathcal{T}_E^d$, and $x \in \cup_{i \in I} A_i$. There exists $i \in I$, such that $x \in A_i$. Since $A_i \in \mathcal{T}_E^d$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq A_i$. In particular, $B(x, \epsilon) \subseteq \cup_{i \in I} A_i$. It follows that $\cup_{i \in I} A_i \in \mathcal{T}_E^d$, and (iii) of definition (13) is satisfied for $\mathcal{T}_E^d$. Having checked (i), (ii) and (iii) of definition (13), we conclude that $\mathcal{T}_E^d$ is indeed a topology on E.

2. Let $y \in B(x, \epsilon)$. Then $d(x, y) < \epsilon$. Let $\eta = \epsilon - d(x, y)$. Then

$\eta > 0$, and for all $z \in B(y, \eta)$, from (iii) of definition (28):

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \eta = \epsilon$$

It follows that $B(y, \eta) \subseteq B(x, \epsilon)$, and we have proved that $B(x, \epsilon) \in \mathcal{T}_E^d$. In other words, the *open ball* $B(x, \epsilon)$ is an open set in E, with respect to the metric topology on E.

Exercise 2

Exercise 3. If $E = \mathbf{R}$ and $d(x,y) = |x-y|$, then for all $x \in \mathbf{R}$ and $\epsilon > 0$, we have $B(x,\epsilon) =]x-\epsilon, x+\epsilon[$. Comparing definition (17) for the usual topology on $\mathbf{R}$, with definition (30), it appears that the usual topology on $\mathbf{R}$, $\mathcal{T}_{\mathbf{R}}$, is nothing but the metric topology $\mathcal{T}^d_{\mathbf{R}}$.

Exercise 3

Exercise 4. Let $\mathcal{P}$ be the property that for all $x \in E$ and $\epsilon > 0$, there exists $\eta > 0$ such that for all $y \in E$:

$$d(x,y) < \eta \quad \Rightarrow \quad \delta(f(x), f(y)) < \epsilon$$

Suppose that property $\mathcal{P}$ is true. Let $B \in \mathcal{T}_F^\delta$ be an open set in F, and $x \in f^{-1}(B)$. Then $f(x) \in B$. Since $B \in \mathcal{T}_F^\delta$, from definition (30) there exists $\epsilon > 0$ such that $B(f(x), \epsilon) \subseteq B$. However, from property $\mathcal{P}$, there exists $\eta > 0$, such that:

$$y \in B(x, \eta) \quad \Rightarrow \quad f(y) \in B(f(x), \epsilon)$$

It follows that if $y \in B(x, \eta)$, then $f(y) \in B$, i.e. $y \in f^{-1}(B)$. Hence, $B(x, \eta) \subseteq f^{-1}(B)$. We have proved that $f^{-1}(B)$ is an open set in E, i.e. $f^{-1}(B) \in \mathcal{T}_E^d$. This being true for all $B \in \mathcal{T}_F^\delta$, from definition (27) we conclude that $f : E \to F$ is continuous.
Conversely, suppose that f is continuous. Let $x \in E$ and $\epsilon > 0$. From exercise (2), the open ball $B(f(x), \epsilon)$ is an open set in F. Since f is continuous, it follows that $f^{-1}(B(f(x), \epsilon))$ is an open set in E, which furthermore contains x. There exists $\eta > 0$, such that

$B(x,\eta) \subseteq f^{-1}(B(f(x),\epsilon))$. In other words, if $y \in B(x,\eta)$, then $f(y) \in B(f(x),\epsilon)$, or equivalently:

$$d(x,y) < \eta \quad \Rightarrow \quad \delta(f(x),f(y)) < \epsilon$$

It follows that property $\mathcal{P}$ is true. We have proved that property $\mathcal{P}$ is equivalent to $f : E \to F$ being continuous.

Exercise 4

Exercise 5.

1. Let $A \in \mathcal{T}_F$. From definition (23) of an induced topology, there exists $B \in \mathcal{T}_E^d$, such that $A = B \cap F$. Let $x \in A$. Then in particular $x \in B$ and from definition (30), there exists $\epsilon > 0$ such that $B(x,\epsilon) \subseteq B$, where $B(x,\epsilon)$ is the open ball in E:

   $$B(x,\epsilon) \stackrel{\triangle}{=} \{y \in E \,:\, d(x,y) < \epsilon\}$$

   If $B'(x,\epsilon)$ denotes the open ball in F:

   $$B'(x,\epsilon) \stackrel{\triangle}{=} \{y \in F \,:\, d_{|F}(x,y) < \epsilon\}$$

   then from $d_{|F}(x,y) = d(x,y)$ for all $(x,y) \in F^2$, we conclude that $B'(x,\epsilon) = B(x,\epsilon) \cap F$, for all $x \in F$. Hence, we see that $B'(x,\epsilon) \subseteq B \cap F = A$. It follows that $A \in \mathcal{T}_F^{d_{|F}} = \mathcal{T}'_F$. We have proved that $\mathcal{T}_F \subseteq \mathcal{T}'_F$.

2. Let $A \in \mathcal{T}'_F$. By definition (30), for all $x \in A$, there exists $\epsilon_x > 0$ such that $B'(x,\epsilon_x) \subseteq A$, where $B'(x,\epsilon_x)$ is the open ball in F.

However, for all $x \in F$, $B'(x, \epsilon_x) = B(x, \epsilon_x) \cap F$, where $B(x, \epsilon_x)$ is the open ball in E. It follows that $x \in B(x, \epsilon_x) \cap F \subseteq A$ for all $x \in A$. Finally, $A = (\cup_{x \in A} B(x, \epsilon_x)) \cap F$.

3. A topology being closed under arbitrary union, and an open ball being open for the metric topology, it follows from 2. that any $A \in \mathcal{T}'_F$ can be expressed as $A = B \cap F$,where B is open for the metric topology on E, i.e. $B \in \mathcal{T}^d_E$. Hence, any $A \in \mathcal{T}'_F$ belongs to $(\mathcal{T}^d_E)_{|F} = \mathcal{T}_F$. We have proved that $\mathcal{T}'_F \subseteq \mathcal{T}_F$. The purpose of this exercise is to prove theorem (12). Given any subset F of a metric space (E, d), the topology on F induced by the metric topology on E is a very *natural* topology for F. However, $(F, d_{|F})$ being itself a metric space, the corresponding metric topology is also a very *natural* topology for F. Fortunately, theorem (12) states that these two topologies do in fact coincide.

Exercise 5

Exercise 6.

1. If $[-1,0[$ was open in $\mathbf{R}$, there would exist $\epsilon > 0$ such that $]-1-\epsilon,-1+\epsilon[\subseteq [-1,0[$. This is obviously not the case.

2. $[-1,0[=]-2,0[\cap[-1,1]$. Since $]-2,0[$ is open in $\mathbf{R}$, $[-1,0[$ is of the form $[-1,0[= A\cap[-1,1]$ with $A \in \mathcal{T}_{\mathbf{R}}$. $[-1,0[$ is therefore an element of the induced topology on $[-1,1]$. In other words, $[-1,0[$ is an open set in $[-1,1]$.

3. Let $\psi :]-1,1[\to \mathbf{R}$ be defined by $\psi(y) = y/(1-|y|)$. It is easy to check that $\psi\circ\phi(x) = x$ for all $x \in \mathbf{R}$, and $\phi\circ\psi(y) = y$ for all $y \in]-1,1[$. It follows that ϕ is a bijection and $\phi^{-1} = \psi$. The fact that ϕ and ψ are continuous, may be regarded as an obvious point. However, if one wants to prove it from principles contained in these tutorials, the following argument can be used: from exercise (3), the usual topology on $\mathbf{R}$ is in fact the metric topology associated with $d(x,y) = |x-y|$. From theorem (12), the induced topology on $]-1,1[$ is also the metric

topology associated with $d(x, y) = |x - y|$. Consequently, the two topologies being metric, we can prove the continuity of ϕ and ψ using exercise (4). For $x \geq 0$ and $y \geq 0$, we have:

$$|\phi(x) - \phi(y)| = \frac{|x - y|}{(1 + x)(1 + y)} \leq |x - y| \tag{1}$$

and:

$$|\phi(x) + \phi(y)| = \frac{x}{1 + x} + \frac{y}{1 + y} \leq |x + y|$$

and since $\phi(-x) = -\phi(x)$ for all $x \in \mathbf{R}$, it is easy to check that equation (1) actually holds for all $x, y \in \mathbf{R}$. The continuity of ϕ is therefore an immediate consequence of exercise (4). Let $x \in]-1, 1[$ and $\epsilon > 0$ be given. For all $y \in]-1, 1[$, we have:

$$|\psi(x) - \psi(y)| = \left| \frac{x - y}{1 - |y|} + \frac{x(|x| - |y|)}{(1 - |x|)(1 - |y|)} \right|$$

Using the fact that $||x|-|y|| \leq |x-y|$ and $|x|<1$, we obtain:

$$|\psi(x)-\psi(y)| \leq \frac{|x-y|}{1-|y|} + \frac{|x-y|}{(1-|x|)(1-|y|)} \tag{2}$$

Let $\eta_1 > 0$ be such that $-1 < x-\eta_1 < x+\eta_1 < 1$. Then, the map $y \to 1/(1-|y|)$ is bounded on $]x-\eta_1, x+\eta_1[$. It follows from (2) that there exists $M \in \mathbf{R}^+$ such that for all $y \in]x-\eta_1, x+\eta_1[$:

$$|\psi(x)-\psi(y)| \leq M|x-y| + \frac{M}{(1-|x|)}|x-y|$$

Consequently, choosing $\eta > 0$ sufficiently small, it is possible to ensure that $|\psi(x)-\psi(y)| < \epsilon$, for all $y \in]x-\eta, x+\eta[$. We conclude from exercise (4) that ψ is continuous. Since ϕ and ψ are continuous, ϕ is a homeomorphism from $\mathbf{R}$ to $]-1,1[$.

4. Given $\epsilon > 0$ and $x \geq \max(1/\epsilon - 1, 0)$, we have:

$$|\phi(x)-1| = \frac{1}{1+x} \leq \epsilon$$

It follows that $\phi(x) \to 1$ as $x \to +\infty$. Since $\phi(-x) = -\phi(x)$ for all $x \in \mathbf{R}$, we conclude that $\phi(x) \to -1$ as $x \to -\infty$.

Exercise 6

Exercise 7.

1. Let $y \in [-1,1]$. If $y = 1$, then $y = \bar{\phi}(+\infty)$. If $y = -1$, then $y = \bar{\phi}(-\infty)$. If $y \in]-1,1[$, ϕ being onto, there exists $x \in \mathbf{R}$ such that $y = \phi(x) = \bar{\phi}(x)$. In any case, there exists $x \in \bar{\mathbf{R}}$ such that $y = \bar{\phi}(x)$. So $\bar{\phi}$ is onto. Suppose $x_1, x_2 \in \bar{\mathbf{R}}$ are such that $\bar{\phi}(x_1) = \bar{\phi}(x_2)$. If $\bar{\phi}(x_1) \in]-1,1[$, then $\phi(x_1) = \phi(x_2)$, and ϕ being injective, $x_1 = x_2$. If $\bar{\phi}(x_1) = 1$, then $x_1 = x_2 = +\infty$. If $\bar{\phi}(x_1) = -1$, then $x_1 = x_2 = -\infty$. In any case, $x_1 = x_2$. It follows that $\bar{\phi}$ is injective. Finally, $\bar{\phi}$ is a bijection.

2. $\bar{\phi}(\emptyset) = \emptyset$ is open in $[-1,1]$. So $\emptyset \in \mathcal{T}_{\bar{\mathbf{R}}}$. $\bar{\phi}(\bar{\mathbf{R}}) = [-1,1]$ is open in $[-1,1]$, so $\bar{\mathbf{R}} \in \mathcal{T}_{\bar{\mathbf{R}}}$. Let $A, B \in \mathcal{T}_{\bar{\mathbf{R}}}$. Using exercise (1), any direct image by $\bar{\phi}$ can also be viewed as an inverse image by $\bar{\psi}$. Hence, we have:

$$\bar{\phi}(A \cap B) = \bar{\psi}^{-1}(A \cap B) = \bar{\psi}^{-1}(A) \cap \bar{\psi}^{-1}(B) = \bar{\phi}(A) \cap \bar{\phi}(B)$$

Since A and B lie in $\mathcal{T}_{\bar{\mathbf{R}}}$, both $\bar{\phi}(A)$ and $\bar{\phi}(B)$ are open in $[-1,1]$. It follows that $\bar{\phi}(A \cap B)$ is open in $[-1,1]$, so $A \cap B \in \mathcal{T}_{\bar{\mathbf{R}}}$. Hence, we see that $\mathcal{T}_{\bar{\mathbf{R}}}$ is closed under finite intersection. Let $(A_i)_{i\in I}$ be a family of elements of $\mathcal{T}_{\bar{\mathbf{R}}}$. We have:

$$\bar{\phi}(\cup_{i\in I}A_i) = \bar{\psi}^{-1}(\cup_{i\in I}A_i) = \cup_{i\in I}\bar{\psi}^{-1}(A_i) = \cup_{i\in I}\bar{\phi}(A_i)$$

Each $\bar{\phi}(A_i)$ being open in $[-1,1]$, $\bar{\phi}(\cup_{i\in I}A_i)$ is also open in $[-1,1]$. It follows that $\cup_{i\in I}A_i \in \mathcal{T}_{\bar{\mathbf{R}}}$. Hence, we see that $\mathcal{T}_{\bar{\mathbf{R}}}$ is closed under arbitrary union. we have proved that $\mathcal{T}_{\bar{\mathbf{R}}}$ is indeed a topology on $\bar{\mathbf{R}}$.

3. From 1. we know that $\bar{\phi}$ is a bijection from $\bar{\mathbf{R}}$ to $[-1,1]$. Let B be open in $[-1,1]$. We have:

$$B = (\bar{\phi}\circ\bar{\psi})^{-1}(B) = \bar{\psi}^{-1}(\bar{\phi}^{-1}(B))$$

Using exercise (1), we see that $B = \bar{\phi}(\bar{\phi}^{-1}(B))$. So $\bar{\phi}(\bar{\phi}^{-1}(B))$ is open in $[-1,1]$. From the very definition of $\mathcal{T}_{\bar{\mathbf{R}}}$, it follows that $\bar{\phi}^{-1}(B) \in \mathcal{T}_{\bar{\mathbf{R}}}$. From definition (27) we conclude that $\bar{\phi}$ is continuous. Let A be open in $\bar{\mathbf{R}}$, i.e. $A \in \mathcal{T}_{\bar{\mathbf{R}}}$. By definition, $\bar{\phi}(A)$ is open in $[-1,1]$. Using exercise (1), $\bar{\phi}(A) = \bar{\psi}^{-1}(A)$. Hence,

$\bar{\psi}^{-1}(A)$ is open in $[-1,1]$. From definition (27) we conclude that $\bar{\psi}$ is continuous. Finally, $\bar{\phi}$ is a homeomorphism from $\bar{\mathbf{R}}$ to $[-1,1]$.

4. We have:

$$\begin{array}{rclcl} \bar{\phi}([-\infty,2[) & = & [-1,2/3[& = &]-\infty,2/3[\cap[-1,1] \\ \bar{\phi}(]3,+\infty]) & = &]3/4,1] & = &]3/4,+\infty[\cap[-1,1] \\ \bar{\phi}(]3,+\infty[) & = &]3/4,1[& = &]3/4,1[\cap[-1,1] \end{array}$$

It follows that $\bar{\phi}([-\infty,2[)$, $\bar{\phi}(]3,+\infty])$ and $\bar{\phi}(]3,+\infty[)$ are all open sets in $[-1,1]$. Consequently, $[-\infty,2[$, $]3,+\infty]$ and $]3,+\infty[$ are open in $\bar{\mathbf{R}}$.

5. Let $\phi' : \bar{\mathbf{R}} \to [-1,1]$ be an arbitrary homeomorphism, and $\psi' = (\phi')^{-1}$. Suppose $U \subseteq \bar{\mathbf{R}}$ is open in $\bar{\mathbf{R}}$, i.e. $U \in \mathcal{T}_{\bar{\mathbf{R}}}$. Since ψ' is continuous, $(\psi')^{-1}(U)$ is open in $[-1,1]$. Using exercise (1), $(\psi')^{-1}(U) = \phi'(U)$. So $\phi'(U)$ is open in $[-1,1]$. Conversely, suppose $\phi'(U)$ is open in $[-1,1]$ for $U \subseteq \bar{\mathbf{R}}$. Since ϕ' is continuous,

$(\phi')^{-1}(\phi'(U))$ is open in $\bar{\mathbf{R}}$. However, using exercise (1):

$$(\phi')^{-1}(\phi'(U)) = (\phi')^{-1}((\psi')^{-1}(U)) = (\psi' \circ \phi')^{-1}(U) = U$$

Hence, U is open in $\bar{\mathbf{R}}$. The purpose of this exercise is to give a formal description of the *usual topology* on $\bar{\mathbf{R}}$, leading to definition (34).

Exercise 7

Exercise 8.

1. From definition (23), $\mathcal{T}'$ is the topology on $\mathbf{R}$ induced by $\mathcal{T}_{\bar{\mathbf{R}}}$.

2. Let $U \subseteq \bar{\mathbf{R}}$. Let $y \in \phi(U \cap \mathbf{R})$. There exists $x \in U \cap \mathbf{R}$ such that $y = \phi(x)$. In particular, $y \in]-1,1[$ and $y = \bar{\phi}(x)$ with $x \in U$. So $y \in \bar{\phi}(U) \cap]-1,1[$. Conversely, suppose that $y \in \bar{\phi}(U) \cap]-1,1[$. There exists $x \in U$ such that $y = \bar{\phi}(x)$. But $\bar{\phi}(x) \in]-1,1[$ implies that that $x \in \mathbf{R}$, and therefore $\bar{\phi}(x) = \phi(x) = y$. So $x \in U \cap \mathbf{R}$ and $\phi(x) = y$. It follows that $y \in \phi(U \cap \mathbf{R})$. We have proved that $\phi(U \cap \mathbf{R}) = \bar{\phi}(U) \cap]-1,1[$.

3. Let $U \in \mathcal{T}_{\bar{\mathbf{R}}}$. By definition, $\bar{\phi}(U)$ is open in $[-1,1]$. There exists B open in $\mathbf{R}$, such that $\bar{\phi}(U) = B \cap [-1,1]$. Hence, $\bar{\phi}(U) \cap]-1,1[= B \cap]-1,1[$. From 2., $\phi(U \cap \mathbf{R}) = B \cap]-1,1[$. We conclude that $\phi(U \cap \mathbf{R})$ is open in $]-1,1[$.

4. Let $V \in \mathcal{T}'$. By definition, there exists $U \in \mathcal{T}_{\bar{\mathbf{R}}}$ such that $V = U \cap \mathbf{R}$. From 3., we see that $\phi(V)$ is open in $]-1,1[$.

ϕ being continuous, $\phi^{-1}(\phi(V))$ is therefore open in $\mathbf{R}$. However, using exercise (1):

$$\phi^{-1}(\phi(V)) = \phi^{-1}(\psi^{-1}(V)) = (\psi \circ \phi)^{-1}(V) = V$$

It follows that V is open in $\mathbf{R}$, i.e. $V \in \mathcal{T}_{\mathbf{R}}$. We have proved that $\mathcal{T}' \subseteq \mathcal{T}_{\mathbf{R}}$

5. Let $U \in \mathcal{T}_{\mathbf{R}}$. Since $U \subseteq \mathbf{R}$, it is easy to check that $\bar{\phi}(U) = \phi(U)$. Using exercise (1), $\phi(U) = \psi^{-1}(U)$, and ψ being continuous, $\psi^{-1}(U)$ is open in $]-1,1[$. It follows that $\bar{\phi}(U)$ is open in $]-1,1[$. There exists B open in $\mathbf{R}$, such that $\bar{\phi}(U) = B\cap]-1,1[$. In particular $\bar{\phi}(U)$ is also open in $\mathbf{R}$, with $\bar{\phi}(U) = \bar{\phi}(U)\cap[-1,1]$. We conclude that $\bar{\phi}(U)$ is open in $[-1,1]$.

6. For all $U \in \mathcal{T}_{\mathbf{R}}$, from 5., $\bar{\phi}(U)$ is open in $[-1,1]$. It follows that $U \in \mathcal{T}_{\bar{\mathbf{R}}}$. We have proved that $\mathcal{T}_{\mathbf{R}} \subseteq \mathcal{T}_{\bar{\mathbf{R}}}$.

7. Let $U \in \mathcal{T}_{\mathbf{R}}$. From 6., $U \in \mathcal{T}_{\bar{\mathbf{R}}}$. However, since $U \subseteq \mathbf{R}$, we have $U = U \cap \mathbf{R}$. From $U \in \mathcal{T}_{\bar{\mathbf{R}}}$ we conclude that $U \in \mathcal{T}'$. We

have proved that $\mathcal{T}_{\mathbf{R}} \subseteq \mathcal{T}'$. From 4., $\mathcal{T}' \subseteq \mathcal{T}_{\mathbf{R}}$. It follows that $\mathcal{T}_{\mathbf{R}} = \mathcal{T}'$. In other words, the topology on $\mathbf{R}$ induced by the usual topology on $\bar{\mathbf{R}}$, is nothing but the usual topology on $\mathbf{R}$.

8. Using the trace theorem (10), we have:

$$\mathcal{B}(\bar{\mathbf{R}})_{|\mathbf{R}} = \sigma(\mathcal{T}_{\bar{\mathbf{R}}})_{|\mathbf{R}} = \sigma((\mathcal{T}_{\bar{\mathbf{R}}})_{|\mathbf{R}}) = \sigma(\mathcal{T}_{\mathbf{R}}) = \mathcal{B}(\mathbf{R})$$

Exercise 8

Exercise 9.

1. $d(x, y) = 0$ is equivalent to $\phi(x) = \phi(y)$, which is in turn equivalent to $x = y$. So (i) of definition (28) is satisfied for d. The fact that (ii) is also satisfied is completely obvious. Given $x, y, z \in \bar{\mathbf{R}}$, we have:

$$|\phi(x) - \phi(y)| \leq |\phi(x) - \phi(z)| + |\phi(z) - \phi(y)|$$

   It follows that (iii) of definition (28) is also satisfied for d. We have proved that d is indeed a metric on $\bar{\mathbf{R}}$.

2. Let $U \in \mathcal{T}_{\bar{\mathbf{R}}}$ and $\psi = \phi^{-1}$. Since, $\phi(U) = \psi^{-1}(U)$, ψ being continuous, $\phi(U)$ is open in $[-1, 1]$.

3. Let $U \in \mathcal{T}_{\bar{\mathbf{R}}}$ and $y \in \phi(U)$. From 2., $\phi(U)$ is open in $[-1, 1]$. From theorem (12), the induced topology on $[-1, 1]$ is also the metric topology associated with $d(x, y) = |x - y|$ on $[-1, 1]^2$. Hence, there exists $\epsilon > 0$ such that $B'(y, \epsilon) \subseteq \phi(U)$, where $B'(y, \epsilon)$ is the open ball in $[-1, 1]$. Equivalently, there exists

$\epsilon > 0$, such that:

$$\forall z \in [-1,1] \ , \ |z - y| < \epsilon \ \Rightarrow \ z \in \phi(U) \tag{3}$$

4. Let $U \in \mathcal{T}_{\bar{\mathbf{R}}}$. Let $x \in U$ and $y = \phi(x)$. Then $y \in \phi(U)$. From 3., there exists $\epsilon > 0$ such that property (3) holds. Let $x' \in B(x,\epsilon)$ where $B(x,\epsilon)$ is the open ball in $\bar{\mathbf{R}}$. Then $d(x,x') < \epsilon$, i.e. $|\phi(x') - y| < \epsilon$. Since $\phi(x') \in [-1,1]$, from property (3), we see that $\phi(x') \in \phi(U)$. There exists $x'' \in U$ such that $\phi(x') = \phi(x'')$. ϕ being injective, $x' = x''$ and in particular $x' \in U$. We have proved that $B(x,\epsilon) \subseteq U$. It follows that $U \in \mathcal{T}^d_{\bar{\mathbf{R}}}$. This being true for all $U \in \mathcal{T}_{\bar{\mathbf{R}}}$, we conclude that $\mathcal{T}_{\bar{\mathbf{R}}} \subseteq \mathcal{T}^d_{\bar{\mathbf{R}}}$.

5. Let $U \in \mathcal{T}^d_{\bar{\mathbf{R}}}$ and $x \in U$. From definition (30), there exists $\epsilon > 0$ such that $B(x,\epsilon) \subseteq U$. In other words, there exists $\epsilon > 0$ such that:

$$\forall y \in \bar{\mathbf{R}} \ , \ |\phi(x) - \phi(y)| < \epsilon \ \Rightarrow \ y \in U \tag{4}$$

6. Let $U \in \mathcal{T}^d_{\bar{\mathbf{R}}}$ and $z \in \phi(U)$. There exists $x \in U$ such that

$z = \phi(x)$. Let $\epsilon > 0$ be such that property (4) holds. Let $z' \in B'(z, \epsilon)$, where $B'(z, \epsilon)$ is the open ball in $[-1, 1]$. ϕ being onto, there exists $y \in \bar{\mathbf{R}}$ such that $z' = \phi(y)$. Since $|z - z'| < \epsilon$, we have $|\phi(x) - \phi(y)| < \epsilon$. Using property (4), $y \in U$. It follows that $z' \in \phi(U)$. We have proved that $B'(z, \epsilon) \subseteq \phi(U)$. So $\phi(U)$ is open in $[-1, 1]$ with respect to the metric topology on $[-1, 1]$. From theorem (12), this topology coincide with the induced topology on $[-1, 1]$. Finally, $\phi(U)$ is open in $[-1, 1]$.

7. Let $U \in \mathcal{T}^d_{\bar{\mathbf{R}}}$, and $\psi = \phi^{-1}$. From 6., $\phi(U) = \psi^{-1}(U)$ is open in $[-1, 1]$. ϕ being continuous $\phi^{-1}(\psi^{-1}(U)) = (\psi \circ \phi)^{-1}(U) = U$ is open in $\bar{\mathbf{R}}$. We have proved that $\mathcal{T}^d_{\bar{\mathbf{R}}} \subseteq \mathcal{T}_{\bar{\mathbf{R}}}$.

8. We have $\mathcal{T}^d_{\bar{\mathbf{R}}} = \mathcal{T}_{\bar{\mathbf{R}}}$. d is a metric on $\bar{\mathbf{R}}$, for which the associated metric topology coincide with the usual topology on $\bar{\mathbf{R}}$. From definition (32), $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is metrizable. This proves theorem (13).

Exercise 9

Exercise 10.

1. Let $B \subseteq S$. For all $x \in \Omega$, since $f(\Omega) \subseteq S'$, $f(x) \in B$ is equivalent to $f(x) \in B \cap S'$. Hence, $f^{-1}(B) = f^{-1}(B \cap S')$.

2. From definition (35), $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ is measurable, if and only if $f^{-1}(B) \in \mathcal{F}$, for all $B \in \Sigma$. From 1., this is equivalent to $f^{-1}(B \cap S') \in \mathcal{F}$, for all $B \in \Sigma$, or in other words, $f^{-1}(B') \in \mathcal{F}$, for all $B' \in \Sigma_{|S'} = \Sigma'$. It follows that the measurability of f viewed as a function with values in (S, Σ), is equivalent to the measurability of f viewed as a function with values in (S', Σ').

3. From the trace theorem (10) and the fact that the topologies on $\mathbf{R}$ and $\mathbf{R}^+$ are induced from the topology on $\bar{\mathbf{R}}$, $\mathcal{B}(\mathbf{R}) = \mathcal{B}(\bar{\mathbf{R}})_{|\mathbf{R}}$ and $\mathcal{B}(\mathbf{R}^+) = \mathcal{B}(\bar{\mathbf{R}})_{|\mathbf{R}^+}$. So the equivalence between (i), (ii) and (iii) is a direct application of 2.

Exercise 10

Exercise 11.

1. Let $B \subseteq \mathcal{S}_1$. For all $x \in \Omega$, $g \circ f(x) \in B$ is equivalent to $f(x) \in g^{-1}(B)$, which is in turn equivalent to $x \in f^{-1}(g^{-1}(B))$. It follows that $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$. Note that we have used this property on several occasions in the solutions of exercises (7) and (8).

2. Let $B \in \Sigma_1$. Since $g : (S, \Sigma) \to (S_1, \Sigma_1)$ is measurable, we have $g^{-1}(B) \in \Sigma$. Since $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ is measurable, we have $f^{-1}(g^{-1}(B)) \in \mathcal{F}$. Using 1., we see that $(f \circ g)^{-1}(B) \in \mathcal{F}$. It follows that $f \circ g : (\Omega, \mathcal{F}) \to (S_1, \Sigma_1)$ is measurable.

Exercise 11

Exercise 12.

1. f being defined on Ω, any inverse image by f is by definition (26) a subset of Ω. Moreover, for all $x \in \Omega$, $f(x) \in S$. So $x \in f^{-1}(S)$ and $\Omega \subseteq f^{-1}(S)$. We have proved that $\Omega = f^{-1}(S)$.

2. For all $x \in \Omega$, $f(x) \in B^c$ is equivalent to $x \notin f^{-1}(B)$. So $f^{-1}(B^c) = (f^{-1}(B))^c$.

3. Let $(B_i)_{i \in I}$ be a family of subsets of S. $f(x) \in \cup_{i \in I} B_i$ is equivalent to $f(x) \in B_i$ for some $i \in I$, which is in turn equivalent to $x \in \cup_{i \in I} f^{-1}(B_i)$. So $f^{-1}(\cup_{i \in I} B_i) = \cup_{i \in I} f^{-1}(B_i)$. Note that we have used this property in the solution of exercise (7).

4. Σ being a σ-algebra on S, $S \in \Sigma$. From 1., $f^{-1}(S) = \Omega$, and $\mathcal{F}$ being a σ-algebra on Ω, $\Omega \in \mathcal{F}$. So $f^{-1}(S) \in \mathcal{F}$, and $S \in \Gamma$. Let $B \in \Gamma$. In particular $B \in \Sigma$ and therefore $B^c \in \Sigma$. Moreover from 2., $f^{-1}(B^c) = (f^{-1}(B))^c$. Since $B \in \Gamma$, $f^{-1}(B) \in \mathcal{F}$ and therefore $(f^{-1}(B))^c \in \mathcal{F}$. It follows that $f^{-1}(B^c) \in \mathcal{F}$ and we

see that $B^c \in \Gamma$. We have proved that Γ is closed under complementation. Let $(B_n)_{n\geq 1}$ be a sequence of elements of Γ. In particular $(B_n)_{n\geq 1}$ is a sequence of elements of Σ and therefore $\cup_{n=1}^{+\infty} B_n \in \Sigma$. Moreover, $f^{-1}(\cup_{n=1}^{+\infty} B_n) = \cup_{n=1}^{+\infty} f^{-1}(B_n)$. Since $B_n \in \Gamma$, for all $n \geq 1$, $f^{-1}(B_n) \in \mathcal{F}$ for all $n \geq 1$ and therefore $\cup_{n=1}^{+\infty} f^{-1}(B_n) \in \mathcal{F}$. It follows that $f^{-1}(\cup_{n=1}^{+\infty} B_n) \in \mathcal{F}$ and we see that $\cup_{n=1}^{+\infty} B_n \in \Gamma$. We have proved that Γ is closed under countable union. Finally, Γ is a σ-algebra on S.

5. Suppose $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ is measurable. Since $\mathcal{A} \subseteq \Sigma$, for all $B \in \mathcal{A}$, $f^{-1}(B) \in \mathcal{F}$. Conversely, suppose that the weaker condition of $f^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{A}$, is satisfied. Then, $\mathcal{A} \subseteq \Gamma$. From 4., Γ is a σ-algebra on S. Since the σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$ is the smallest σ-algebra on S containing $\mathcal{A}$, we obtain that $\sigma(\mathcal{A}) \subseteq \Gamma$. However $\sigma(\mathcal{A}) = \Sigma$. It follows that $\Sigma \subseteq \Gamma$, and in particular, $f^{-1}(B) \in \mathcal{F}$ for all $B \in \Sigma$. So $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ is measurable. This proves theorem (14).

Exercise 12

Exercise 13. Let $f : (\Omega, \mathcal{T}) \to (S, \mathcal{T}_S)$ be continuous. By definition (16) , the Borel σ-algebra $\mathcal{B}(S)$ is generated by the set of all open sets, i.e. $\mathcal{B}(S) = \sigma(\mathcal{T}_S)$. Since f is continuous, for all $B \in \mathcal{T}_S$, we have $f^{-1}(B) \in \mathcal{T}$. In particular, for all $B \in \mathcal{T}_S$, $f^{-1}(B) \in \mathcal{B}(\Omega)$. Using theorem (14), we conclude that $f : (\Omega, \mathcal{B}(\Omega)) \to (S, \mathcal{B}(S))$ is measurable.

Exercise 13

Exercise 14.

1. Let $\bar{\phi} : \bar{\mathbf{R}} \to [-1,1]$ be defined as in definition (34). Then, for all $c \in \mathbf{R}$, $\bar{\phi}([-\infty, c[) = [-1, \bar{\phi}(c)[$ and $\bar{\phi}(]c, +\infty]) =]\bar{\phi}(c), 1]$. Both sets being open in $[-1,1]$, we conclude that $\mathcal{C}_2 \subseteq \mathcal{T}_{\bar{\mathbf{R}}}$ and $\mathcal{C}_4 \subseteq \mathcal{T}_{\bar{\mathbf{R}}}$.

2. Using 1., for all $c \in \mathbf{R}$, we have $[-\infty, c]^c =]c, +\infty] \in \mathcal{T}_{\bar{\mathbf{R}}}$ and $[c, +\infty]^c = [-\infty, c[\in \mathcal{T}_{\bar{\mathbf{R}}}$. Hence, the complements of any element of $\mathcal{C}_1$ or $\mathcal{C}_3$ is open in $\bar{\mathbf{R}}$. It follows that any element of $\mathcal{C}_1$ or $\mathcal{C}_3$ is closed in $\bar{\mathbf{R}}$.

3. Let $i = 1, \ldots, 4$. From 1. and 2., any element of $\mathcal{C}_i$ is either closed or open in $\bar{\mathbf{R}}$. In any case, it is a Borel set in $\bar{\mathbf{R}}$. Hence, $\mathcal{C}_i \subseteq \mathcal{B}(\bar{\mathbf{R}})$. Since $\sigma(\mathcal{C}_i)$ is the smallest σ-algebra on $\bar{\mathbf{R}}$ containing $\mathcal{C}_i$, we conclude that $\sigma(\mathcal{C}_i) \subseteq \mathcal{B}(\bar{\mathbf{R}})$.

4. From exercise (8), the usual topology on $\bar{\mathbf{R}}$ induces the usual topology on $\mathbf{R}$. Hence, for all $U \in \mathcal{T}_{\bar{\mathbf{R}}}$, $U \cap \mathbf{R} \in (\mathcal{T}_{\bar{\mathbf{R}}})_{|\mathbf{R}} = \mathcal{T}_{\mathbf{R}}$, i.e. $U \cap \mathbf{R}$ is open in $\mathbf{R}$.

5. Let U be open in $\mathbf{R}$. For all $x \in U$, there exists $\epsilon_x > 0$ such that $]x-\epsilon_x, x+\epsilon_x[\subseteq U$. Let $p_x \in]x-\epsilon_x, x[\cap\mathbf{Q}$ and $q_x \in]x, x+\epsilon_x[\cap\mathbf{Q}$. Then, $x \in]p_x, q_x[\subseteq U$. It follows that $U = \cup_{i\in I} A_i$, where I is the countable set $I = \{]p_x, q_x[: x \in U\}$ and $A_i = i$ for all $i \in I$. We have proved that U can be expressed as a countable union of open bounded intervals in $\mathbf{R}$[1].

6. For all $n \geq 1$, $]a, b-1/n] \subseteq]a, b[$ and $[a+1/n, b[\subseteq]a, b[$. Moreover, for all $x \in]a, b[$, there exists $n \geq 1$ with $a + 1/n \leq x \leq b - 1/n$. It follows that:

$$]a, b[= \bigcup_{n=1}^{+\infty}]a, b-1/n] = \bigcup_{n=1}^{+\infty} [a+1/n, b[$$

7. For all $a, b \in \mathbf{R}$, $]a, b] =]a, +\infty]\setminus]b, +\infty] = [-\infty, b] \setminus [-\infty, a]$. So $]a, b] \in \sigma(\mathcal{C}_4) \cap \sigma(\mathcal{C}_1)$. Similarly $[a, b[\in \sigma(\mathcal{C}_2) \cap \sigma(\mathcal{C}_3)$. Using 6., we conclude that $]a, b[\in \sigma(\mathcal{C}_i)$, for all $i = 1, \ldots, 4$.

[1] If you think this proof was a bit quick, see Exercise (7) of the previous tutorial.

8. $\{+\infty\} = \cap_n [n, +\infty] = \cap_n]n, +\infty] = \cap_n [-\infty, n]^c = \cap_n [-\infty, n[^c$. We conclude that $\{+\infty\} \in \sigma(\mathcal{C}_i)$, and similarly $\{-\infty\} \in \sigma(\mathcal{C}_i)$, for all $i = 1, \dots, 4$.

9. Let $i = 1, \dots, 4$. Let $U \in \mathcal{T}_{\bar{\mathbf{R}}}$. From 4., $U \cap \mathbf{R} \in \mathcal{T}_{\mathbf{R}}$. From 5., $U \cap \mathbf{R}$ can be expressed as a countable union of open bounded intervals in $\mathbf{R}$. From 7., any such interval is an element of $\sigma(\mathcal{C}_i)$. It follows that $U \cap \mathbf{R} \in \sigma(\mathcal{C}_i)$. However, $U = (U \cap \mathbf{R}) \uplus A$, where A is either $\emptyset$, $\{-\infty\}$, $\{+\infty\}$ or $\{-\infty, +\infty\}$. We conclude from 8. that in any case, $U \in \sigma(\mathcal{C}_i)$. We have proved that $\mathcal{T}_{\bar{\mathbf{R}}} \subseteq \sigma(\mathcal{C}_i)$, and therefore $\mathcal{B}(\bar{\mathbf{R}}) \subseteq \sigma(\mathcal{C}_i)$. From 3., $\sigma(\mathcal{C}_i) \subseteq \mathcal{B}(\bar{\mathbf{R}})$. Finally $\sigma(\mathcal{C}_i) = \mathcal{B}(\bar{\mathbf{R}})$.

10. Given $B \subseteq \bar{\mathbf{R}}$, $\{f \in B\}$ denotes $f^{-1}(B)$. $(i) \Leftrightarrow (ii)$ is just definition (35). Similarly, $\{f \leq c\} = f^{-1}([-\infty, c])$, etc... and the equivalence between (i), and (iii), (iv), (v) and (vi), stems from a direct application of theorem (14), using $\sigma(\mathcal{C}_i) = \mathcal{B}(\bar{\mathbf{R}})$.

Exercise 14

Exercise 15.

1. Let $\omega \in \{c \leq g\} = g^{-1}([c, +\infty])$. Then $c \leq g(\omega) = \inf_{n\geq 1} f_n(\omega)$. In particular, for all $n \geq 1$, $c \leq f_n(\omega)$. So $\omega \in \cap_{n=1}^{+\infty}\{c \leq f_n\}$. Conversely, suppose that $c \leq f_n(\omega)$ for all $n \geq 1$. Then c is a lower-bound of all $f_n(\omega)$'s for $n \geq 1$. $g(\omega)$ being the greatest of such lower-bound, we have $c \leq g(\omega)$. We have proved that $\{c \leq g\} = \cap_{n=1}^{+\infty}\{c \leq f_n\}$.

2. Let $\omega \in \{h \leq c\} = h^{-1}([-\infty, c])$. Then $\sup_{n\geq 1} f_n(\omega) = h(\omega) \leq c$. In particular, for all $n \geq 1$, $f_n(\omega) \leq c$. So $\omega \in \cap_{n=1}^{+\infty}\{f_n \leq c\}$. Conversely, suppose that $f_n(\omega) \leq c$ for all $n \geq 1$. Then c is an upper-bound of all $f_n(\omega)$'s for $n \geq 1$. $h(\omega)$ being the smallest of such upper-bound, we have $h(\omega) \leq c$. We have proved that $\{h \leq c\} = \cap_{n=1}^{+\infty}\{f_n \leq c\}$.

3. All f_n's being measurable, using theorem (15), we conclude from 1. and 2. that $g, h : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are measurable.

Exercise 15

Exercise 16.

1. Let $n \geq 1$. For all $k \geq n$, $u_n = \inf_{k\geq n} v_k \leq v_k$. In particular, u_n is a lower-bound of all v_k's for $k \geq n+1$. u_{n+1} being the greatest of such lower-bound, we see that $u_n \leq u_{n+1}$. From definition (36), we have $u = \sup_{n\geq 1} u_n$. In particular, u is an upper-bound of all u_n's. We have proved that $u_n \leq u_{n+1} \leq u$.

2. Let $n \geq 1$. For all $k \geq n$, $v_k \leq \sup_{k\geq n} v_k = w_n$. In particular, w_n is an upper-bound of all v_k's for $k \geq n+1$. w_{n+1} being the smallest of such upper-bound, we see that $w_{n+1} \leq w_n$. From definition (36), we have $w = \inf_{n\geq 1} w_n$. In particular, w is a lower-bound of all w_n's. We have proved that $w \leq w_{n+1} \leq w_n$.

3. From 1., $(u_n)_{n\geq 1}$ is a non-decreasing sequence in $\bar{\mathbf{R}}$. It therefore converges to $\sup_{n\geq 1} u_n = u$. Indeed, suppose $u = +\infty$. Then, u being the smallest of all u_n's upper-bounds, for all $A \in \mathbf{R}^+$, there exists $N \geq 1$ such that $A < u_N$. Since $(u_n)_{n\geq 1}$ is non-decreasing, we have $A < u_n$ for all $n \geq N$. It follows that

$u_n \uparrow +\infty$. If $u = -\infty$, then $u_n = -\infty$ for all $n \geq 1$ and $u_n \uparrow -\infty$. If $u \in \mathbf{R}$, then given $\epsilon > 0$, $u - \epsilon < u$. So $u - \epsilon$ cannot be an upper-bound of all u_n's. There exists $N \geq 1$ such that $u - \epsilon < u_N \leq u$. Since $(u_n)_{n\geq 1}$ is non-decreasing, we have $u - \epsilon < u_n \leq u$ for all $n \geq N$. It follows that $u_n \uparrow u$. Similarly, $(w_n)_{n\geq 1}$ being a non-increasing sequence in $\bar{\mathbf{R}}$, it converges to $\inf_{n\geq 1} w_n = w$. So $w_n \downarrow w$.

4. For all $n \geq 1$, $u_n = \inf_{k\geq n} v_k \leq v_n \leq \sup_{k\geq n} v_k = w_n$.

5. From $u_n \leq w_n$, taking the limit as $n \to +\infty$, we obtain $u \leq w$.

6. From 5., for all $n \geq 1$, $u_n \leq v_n \leq w_n$. If $u = w$, then $(u_n)_{n\geq 1}$ and $(w_n)_{n\geq 1}$ converge to the same limit $u \in \bar{\mathbf{R}}$. It follows that $(v_n)_{n\geq 1}$ also converges to $u \in \bar{\mathbf{R}}$.

7. Let $a, b \in \mathbf{R}$, with $u < a < b < w$. Let $n \geq 1$. In particular, we have $u_n < a < b < w_n$. Since $u_n = \inf_{k\geq n} v_k$, u_n is the greatest lower-bound of all v_k's for $k \geq n$. It follows that a

cannot be such lower-bound. There exists $N_1 \geq n$ such that $v_{N_1} < a$. Similarly, b cannot be an upper-bound of all v_k's for $k \geq n$. There exists $N_2 \geq n$ such that $b < v_{N_2}$.

8. From 7., there exist $n_1, m_1 \geq 1$, such that $v_{n_1} < a < b < v_{m_1}$. Let $n = \max(n_1 + 1, m_1 + 1)$. Using 7. once more, there exist $n_2, m_2 \geq n$ such that $v_{n_2} < a < b < v_{m_2}$. In particular, we have $n_1 < n_2$ and $m_1 < m_2$. By induction, we can therefore construct two strictly increasing sequences of integers $(n_k)_{k\geq 1}$ and $(m_k)_{k\geq 1}$ such that $v_{n_k} < a < b < v_{m_k}$ for all $k \geq 1$.

9. Suppose that $(v_n)_{n\geq 1}$ converges to some $v \in \bar{\mathbf{R}}$. From 5., $u \leq w$. Suppose $u < w$, and let $a, b \in \mathbf{R}$, $u < a < b < w$. Using 8., let $(n_k)_{k\geq 1}$ and $(m_k)_{k\geq 1}$ be two strictly increasing sequences of integers such that $v_{n_k} < a < b < v_{m_k}$. Taking the limit as $k \to +\infty$, we obtain $v \leq a < b \leq v$ which is a contradiction. It follows that if $(v_n)_{n\geq 1}$ converges to some $v \in \bar{\mathbf{R}}$, then $u = w$.

Exercise 16

Exercise 17.

1. Let $\omega \in \{f < g\}$. Then $f(\omega) < g(\omega)$. There exists a rational number $r \in \mathbf{Q}$ such that $f(\omega) < r < g(\omega)$. It follows that $\omega \in \{f < r\} \cap \{r < g\}$. So $\{f < g\} \subseteq \cup_{r \in \mathbf{Q}} \{f < r\} \cap \{r < g\}$. The reverse inclusion is clear.

2. Since f and g are measurable, $\{f < r\} = f^{-1}([-\infty, r[)$ and $\{r < g\} = g^{-1}(]r, +\infty])$ are both elements of $\mathcal{F}$, for all $r \in \mathbf{Q}$. Using 1., and the fact that $\mathbf{Q}$ is a countable set, it follows that $\{f < g\} \in \mathcal{F}$. Similarly, $\{g < f\} \in \mathcal{F}$. Moreover, we have $\{f \leq g\} = \{g < f\}^c \in \mathcal{F}$ and $\{g \leq f\} = \{f < g\}^c \in \mathcal{F}$. Finally, $\{f = g\} = \{f \leq g\} \cap \{g \leq f\} \in \mathcal{F}$.

Exercise 17

Exercise 18.

1. Let $g_n = \inf_{k\geq n} f_k$ and $h_n = \sup_{k\geq n} f_k$, for all $n \geq 1$. Being a countable infimum and supremum of measurable maps, using exercise (15), we see that g_n and h_n are measurable for all $n \geq 1$. Since $g = \sup_{n\geq 1} g_n$ and $h = \inf_{n\geq 1} h_n$, we conclude also from exercise (15), that $g, h : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are measurable.

2. Using 5. of exercise (16), $g(\omega) \leq h(\omega)$, for all $\omega \in \Omega$. So $g \leq h$.

3. Since $f, g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are measurable, using exercise (17), we conclude that $\{g = h\} \in \mathcal{F}$.

4. The set $\{\omega : \omega \in \Omega\ ,\ \lim_{n\to+\infty} f_n(\omega)$ exists in $\bar{\mathbf{R}}\}$ is by virtue of theorem (16), equal to $\{g = h\}$. From 3., it is therefore an element of $\mathcal{F}$.

5. If $f_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$, using theorem (16), $f = g = h$. From 1., $f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is itself measurable.

Exercise 18

Exercise 19.

1. For all $c \in \mathbf{R}$, $\{-f < c\} = \{-c < f\}$. From theorem (15), we see that $-f$ is measurable. From $\{|f| < c\} = \{-c < f\} \cap \{f < c\}$, $|f|$ is measurable. If $c \leq 0$, then $\{f^+ < c\} = \emptyset$. If $c > 0$, then $\{f^+ < c\} = \{f < c\}$. In any case $\{f^+ < c\} \in \mathcal{F}$ and it follows that f^+ is measurable. Similarly, f^- is measurable.

2. An expression of the form $(+\infty) + (-\infty)$ is meaningless. Since f takes values in $\bar{\mathbf{R}}$, given $a \in \bar{\mathbf{R}}$ and $\omega \in \Omega$, the sum $a + f(\omega)$ may not be meaningful.

3. Let $a \in \mathbf{R}$. Then $a + f$ is meaningful as a map defined on Ω. Given $c \in \mathbf{R}$, we have $\{a + f < c\} = \{f < c - a\}$. We conclude from theorem (15) that $a + f$ is measurable.

4. Let $a \in \bar{\mathbf{R}}$. From 1., $-f$ is measurable whenever f is measurable. Without loss of generality, we can therefore assume that $a \geq 0$. If $0 < a < +\infty$, then for all $c \in \mathbf{R}$, $\{a.f < c\} = \{f < c/a\}$. It

follows from theorem (15) that $a.f$ is measurable. If $a = 0$, since by convention $0.(+\infty) = 0.(-\infty) = 0$, we have $a.f = 0$. Given $c \in \mathbf{R}$, $\{a.f < c\}$ is either $\emptyset$ or Ω. In any case $\{a.f < c\} \in \mathcal{F}$, and $a.f$ is measurable. If $a = +\infty$, then for all $c \in \mathbf{R}$, we have $\{a.f < c\} = \{f < 0\}$ if $c \leq 0$, and $\{a.f < c\} = \{f < 0\} \uplus \{f = 0\}$ if $c > 0$. In any case, $\{a.f < c\} \in \mathcal{F}$ and $a.f$ is measurable.

5. Given $\omega \in \Omega$, the sum $f(\omega) + g(\omega)$ may not be meaningful.

6. If $f \geq 0$ and $g \geq 0$, the sum $f + g$ is meaningful as a map defined on Ω. Let $\omega \in \{f + g < c\}$ where $c \in \mathbf{R}$. In particular, $g(\omega) < +\infty$. Subtracting $g(\omega)$ from both side of the inequality, we obtain $f(\omega) < c - g(\omega)$, i.e. $\omega \in \{f < c - g\}$. Conversely, if $f(\omega) < c - g(\omega)$, then $g(\omega)$ is again finite, and $f(\omega) + g(\omega) < c$. So $\{f + g < c\} = \{f < c - g\}$. This equality may have looked obvious in the first place. However, it is easy to make mistake with algebra and inequalities involving $+\infty$ and $-\infty$... From 1., $-g$ is a measurable map. Using 3., for all $c \in \mathbf{R}$, $c - g$ is also measurable. From exercise (17), $\{f < c - g\} \in \mathcal{F}$. Finally, using

theorem (15), we conclude that $f + g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable. The sum of two non-negative and measurable maps, is itself a non-negative and measurable map.

7. Suppose we have:

$$(\{f = +\infty\} \cap \{g = -\infty\}) \cup (\{f = -\infty\} \cap \{g = +\infty\}) = \emptyset$$

Then $f + g$ is meaningful as a map defined on Ω. As in 6., given $c \in \mathbf{R}$ we wish to argue that $\{f + g < c\} = \{f < c - g\}$. Given $\omega \in \Omega$, this amounts to checking the equivalence between the two inequalities $f(\omega) + g(\omega) < c$ and $f(\omega) < c - g(\omega)$, which is obviously true in the case when $f(\omega), g(\omega) \in \mathbf{R}$. Since the only other possible case is $f(\omega) = g(\omega) = +\infty$ or $f(\omega) = g(\omega) = -\infty$, such equivalence is clear and we have proved that the equality $\{f + g < c\} = \{f < c - g\}$ holds. As in 6. we conclude that $f + g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable. The sum of two $\bar{\mathbf{R}}$-valued measurable maps is itself measurable, provided it is well-defined.

8. If $f(\Omega) \subseteq \mathbf{R} \setminus \{0\}$, then $1/f$ is meaningful as a map defined on Ω. Let $c \in \mathbf{R}$. If $c > 0$, then $\{1/f < c\} = \{f < 0\} \uplus \{f > 1/c\}$. If $c = 0$, then $\{1/f < c\} = \{f < 0\}$. In the final case when $c < 0$, we have $\{1/f < c\} = \{1/c < f\} \cap \{f < 0\}$. In any case, $\{1/f < c\} \in \mathcal{F}$, and we conclude from theorem (15) that $1/f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

9. Let $B \in \mathcal{B}(\bar{\mathbf{R}})$. Then $\{\bar{f} \in B\} = (\{f \in B\} \cap \{f = 0\}^c) \uplus \{f = 0\}$, if $1 \in B$. Otherwise, $\{\bar{f} \in B\} = \{f \in B\} \cap \{f = 0\}^c$. In any case, $\{\bar{f} \in B\} \in \mathcal{F}$ and $\bar{f} : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

10. We have $\Omega = \{f > 0\} \uplus \{f < 0\} \uplus \{f = 0\}$. If $f(\omega) > 0$, then $f(\omega)g(\omega) < c$ is equivalent to $g(\omega) < c/\bar{f}(\omega)$. If $f(\omega) < 0$, then $f(\omega)g(\omega) < c$ is equivalent to $g(\omega) > c/\bar{f}(\omega)$. Finally, if $f(\omega) = 0$, then $f(\omega)g(\omega) < c$ is equivalent to $f(\omega) < c$. It follows that $\{fg < c\}$ can be expressed as:

$$(\{f>0\} \cap \{g < c/\bar{f}\}) \uplus (\{f<0\} \cap \{g > c/\bar{f}\}) \uplus (\{f = 0\} \cap \{f < c\})$$

11. Whether or not f and g take values in $\mathbf{R}$, the product fg is meaningful as a map defined on Ω. In the case when $f(\Omega) \subseteq \mathbf{R}$ and $g(\Omega) \subseteq \mathbf{R}$, given $c \in \mathbf{R}$, we can use the decomposition of $\{fg < c\}$ obtained in 10. Furthermore, from 9., $\bar{f}$ is a measurable map with values in $\mathbf{R} \setminus \{0\}$. Using 8., $1/\bar{f}$ is measurable. From 4., $c/\bar{f}$ is also measurable. It follows from exercise (17), that $\{g < c/\bar{f}\} \in \mathcal{F}$ and $\{g > c/\bar{f}\} \in \mathcal{F}$. Hence, all sets involved in 10. are elements of $\mathcal{F}$. So $\{fg < c\} \in \mathcal{F}$. We conclude from theorem (15) that $fg : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable. In the following exercise, we shall extend this result to the more general case when f and g have arbitrary values in $\bar{\mathbf{R}}$.

Exercise 19

Exercise 20.

1. For all $B \in \mathcal{B}(\bar{\mathbf{R}})$, the inverse image $\bar{f}^{-1}(B)$ can be written as:

   $$\bar{f}^{-1}(B) = (f^{-1}(B) \cap f^{-1}(\mathbf{R})) \uplus (A \cap (\{f = +\infty\} \uplus \{f = -\infty\}))$$

   where $A = \Omega$ if $1 \in B$, and $A = \emptyset$ otherwise. It follows that $\bar{f}^{-1}(B) \in \mathcal{F}$, and $\bar{f}$ is measurable. Similarly, $\bar{g}$ is measurable.

2. All A_i's and B_j's are inverse images of Borel sets in $\bar{\mathbf{R}}$, by measurable maps. They are therefore elements of $\mathcal{F}$.

3. Since $\Omega = \uplus_{i,j=1}^{5} A_i \cap B_j$, for all $B \in \mathcal{B}(\bar{\mathbf{R}})$, we have:

   $$\{fg \in B\} = \biguplus_{i,j=1}^{5} (A_i \cap B_j \cap \{fg \in B\})$$

4. For all $1 \leq i, j \leq 3$ and $\omega \in A_i \cap B_j$, $f(\omega) \in \mathbf{R}$ and $g(\omega) \in \mathbf{R}$. In particular, $f(\omega) = \bar{f}(\omega)$, and $g(\omega) = \bar{g}(\omega)$. Hence, we conclude that $A_i \cap B_j \cap \{fg \in B\} = A_i \cap B_j \cap \{\bar{f}\bar{g} \in B\}$.

5. Suppose $i \geq 4$ or $j \geq 4$. Then, for all $\omega \in A_i \cap B_j$, $f(\omega)g(\omega)$ is either $-\infty$, 0 or $+\infty$. More specifically, $f(\omega)g(\omega) = a$, with:

$$a = \begin{cases} -\infty & \text{if} \quad (i,j) \in \{(1,4),(2,5),(4,5),(5,4),(5,2),(4,1)\} \\ 0 & \text{if} \quad (i,j) \in \{(3,4),(3,5),(4,3),(5,3)\} \\ +\infty & \text{if} \quad (i,j) \in \{(1,5),(2,4),(4,4),(5,5),(5,1),(4,2)\} \end{cases}$$

Hence, given $B \in \mathcal{B}(\bar{\mathbf{R}})$, $A_i \cap B_j \cap \{fg \in B\} = \emptyset$ if $a \notin B$, and $A_i \cap B_j \cap \{fg \in B\} = A_i \cap B_j$ if $a \in B$.

6. Let $B \in \mathcal{B}(\bar{\mathbf{R}})$. From 1., $\bar{f}$ and $\bar{g}$ are measurable. Moreover, by construction, both $\bar{f}$ and $\bar{g}$ take values in $\mathbf{R}$. From exercise (19), it follows that $\bar{f}\bar{g}$ is measurable. Hence, $\{\bar{f}\bar{g} \in B\} \in \mathcal{F}$. From 2., all A_i's and B_j's are elements of $\mathcal{F}$. Using 4., whenever $1 \leq i,j \leq 3$, $A_i \cap B_j \cap \{fg \in B\} \in \mathcal{F}$. However, from 5., we also have $A_i \cap B_j \cap \{fg \in B\} \in \mathcal{F}$, for all $i \geq 4$ or $j \geq 4$. We conclude from 3. that $\{fg \in B\} \in \mathcal{F}$. We have proved that $fg : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

Exercise 20

Exercise 21.

1. Let $x \in A$. Suppose $U \in \mathcal{T}$ is such that $x \in U$. Then $x \in U \cap A$. In particular, $U \cap A \neq \emptyset$. So $x \in \bar{A}$. We have proved that $A \subseteq \bar{A}$. Suppose $x \notin \bar{A}$. From definition (37), there exists an open set $U_x \in \mathcal{T}$ such that $x \in U_x$ and $U_x \cap A = \emptyset$. Moreover, for all $y \in U_x$, from $U_x \in \mathcal{T}$, $U_x \cap A = \emptyset$ and definition (37), we see that $y \notin \bar{A}$. Hence, for all $x \in \bar{A}^c$, there exists $U_x \in \mathcal{T}$, such that $x \in U_x \subseteq \bar{A}^c$. It follows that $\bar{A}^c = \cup_{x \notin \bar{A}} U_x$, and $\bar{A}^c$ is therefore an open set in E. Hence, $\bar{A}$ is closed in E.

2. Suppose that B is closed and $A \subseteq B$. Then $B^c \in \mathcal{T}$. Suppose that $\bar{A} \subseteq B$ is false. There exists $x \in \bar{A} \cap B^c$. From $x \in B^c \in \mathcal{T}$ and definition (37), we see that $B^c \cap A \neq \emptyset$. This contradicts the assumption that $A \subseteq B$. It follows that $\bar{A} \subseteq B$.

3. From 1., $\bar{A}$ is indeed a closed set containing A. From 2., $\bar{A}$ is the smallest closed set containing A.

4. Suppose $A = \bar{A}$. Then from 1., A is closed. Conversely, suppose

that A is closed. Since $A \subseteq A$, using 2., $\bar{A} \subseteq A$. However from 1., $A \subseteq \bar{A}$. So $A = \bar{A}$. We have proved that A is closed, if and only if $A = \bar{A}$.

5. Suppose $\mathcal{T}$ is the metric topology associated with some metric d on E. Let A' be defined by:

$$A' = \{x \in E \ : \ \forall \epsilon > 0 \ , \ B(x, \epsilon) \cap A \neq \emptyset\}$$

Let $x \in \bar{A}$. For all $\epsilon > 0$, from exercise (2), $B(x, \epsilon)$ is an open set in E, which furthermore contains x. Hence, from definition (37), $B(x, \epsilon) \cap A \neq \emptyset$ and we see that $x \in A'$. So $\bar{A} \subseteq A'$. Conversely, suppose $x \in A'$. Let $U \in \mathcal{T}$ be such that $x \in U$. $\mathcal{T}$ being the metric topology, from definition (30), there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. However, since $x \in A'$, $B(x, \epsilon) \cap A \neq \emptyset$. In particular, $U \cap A \neq \emptyset$. It follows that $x \in \bar{A}$, and $A' \subseteq \bar{A}$. We have proved that $\bar{A} = A'$.

Exercise 21

Exercise 22.

1. By definition, for all $y \in \bar{A}$, $d(x, \bar{A}) \leq d(x, y)$. From exercise (21), $A \subseteq \bar{A}$. It follows that $d(x, \bar{A})$ is a lower-bound of all $d(x, y)$ for $y \in A$. $d(x, A)$ being the greatest of such lower-bound, we have $d(x, \bar{A}) \leq d(x, A)$. Suppose $d(x, \bar{A}) < d(x, A)$. Let $\alpha \in \mathbf{R}$ be such that $d(x, \bar{A}) < \alpha < d(x, A)$. It follows from $d(x, \bar{A}) < \alpha$, that α cannot be a lower-bound of all $d(x, y)$ for $y \in \bar{A}$. There exists $y \in \bar{A}$ such that $d(x, y) < \alpha$. Since $y \in \bar{A}$, from exercise (21), for all $\epsilon > 0$, $B(y, \epsilon) \cap A \neq \emptyset$. There exists $z \in A$ such that $d(y, z) < \epsilon$. In particular:

$$d(x, A) \leq d(x, z) \leq d(x, y) + d(y, z) < \alpha + \epsilon$$

   $\epsilon > 0$ being arbitrary, it follows that $d(x, A) \leq \alpha$. This is a contradiction. We conclude that $d(x, \bar{A}) = d(x, A)$.

2. Suppose that $d(x, A) = 0$. For all $\epsilon > 0$, ϵ cannot be a lower-bound of all $d(x, y)$ for $y \in A$. There exists $y \in A$, such that $d(x, y) < \epsilon$. In other words, $B(x, \epsilon) \cap A \neq \emptyset$. Hence, from

exercise (21), $x \in \bar{A}$. Conversely, suppose $x \in \bar{A}$. Then for all $\epsilon > 0$, $B(x, \epsilon) \cap A \neq \emptyset$. Let $y \in B(x, \epsilon) \cap A$. We have $d(x, A) \leq d(x, y) < \epsilon$. $\epsilon > 0$ being arbitrary, it follows that $d(x, A) \leq 0$. However, 0 is a lower-bound of all $d(x, y)$ for $y \in A$. So $0 \leq d(x, A)$. Hence $d(x, A) = 0$. We have proved that $d(x, A) = 0$, if and only if $x \in \bar{A}$.

3. Let $x, y \in E$. For all $z \in A$, we have:

$$d(x, A) \leq d(x, z) \leq d(x, y) + d(y, z)$$

   Subtracting $d(x, y) \in \mathbf{R}^+$ from both side of the inequality, the difference $d(x, A) - d(x, y)$ appears as a lower-bound of all $d(y, z)$ for $z \in A$. $d(y, A)$ being the greatest of such lower-bound, $d(x, A) - d(x, y) \leq d(y, A)$. Hence, $d(x, A) \leq d(x, y) + d(y, A)$.

4. Let $x, y \in E$. If $A \neq \emptyset$, there exists $z \in A$. From the inequality $d(x, A) \leq d(x, z)$, we have in particular $d(x, A) < +\infty$ and similarly $d(y, A) < +\infty$. The difference $d(x, A) - d(y, A)$ is therefore meaningful. $d(x, A) \leq d(x, y) + d(y, A)$ is obtained

from 3. Similarly, $d(y, A) \leq d(y, x) + d(x, A)$. It follows that $|d(x, A) - d(y, A)| \leq d(x, y)$.

5. If $A = \emptyset$, then for all $x \in E$, $\Phi_A(x) = +\infty$. The map Φ_A is therefore continuous. If $A \neq \emptyset$, then from 4., for all $x, y \in E$, $|\Phi_A(x) - \Phi_A(y)| \leq d(x, y)$. From theorem (12), the induced topology on $\mathbf{R}^+$ coincide with the metric topology. Using exercise (4), it follows that $\Phi_A : (E, \mathcal{T}_E^d) \to (\mathbf{R}^+, \mathcal{T}_{\mathbf{R}^+})$ is continuous. However, for all $U \in \mathcal{T}_{\bar{\mathbf{R}}}$, $U \cap \mathbf{R}^+ \in \mathcal{T}_{\mathbf{R}^+}$ and therefore, $\Phi_A^{-1}(U) = \Phi_A^{-1}(U \cap \mathbf{R}^+) \in \mathcal{T}_E^d$. So $\Phi_A : (E, \mathcal{T}_E^d) \to (\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is also continuous. Note that $\delta(u, v) = |u - v|$ is not a metric on $\bar{\mathbf{R}}$. Hence, we could not use exercise (4) to prove directly the continuity of Φ_A, viewed as a map with values in $\bar{\mathbf{R}}$.

6. Suppose that A is closed. From exercise (21), $A = \bar{A}$. Hence, from 2., $d(x, A) = 0$ is equivalent to $x \in A$. So $A = \Phi_A^{-1}(\{0\})$.

Exercise 22

Exercise 23.

1. The upper and lower limits as defined in definition (36), require the notions of infimums and supremums. Such notions may not be meaningful on an arbitrary metric space (E, d).

2. Let $\mathcal{A}$ be the set of all closed sets in E. $\mathcal{T}_E^d$ being the metric topology on E, the Borel σ-algebra on E is generated by $\mathcal{T}_E^d$, i.e. $\mathcal{B}(E) = \sigma(\mathcal{T}_E^d)$. In fact, $\mathcal{B}(E)$ is also generated by $\mathcal{A}$. Indeed, for all $A \in \mathcal{A}$, $A^c \in \mathcal{T}_E^d$. In particular $A^c \in \mathcal{B}(E)$, and therefore we have $A \in \mathcal{B}(E)$. So $\mathcal{A} \subseteq \mathcal{B}(E)$ and consequently, $\sigma(\mathcal{A}) \subseteq \mathcal{B}(E)$. However, for all $U \in \mathcal{T}_E^d$, $U^c \in \mathcal{A}$. In particular, $U^c \in \sigma(\mathcal{A})$, and therefore $U \in \sigma(\mathcal{A})$. So $\mathcal{T}_E^d \subseteq \sigma(\mathcal{A})$ and consequently, we have $\mathcal{B}(E) \subseteq \sigma(\mathcal{A})$. We have proved that $\mathcal{B}(E) = \sigma(\mathcal{A})$. From theorem (14), we conclude that a map $f : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is measurable, if and only if $f^{-1}(A) \in \mathcal{F}$, for all $A \in \mathcal{A}$.

3. Let A be closed in E. From exercise (22), $A = \Phi_A^{-1}(\{0\})$. Hence, $f^{-1}(A) = f^{-1}(\Phi_A^{-1}(\{0\})) = (\Phi_A \circ f)^{-1}(\{0\})$.

4. Let $n \geq 1$. By assumption, $f_n : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is measurable. From exercise (22), $\Phi_A : (E, \mathcal{T}_E^d) \to (\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is continuous. Using exercise (13), it follows that $\Phi_A : (E, \mathcal{B}(E)) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable. We conclude from exercise (11) that the map $\Phi_A \circ f_n : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable. Note that this is true for all $A \subseteq E$, irrespective of whether or not A is closed.

5. Let $A \subseteq E$. By assumption, for all $\omega \in \Omega$, $f_n(\omega) \to f(\omega)$. Since Φ_A is continuous, it follows that $\Phi_A \circ f_n(\omega) \to \Phi_A \circ f(\omega)$. A more direct justification of this fact is as follows: $f_n(\omega) \to f(\omega)$ is a short way of saying that given $\epsilon > 0$, there exists $N \geq 1$, such that $n \geq N$ implies that $d(f_n(\omega), f(\omega)) < \epsilon$. In the case when $A \neq \emptyset$, from exercise (22), we see that $n \geq N$ also implies that $|\Phi_A(f_n(\omega)) - \Phi_A(f(\omega))| \leq d(f_n(\omega), f(\omega)) < \epsilon$. Hence, $\Phi_A \circ f_n(\omega) \to \Phi_A \circ f(\omega)$. The fact that this is still true when $A = \emptyset$ is clear. Since $\Phi_A \circ f_n$ is a measurable map for all $n \geq 1$, we see from exercise (18) that $\Phi_A \circ f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable. In particular, $(\Phi_A \circ f)^{-1}(\{0\}) \in \mathcal{F}$. However, from

3., $(\Phi_A \circ f)^{-1}(\{0\}) = f^{-1}(A)$, whenever A is closed in E. We have proved that $f^{-1}(A) \in \mathcal{F}$, for all A closed in E. From 2., we conclude that $f : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is measurable. The purpose of this exercise is to prove theorem (17).

Exercise 23

Exercise 24. For all $z, z' \in \mathbf{C}$, we have $|Re(z) - Re(z')| \leq |z - z'|$, $|Im(z) - Im(z')| \leq |z - z'|$ and $||\,z\,| - |z'|| \leq |z - z'|$. From exercise (4), it follows that $Re, Im, |\,.\,| : (\mathbf{C}, \mathcal{T}_{\mathbf{C}}) \to (\mathbf{R}, \mathcal{T}_{\mathbf{R}})$ are all continuous maps. From exercise (13), $Re, Im, |\,.\,| : (\mathbf{C}, \mathcal{B}(\mathbf{C})) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ are therefore measurable. Since $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is measurable, using exercise (11), we conclude that $u = Re \circ f$, $v = Im \circ f$ and $|f| = |\,.\,| \circ f$ are all measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\mathbf{R})$. In fact, using exercise (10), they are also measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\bar{\mathbf{R}})$. Essentially, this last point is a direct consequence of the fact that given $B \in \mathcal{B}(\bar{\mathbf{R}})$, $B \cap \mathbf{R} \in \mathcal{B}(\mathbf{R})$.

Exercise 24

Exercise 25.

1. Let $A =]a,b[\times]c,d[\in \mathcal{C}$, and $z = x + iy \in A$. Then $x \in]a,b[$ and $y \in]c,d[$. Let $\epsilon > 0$ be such that $|x - x'| < \epsilon \Rightarrow x' \in]a,b[$, and $|y - y'| < \epsilon \Rightarrow y' \in]c,d[$. Then $|z - z'| < \epsilon \Rightarrow z' \in A$, for all $z' \in \mathbf{C}$. Hence, there exists $\epsilon > 0$ such that $B(z,\epsilon) \subseteq A$. We have proved that A is open in $\mathbf{C}$.

2. From 1., $\mathcal{C} \subseteq \mathcal{T}_{\mathbf{C}}$. In particular, $\mathcal{C} \subseteq \mathcal{B}(\mathbf{C})$. The σ-algebra $\sigma(\mathcal{C})$ generated by $\mathcal{C}$ being the smallest σ-algebra on $\mathbf{C}$ containing $\mathcal{C}$, we conclude that $\sigma(\mathcal{C}) \subseteq \mathcal{B}(\mathbf{C})$.

3. If $|x| < \eta$ and $|y| < \eta$, then $|z| \leq \sqrt{x^2 + y^2} < \sqrt{2}\eta$.

4. Let U be open in $\mathbf{C}$, and $z = x + iy \in U$. There exists $\epsilon > 0$, such that $B(z,\epsilon) \subseteq U$. Let $\eta = \epsilon/\sqrt{2}$. Using 3., we have $]x - \eta, x + \eta[\times]y - \eta, y + \eta[\subseteq U$. Let $a_z \in]x - \eta, x[\cap\mathbf{Q}$, and $b_z \in]x, x + \eta[\cap\mathbf{Q}$. Let $c_z \in]y - \eta, y[\cap\mathbf{Q}$ and $d_z \in]y, y + \eta[\cap\mathbf{Q}$. Then, we have $z \in]a_z, b_z[\times]c_z, d_z[\subseteq U$.

5. Let I be the set $I = \{]a_z, b_z[\times]c_z, d_z[, z \in U\}$. Then I is finite or countable, and $U = \cup_{i\in I} B_i$ where $B_i = i \in \mathcal{C}$, for all $i \in I$. In order to express U as a union indexed by the set of positive integers $\mathbf{N}^*$, the following can be done: Let $\psi : I \to \mathbf{N}^*$ be an arbitrary injection. For all $n \geq 1$, define A_n as $A_n = B_i$ if $n \in \psi(I)$ and $n = \psi(i)$, and $A_n = \emptyset$ if $n \notin \psi(I)$. Then, $A_n \in \mathcal{C}$ for all $n \geq 1$, and we have $U = \cup_{n=1}^{+\infty} A_n$.

6. It follows from 5. that $\mathcal{T}_{\mathbf{C}} \subseteq \sigma(\mathcal{C})$. The Borel σ-algebra $\mathcal{B}(\mathbf{C})$ being the smallest σ-algebra on $\mathbf{C}$ containing all open sets, we see that $\mathcal{B}(\mathbf{C}) \subseteq \sigma(\mathcal{C})$. Hence, from 2., $\sigma(\mathcal{C}) = \mathcal{B}(\mathbf{C})$.

7. Let $f = u + iv$. Then, $f^{-1}(A) = u^{-1}(]a, b[) \cap v^{-1}(]c, d[)$, for all $A =]a, b[\times]c, d[\in \mathcal{C}$. Since u and v are assumed to be measurable, $u^{-1}(]a, b[) \in \mathcal{F}$ and $v^{-1}(]c, d[) \in \mathcal{F}$. It follows that $f^{-1}(A) \in \mathcal{F}$. Using 6., we conclude from theorem (14) that f is measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\mathbf{C})$.

Exercise 25