9. L^p-spaces, $p \in [1, +\infty]$

In the following, $(\Omega, \mathcal{F}, \mu)$ is a measure space.

Exercise 1. Let $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be non-negative and measurable maps. Let $p, q \in \mathbf{R}^+$, such that $1/p + 1/q = 1$.

1. Show that $1 < p < +\infty$ and $1 < q < +\infty$.

2. For all $\alpha \in]0, +\infty[$, we define $\phi^\alpha : [0, +\infty] \to [0, +\infty]$ by:

$$\phi^\alpha(x) \triangleq \begin{cases} x^\alpha & \text{if} \quad x \in \mathbf{R}^+ \\ +\infty & \text{if} \quad x = +\infty \end{cases}$$

   Show that ϕ^α is a continuous map.

3. Define $A = (\int f^p d\mu)^{1/p}$, $B = (\int g^q d\mu)^{1/q}$ and $C = \int f g d\mu$. Explain why A, B and C are well defined elements of $[0, +\infty]$.

4. Show that if $A = 0$ or $B = 0$ then $C \leq AB$.

5. Show that if $A = +\infty$ or $B = +\infty$ then $C \leq AB$.

6. We assume from now on that $0 < A < +\infty$ and $0 < B < +\infty$. Define $F = f/A$ and $G = g/B$. Show that:

$$\int_\Omega F^p d\mu = \int_\Omega G^p d\mu = 1$$

7. Let $a, b \in]0, +\infty[$. Show that:

$$\ln(a) + \ln(b) \leq \ln\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right)$$

and:

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

Prove this last inequality for all $a, b \in [0, +\infty]$.

8. Show that for all $\omega \in \Omega$, we have:

$$F(\omega)G(\omega) \leq \frac{1}{p}(F(\omega))^p + \frac{1}{q}(G(\omega))^q$$

9. Show that $C \leq AB$.

Theorem 41 (Hölder's inequality) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Let $p, q \in \mathbf{R}^+$ be such that $1/p + 1/q = 1$. Then:*

$$\int_\Omega fg d\mu \leq \left(\int_\Omega f^p d\mu\right)^{\frac{1}{p}} \left(\int_\Omega g^q d\mu\right)^{\frac{1}{q}}$$

Theorem 42 (Cauchy-Schwarz's inequality:first)
Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Then:

$$\int_\Omega fg d\mu \leq \left(\int_\Omega f^2 d\mu\right)^{\frac{1}{2}} \left(\int_\Omega g^2 d\mu\right)^{\frac{1}{2}}$$

Exercise 2. Let $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Let $p \in]1, +\infty[$. Define $A = (\int f^p d\mu)^{1/p}$ and

$B = (\int g^p d\mu)^{1/p}$ and $C = (\int (f+g)^p d\mu)^{1/p}$.

1. Explain why A, B and C are well defined elements of $[0, +\infty]$.

2. Show that for all $a, b \in]0, +\infty[$, we have:

$$(a+b)^p \leq 2^{p-1}(a^p + b^p)$$

   Prove this inequality for all $a, b \in [0, +\infty]$.

3. Show that if $A = +\infty$ or $B = +\infty$ or $C = 0$ then $C \leq A + B$.

4. Show that if $A < +\infty$ and $B < +\infty$ then $C < +\infty$.

5. We assume from now that $A, B \in [0, +\infty[$ and $C \in]0, +\infty[$. Show the existence of some $q \in \mathbf{R}^+$ such that $1/p + 1/q = 1$.

6. Show that for all $a, b \in [0, +\infty]$, we have:

$$(a+b)^p = (a+b).(a+b)^{p-1}$$

7. Show that:

$$\begin{aligned} \int_\Omega f.(f+g)^{p-1} d\mu &\leq AC^{\frac{p}{q}} \\ \int_\Omega g.(f+g)^{p-1} d\mu &\leq BC^{\frac{p}{q}} \end{aligned}$$

8. Show that:

$$\int_\Omega (f+g)^p d\mu \leq C^{\frac{p}{q}}(A+B)$$

9. Show that $C \leq A + B$.

10. Show that the inequality still holds if we assume that $p = 1$.

Theorem 43 (Minkowski's inequality) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Let $p \in [1, +\infty[$. Then:*

$$\left(\int_\Omega (f+g)^p d\mu\right)^{\frac{1}{p}} \leq \left(\int_\Omega f^p d\mu\right)^{\frac{1}{p}} + \left(\int_\Omega g^p d\mu\right)^{\frac{1}{p}}$$

Definition 73 *The* **L^p-spaces**, *$p \in [1, +\infty[$, on $(\Omega, \mathcal{F}, \mu)$, are:*

$$L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} \left\{ f : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R})) \text{ measurable}, \int_\Omega |f|^p d\mu < +\infty \right\}$$

$$L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} \left\{ f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C})) \text{ measurable}, \int_\Omega |f|^p d\mu < +\infty \right\}$$

For all $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we put:

$$\|f\|_p \stackrel{\triangle}{=} \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}}$$

Exercise 3. Let $p \in [1, +\infty[$. Let $f, g \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

1. Show that $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) = \{f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \ , \ f(\Omega) \subseteq \mathbf{R}\}$.
2. Show that $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbf{R}$-linear combinations.
3. Show that $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbf{C}$-linear combinations.
4. Show that $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.
5. Show that $\|f\|_p = 0 \ \Leftrightarrow \ f = 0$ μ-a.s.
6. Show that for all $\alpha \in \mathbf{C}$, $\|\alpha f\|_p = |\alpha|.\|f\|_p$.
7. Explain why $(f, g) \to \|f - g\|_p$ is not a metric on $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$

Definition 74 *For all $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ measurable, Let:*

$$\|f\|_\infty \stackrel{\triangle}{=} \inf\{M \in \mathbf{R}^+ \ , \ |f| \leq M \ \ \mu\text{-}a.s.\}$$

The **L^∞-spaces** *on a measure space* $(\Omega, \mathcal{F}, \mu)$ *are:*

$$L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} \{f : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R})) \text{ measurable}, \|f\|_\infty < +\infty\}$$

$$L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} \{f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C})) \text{ measurable}, \|f\|_\infty < +\infty\}$$

Exercise 4. Let $f, g \in L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

1. Show that $L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) = \{f \in L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \ , \ f(\Omega) \subseteq \mathbf{R}\}$.
2. Show that $|f| \leq \|f\|_\infty$ μ-a.s.
3. Show that $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.
4. Show that $L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbf{R}$-linear combinations.
5. Show that $L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbf{C}$-linear combinations.
6. Show that $\|f\|_\infty = 0 \ \Leftrightarrow \ f = 0$ μ-a.s..
7. Show that for all $\alpha \in \mathbf{C}$, $\|\alpha f\|_\infty = |\alpha|.\|f\|_\infty$.

8. Explain why $(f, g) \to \|f - g\|_\infty$ is not a metric on $L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$

Definition 75 *Let $p \in [1, +\infty]$. Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. For all $\epsilon > 0$ and $f \in L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$, we define the so-called* **open ball** *in $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$:*

$$B(f, \epsilon) \stackrel{\triangle}{=} \{g : g \in L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu), \|f - g\|_p < \epsilon\}$$

We call **usual topology** *in $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$, the set $\mathcal{T}$ defined by:*

$$\mathcal{T} \stackrel{\triangle}{=} \{U : U \subseteq L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu), \forall f \in U, \exists \epsilon > 0, B(f, \epsilon) \subseteq U\}$$

Note that if $(f, g) \to \|f - g\|_p$ was a metric, the usual topology in $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$, would be nothing but the *metric* topology.

EXERCISE 5. Let $p \in [1, +\infty]$. Suppose there exists $N \in \mathcal{F}$ with $\mu(N) = 0$ and $N \neq \emptyset$. Put $f = 1_N$ and $g = 0$

1. Show that $f, g \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $f \neq g$.

2. Show that any open set containing f also contains g.
3. Show that $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ are not Hausdorff.

Exercise 6. Show that the usual topology on $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ is induced by the usual topology on $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, where $p \in [1, +\infty]$.

Definition 76 *Let $(E, \mathcal{T})$ be a topological space. A sequence $(x_n)_{n\geq 1}$ in E is said to* **converge** *to $x \in E$, and we write $x_n \stackrel{\mathcal{T}}{\to} x$, if and only if, for all $U \in \mathcal{T}$ such that $x \in U$, there exists $n_0 \geq 1$ such that:*

$$n \geq n_0 \;\Rightarrow\; x_n \in U$$

When $E = L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ or $E = L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, we write $x_n \stackrel{L^p}{\to} x$.

Exercise 7. Let $(E, \mathcal{T})$ be a topological space and $E' \subseteq E$. Let $\mathcal{T}' = \mathcal{T}_{|E'}$ be the induced topology on E'. Show that if $(x_n)_{n\geq 1}$ is a sequence in E' and $x \in E'$, then $x_n \stackrel{\mathcal{T}}{\to} x$ is equivalent to $x_n \stackrel{\mathcal{T}'}{\to} x$.

Exercise 8. Let $f, g, (f_n)_{n\geq 1}$ be in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $p \in [1, +\infty]$.

1. Recall what the notation $f_n \to f$ means.

2. Show that $f_n \stackrel{L^p}{\to} f$ is equivalent to $\|f_n - f\|_p \to 0$.

3. Show that if $f_n \stackrel{L^p}{\to} f$ and $f_n \stackrel{L^p}{\to} g$ then $f = g$ μ-a.s.

Exercise 9. Let $p \in [1, +\infty]$. Suppose there exists some $N \in \mathcal{F}$ such that $\mu(N) = 0$ and $N \neq \emptyset$. Find a sequence $(f_n)_{n\geq 1}$ in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and f, g in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, $f \neq g$ such that $f_n \stackrel{L^p}{\to} f$ and $f_n \stackrel{L^p}{\to} g$.

Definition 77 *Let $(f_n)_{n\geq 1}$ be a sequence in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, where $(\Omega, \mathcal{F}, \mu)$ is a measure space and $p \in [1, +\infty]$. We say that $(f_n)_{n\geq 1}$ is a* **Cauchy sequence**, *if and only if, for all $\epsilon > 0$, there exists $n_0 \geq 1$ such that:*

$$n, m \geq n_0 \ \Rightarrow \ \|f_n - f_m\|_p \leq \epsilon$$

Exercise 10. Let $f, (f_n)_{n \geq 1}$ be in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $p \in [1, +\infty]$. Show that if $f_n \stackrel{L^p}{\to} f$, then $(f_n)_{n \geq 1}$ is a Cauchy sequence.

Exercise 11. Let $(f_n)_{n \geq 1}$ be Cauchy in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, $p \in [1, +\infty]$.

1. Show the existence of $n_1 \geq 1$ such that:

$$n \geq n_1 \ \Rightarrow \ \|f_n - f_{n_1}\|_p \leq \frac{1}{2}$$

2. Suppose we have found $n_1 < n_2 < \ldots < n_k,\ k \geq 1$, such that:

$$\forall j \in \{1, \ldots, k\}\ ,\ n \geq n_j \ \Rightarrow \ \|f_n - f_{n_j}\|_p \leq \frac{1}{2^j}$$

   Show the existence of n_{k+1}, $n_k < n_{k+1}$ such that:

$$n \geq n_{k+1} \ \Rightarrow \ \|f_n - f_{n_{k+1}}\|_p \leq \frac{1}{2^{k+1}}$$

3. Show that there exists a subsequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$ with:

$$\sum_{k=1}^{+\infty} \|f_{n_{k+1}} - f_{n_k}\|_p < +\infty$$

EXERCISE 12. Let $p \in [1, +\infty]$, and $(f_n)_{n\geq 1}$ be in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, with:

$$\sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p < +\infty$$

We define:

$$g \stackrel{\triangle}{=} \sum_{n=1}^{+\infty} |f_{n+1} - f_n|$$

1. Show that $g : (\Omega, \mathcal{F}) \to [0, +\infty]$ is non-negative and measurable.
2. If $p = +\infty$, show that $g \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_\infty$ μ-a.s.

3. If $p \in [1, +\infty[$, show that for all $N \geq 1$, we have:

$$\left\| \sum_{n=1}^{N} |f_{n+1} - f_n| \right\|_p \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p$$

4. If $p \in [1, +\infty[$, show that:

$$\left(\int_\Omega g^p d\mu \right)^{\frac{1}{p}} \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p$$

5. Show that for $p \in [1, +\infty]$, we have $g < +\infty$ μ-a.s.

6. Define $A = \{g < +\infty\}$. Show that for all $\omega \in A$, $(f_n(\omega))_{n \geq 1}$ is a Cauchy sequence in $\mathbf{C}$. We denote $z(\omega)$ its limit.

7. Define $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$, by:

$$f(\omega) \stackrel{\triangle}{=} \begin{cases} z(\omega) & \text{if} \quad \omega \in A \\ 0 & \text{if} \quad \omega \notin A \end{cases}$$

Show that f is measurable and $f_n \to f$ μ-a.s.

8. if $p = +\infty$, show that for all $n \geq 1$, $|f_n| \leq |f_1| + g$ and conclude that $f \in L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

9. If $p \in [1, +\infty[$, show the existence of $n_0 \geq 1$, such that:

$$n \geq n_0 \ \Rightarrow \ \int_\Omega |f_n - f_{n_0}|^p d\mu \leq 1$$

Deduce from Fatou's lemma that $f - f_{n_0} \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

10. Show that for $p \in [1, +\infty]$, $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

11. Suppose that $f_n \in L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, for all $n \geq 1$. Show the existence of $f \in L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, such that $f_n \to f$ μ-a.s.

EXERCISE 13. Let $p \in [1, +\infty]$, and $(f_n)_{n \geq 1}$ be in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, with:

$$\sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p < +\infty$$

1. Does there exist $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f_n \to f$ μ-a.s.
2. Suppose $p = +\infty$. Show that for all $n < m$, we have:

$$|f_{m+1} - f_n| \leq \sum_{k=n}^{m} \|f_{k+1} - f_k\|_\infty \ \mu\text{-a.s.}$$

3. Suppose $p = +\infty$. Show that for all $n \geq 1$, we have:

$$\|f - f_n\|_\infty \leq \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_\infty$$

4. Suppose $p \in [1, +\infty[$. Show that for all $n < m$, we have:

$$\left(\int_\Omega |f_{m+1} - f_n|^p d\mu\right)^{\frac{1}{p}} \leq \sum_{k=n}^{m} \|f_{k+1} - f_k\|_p$$

5. Suppose $p \in [1, +\infty[$. Show that for all $n \geq 1$, we have:

$$\|f - f_n\|_p \leq \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_p$$

6. Show that for $p \in [1, +\infty]$, we also have $f_n \stackrel{L^p}{\rightarrow} f$.

7. Suppose conversely that $g \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is such that $f_n \stackrel{L^p}{\rightarrow} g$. Show that $f = g$ μ-a.s.. Conclude that $f_n \rightarrow g$ μ-a.s..

Theorem 44 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $p \in [1, +\infty]$, and $(f_n)_{n\geq 1}$ be a sequence in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that:*

$$\sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p < +\infty$$

Then, there exists $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f_n \to f$ μ-a.s. Moreover, for all $g \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, the convergence $f_n \to g$ μ-a.s. and $f_n \stackrel{L^p}{\to} g$ are equivalent.

EXERCISE 14. Let $f, (f_n)_{n\geq 1}$ be in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f_n \stackrel{L^p}{\to} f$, where $p \in [1, +\infty]$.

1. Show that there exists a sub-sequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$, with:

$$\sum_{k=1}^{+\infty} \|f_{n_{k+1}} - f_{n_k}\|_p < +\infty$$

2. Show that there exists $g \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f_{n_k} \to g$ μ-a.s.

3. Show that $f_{n_k} \xrightarrow{L^p} g$ and $g = f$ μ-a.s.

4. Conclude with the following:

Theorem 45 *Let $(f_n)_{n\geq 1}$ be in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f_n \xrightarrow{L^p} f$, where $p \in [1, +\infty]$. Then, we can extract a subsequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$ such that $f_{n_k} \to f$ μ-a.s.*

Exercise 15. Prove the last theorem for $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

Exercise 16. Let $(f_n)_{n\geq 1}$ be Cauchy in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, $p \in [1, +\infty]$.

1. Show that there exists a subsequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$ and f belonging to $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, such that $f_{n_k} \xrightarrow{L^p} f$.

2. Using the fact that $(f_n)_{n\geq 1}$ is Cauchy, show that $f_n \stackrel{L^p}{\to} f$.

Theorem 46 *Let $p \in [1, +\infty]$. Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Then, there exists $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f_n \stackrel{L^p}{\to} f$.*

Exercise 17. Prove the last theorem for $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

Solutions to Exercises

Exercise 1.

1. Since $p, q \in \mathbf{R}^+$, we have $p < +\infty$ and $q < +\infty$. From the inequality $1/p \leq 1/p + 1/q = 1$, we obtain $p \geq 1$. If $p = 1$, then $1/q = 0$, contradicting $q < +\infty$. So $p > 1$, and similarly $q > 1$. We have proved that $1 < p < +\infty$ and $1 < q < +\infty$.

2. Let $\alpha \in]0, +\infty[$ and $\phi = \phi^\alpha$. We want to prove that ϕ is continuous. For all $a \in \mathbf{R}^+$, it is clear that $\lim_{x \to a} \phi(x) = \phi(a)$. So ϕ is continuous at $x = a$. Furthermore, $\lim_{x \to +\infty} \phi(x) = \phi(+\infty)$. So ϕ is also continuous at $+\infty$. For many of us, this is sufficient proof of the fact that ϕ is a continuous map. However, for those who want to apply definition (27), the following can be said: let V be open in $[0, +\infty]$. We want to show that $\phi^{-1}(V)$ is open in $[0, +\infty]$. Let $a \in \phi^{-1}(V)$. Then $\phi(a) \in V$. Since ϕ is continuous at $x = a$, there exists U_a open in $[0, +\infty]$, containing a, such that $\phi(U_a) \subseteq V$. So $a \in U_a \subseteq \phi^{-1}(V)$. It follows that

$\phi^{-1}(V)$ can be written as $\phi^{-1}(V) = \cup_{a\in\phi^{-1}(V)} U_a$, and $\phi^{-1}(V)$ is therefore open in $[0, +\infty]$. From definition (27), we conclude that $\phi : [0, +\infty] \to [0, +\infty]$ is a continuous map.

3. f^p can be viewed as $f^p = \phi^p \circ f$, where ϕ^p is defined as in 2. We proved that ϕ^p is a continuous map. It is therefore measurable with respect to the Borel σ-algebra $B([0, +\infty])$ on $[0, +\infty]$. It follows that $f^p : (\Omega, \mathcal{F}) \to [0, +\infty]$ is a measurable map, which is also non-negative. Hence, the integral $\int f^p d\mu$ is a well-defined element of $[0, +\infty]$, and $A = (\int f^p d\mu)^{1/p}$ is also well-defined, being understood that $(+\infty)^{1/p} = +\infty$. Similarly, $B = (\int f^q d\mu)^{1/q}$ is a well-defined element of $[0, +\infty]$. Finally, the map $fg : (\Omega, \mathcal{F}) \to [0, +\infty]$ is non-negative and measurable, and $C = \int fg d\mu$ is a well-defined element of $[0 + \infty]$.

4. Suppose $A = 0$. Then $\int f^p d\mu = 0$, and since f^p is non-negative, we see that $f^p = 0$ μ-a.s., and consequently $f = 0$ μ-a.s. So $fg = 0$ μ-a.s., and finally $C = \int fg d\mu = 0$. So $C \leq AB$. Similarly, $B = 0$ implies $C = 0$, and therefore $C \leq AB$.

5. Suppose $A = +\infty$. Then, either $B = 0$ or $B > 0$. If $B = 0$, then $C \leq AB$ is true from 4. If $B > 0$, then $AB = +\infty$, and consequently $C \leq AB$. In any case, we see that $C \leq AB$. Similarly, $B = +\infty$ implies $C \leq AB$.

6. Suppose $A, B \in]0, +\infty[$. Let $F = f/A$ and $G = g/B$. We have:

$$\int F^p d\mu = \int (f/A)^p d\mu = \frac{1}{A^p} \int f^p d\mu = 1$$

and similarly, $\int G^p d\mu = 1$.

7. Let $a, b \in]0, +\infty[$. The map $x \to -\ln(x)$ being convex on $]0, +\infty[$, since $1/p + 1/q = 1$, we have:

$$-\ln(\frac{1}{p}a^p + \frac{1}{q}b^q) \leq -\frac{1}{p}\ln(a^p) - \frac{1}{q}\ln(b^q) = -\ln(ab)$$

and consequently $\ln(ab) \leq \ln(a^p/p + b^q/q)$. The map $x \to e^x$

being non-decreasing, we conclude that:

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q \tag{1}$$

It is easy to check that inequality (1) is in fact true for all $a, b \in [0, +\infty]$.

8. For all $\omega \in \Omega$, $F(\omega)$ and $G(\omega)$ are elements of $[0, +\infty]$. From 7.:

$$F(\omega)G(\omega) \leq \frac{1}{p}F(\omega)^p + \frac{1}{q}G(\omega)^q$$

9. Integrating on both side of 8., we obtain:

$$\int FGd\mu \leq \frac{1}{p}\int F^p d\mu + \frac{1}{q}\int G^q d\mu = 1$$

where we have used the fact that $\int F^p d\mu = \int G^q d\mu = 1$. Since $\int FGd\mu = (\int fgd\mu)/AB = C/AB$, we conclude that $C \leq AB$.

Exercise 1

Exercise 2.

1. f^p, g^p and $(f+g)^p$ are all non-negative and measurable. All three integrals $\int f^p d\mu$, $\int g^p d\mu$ and $\int (f+g)^p d\mu$ are therefore well-defined. It follows that A, B and C are well-defined elements of $[0,+\infty]$.

2. Since $p>1$, the map $x \to x^p$ is convex on $]0,+\infty[$. In particular, for all $a,b \in]0,+\infty[$, we have $((a+b)/2)^p \leq (a^p+b^p)/2$. We conclude that:
$$(a+b)^p \leq 2^{p-1}(a^p+b^p) \tag{2}$$
In fact, it is easy to check that (2) holds for all $a,b \in [0,+\infty]$.

3. If $A=+\infty$ or $B=+\infty$, then $A+B=+\infty$, and $C \leq A+B$. If $C=0$, then clearly $C \leq A+B$.

4. Using 2., for all $\omega \in \Omega$, we have:
$$(f(\omega)+g(\omega))^p \leq 2^{p-1}(f(\omega)^p + g(\omega)^p)$$

Integrating on both side of the inequality, we obtain:

$$\int (f+g)^p d\mu \leq 2^{p-1}\left(\int f^p d\mu + \int g^p d\mu\right) \tag{3}$$

If $A < +\infty$ and $B < +\infty$, then both integrals $\int f^p d\mu$ and $\int g^p d\mu$ are finite, and we see from (3) that $\int (f+g)^p d\mu$ is itself finite. So $C < +\infty$.

5. Take $q = p/(p-1)$. Since $p \in]1, +\infty[$, q is a well-defined element of $\mathbf{R}^+$, and $1/p + 1/q = 1$.

6. Let $a, b \in [0, +\infty]$. If $a, b \in \mathbf{R}^+$, then:

$$(a+b)^p = (a+b).(a+b)^{p-1} \tag{4}$$

If $a = +\infty$ or $b = +\infty$, then $a + b = +\infty$ and both sides of (4) are equal to $+\infty$. So (4) is true for all $a, b \in [0, +\infty]$.

7. Using holder's inequality (41), since $q(p-1) = p$, we have:

$$\int f.(f+g)^{p-1} d\mu \leq \left(\int f^p d\mu\right)^{\frac{1}{p}} \left(\int (f+g)^{q(p-1)} d\mu\right)^{\frac{1}{q}} = AC^{\frac{p}{q}}$$

and:

$$\int g.(f+g)^{p-1} d\mu \leq \left(\int g^p d\mu\right)^{\frac{1}{p}} \left(\int (f+g)^{q(p-1)} d\mu\right)^{\frac{1}{q}} = BC^{\frac{p}{q}}$$

8. From 6., we have:

$$\int (f+g)^p d\mu = \int f.(f+g)^{p-1} d\mu + \int g.(f+g)^{p-1} d\mu$$

and using 7., we obtain:

$$\int (f+g)^p d\mu \leq C^{\frac{p}{q}}(A+B)$$

9. From 8., we have $C^p \leq C^{\frac{p}{q}}(A+B)$. Having assumed in 5. that $C \in]0,+\infty[$, we can divide both side of this inequality by $C^{\frac{p}{q}}$,

to obtain $C^{p-\frac{p}{q}} \leq A+B$. Since $p - p/q = 1$, we conclude that $C \leq A + B$.

10. If $p = 1$, then $C = A + B$ is equivalent to:

$$\int (f+g)d\mu = \int f d\mu + \int g d\mu$$

which is true by linearity. In particular, $C \leq A+B$. The purpose of this exercise is to prove minkowski's inequality (43).

Exercise 2

Exercise 3.

1. Let $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ be a map. Then, if f has values in $\mathbf{R}$, i.e. $f(\Omega) \subseteq \mathbf{R}$, then the measurability of f with respect to $(\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is equivalent to its measurability with respect to $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$. Hence:

$$L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) = \{f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)\ ,\ f(\Omega) \subseteq \mathbf{R}\}$$

   The equivalence of measurability with respect to $\mathcal{B}(\mathbf{C})$ and $\mathcal{B}(\mathbf{R})$ may be taken for granted by many. It is easily proved from the fact that $\mathcal{B}(\mathbf{R}) = \mathcal{B}(\mathbf{C})_{|\mathbf{R}}$, i.e. the Borel σ-algebra on $\mathbf{R}$ is the trace on $\mathbf{R}$, of the Borel σ-algebra on $\mathbf{C}$. This fact can be seen from the trace theorem (10), and the fact that the usual topology on $\mathbf{R}$ is induced on $\mathbf{R}$, by the usual topology on $\mathbf{C}$.

2. Let $f, g \in L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ and $\alpha \in \mathbf{R}$. The map $f + \alpha g$ is $\mathbf{R}$-valued and measurable. Moreover, we have $|f + \alpha g| \leq |f| + |\alpha|.|g|$. Since $p \geq 1$, (and in particular $p \geq 0$), the map $x \to x^p$ is non-decreasing on $\mathbf{R}^+$, so $|f + \alpha g|^p \leq (|f| + |\alpha|.|g|)^p$. Hence,

we see that $\int |f+\alpha g|^p d\mu \leq \int (|f|+|\alpha|.|g|)^p d\mu$. However, using minkowski's inequality (43), we have:

$$\left(\int (|f|+|\alpha|.|g|)^p d\mu\right)^{\frac{1}{p}} \leq \left(\int |f|^p d\mu\right)^{\frac{1}{p}} + |\alpha|.\left(\int |g|^p d\mu\right)^{\frac{1}{p}}$$

We conclude that $\int |f+\alpha g|^p d\mu < +\infty$. So $f+\alpha g \in L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, and we have proved that $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ is closed under **R**-linear combinations.

3. The fact that $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is closed under **C**-linear combinations, is proved identically to 2., replacing **R** by **C**.

4. Using $|f+g|^p \leq (|f|+|g|)^p$ and minkowski's inequality (43):

$$\left(\int (|f|+|g|)^p d\mu\right)^{\frac{1}{p}} \leq \left(\int |f|^p d\mu\right)^{\frac{1}{p}} + \left(\int |g|^p d\mu\right)^{\frac{1}{p}}$$

we see that $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

5. Suppose $\|f\|_p = 0$. Then $\int |f|^p d\mu = 0$. Since $|f|^p$ is non-negative, $|f|^p = 0$ μ-a.s., and consequently $f = 0$ μ-a.s. Conversely, if $f = 0$ μ-a.s., then $|f|^p = 0$ μ-a.s., so $\int |f|^p d\mu = 0$ and finally $\|f\|_p = 0$.

6. Let $\alpha \in \mathbf{C}$. We have:

$$\|\alpha f\|_p = \left(\int |\alpha f|^p\right)^{\frac{1}{p}} = |\alpha|.\left(\int |f|^p\right)^{\frac{1}{p}} = |\alpha|.\|f\|_p$$

7. $\|f-g\|_p = 0$ only implies $f = g$ μ-.a.s, and not necessarily $f = g$. So $(f,g) \to \|f-g\|_p$, may not be a metric on $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

Exercise 3

Exercise 4.

1. For all $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ with values in $\mathbf{R}$, the measurability of f with respect to $\mathcal{B}(\mathbf{C})$ is equivalent to its measurability with respect to $\mathcal{B}(\mathbf{R})$. Hence:

$$L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) = \{f \in L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)\ ,\ f(\Omega) \subseteq \mathbf{R}\}$$

2. Since $\|f\|_\infty < +\infty$, for all $n \geq 1$, we have $\|f\|_\infty < \|f\|_\infty + 1/n$. $\|f\|_\infty$ being the greatest lower bound of all μ-almost sure upper bounds of $|f|$, $\|f\|_\infty + 1/n$ cannot be such lower bound. There exists $M \in \mathbf{R}^+$, such that $|f| \leq M$ μ-a.s., and $M < \|f\|_\infty + 1/n$. In particular, $|f| < \|f\|_\infty + 1/n$ μ-a.s. Let A_n be the set defined by $A_n = \{\|f\|_\infty + 1/n \leq |f|\}$. Then $A_n \in \mathcal{F}$ and $\mu(A_n) = 0$. Moreover, $A_n \subseteq A_{n+1}$ and $\cup_{n=1}^{+\infty} A_n = \{\|f\|_\infty < |f|\}$. It follows that $A_n \uparrow \{\|f\|_\infty < |f|\}$, and from theorem (7), we see that:

$$\mu(\{\|f\|_\infty < |f|\}) = \lim_{n \to +\infty} \mu(A_n) = 0$$

We conclude that $|f| \leq \|f\|_\infty$ μ-a.s.

3. Since $|f+g| \leq |f| + |g|$, using 2., we have:

$$|f+g| \leq \|f\|_\infty + \|g\|_\infty \ \mu\text{-a.s.}$$

Hence, $\|f\|_\infty + \|g\|_\infty$ is a μ-almost sure upper bound of $|f+g|$. $\|f+g\|_\infty$ being a lower bound of all such upper bounds, we have $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

4. Let $f, g \in L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ and $\alpha \in \mathbf{R}$. Then $f + \alpha g$ is $\mathbf{R}$-valued and measurable. Furthermore, using 2., we have:

$$|f + \alpha g| \leq |f| + |\alpha|.|g| \leq \|f\|_\infty + |\alpha|.\|g\|_\infty \ \mu\text{-a.s.}$$

It follows that $\|f + \alpha g\|_\infty \leq \|f\|_\infty + |\alpha|.\|g\|_\infty < +\infty$. We conclude that $f + \alpha g \in L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, and we have proved that $L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbf{R}$-linear combinations.

5. The fact that $L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbf{C}$-linear combinations can be proved identically, replacing $\mathbf{R}$ by $\mathbf{C}$.

6. Suppose $\|f\|_\infty = 0$. Then $|f| \leq 0$ μ-a.s., and consequently $f = 0$ μ-a.s. Conversely, if $f = 0$ μ-a.s., then $|f| \leq 0$ μ-a.s., and 0 is

therefore a μ-almost sure upper bound of $|f|$. So $\|f\|_\infty \leq 0$. Since $\|f\|_\infty$ is an infimum of a subset of $\mathbf{R}^+$, it is either $+\infty$ (when such subset is empty), or lies in $\mathbf{R}^+$. So $\|f\|_\infty \geq 0$ and finally $\|f\|_\infty = 0$.

7. We have $|\alpha f| \leq |\alpha|.\|f\|_\infty$ μ-a.s., and hence $\|\alpha f\|_\infty \leq |\alpha|.\|f\|_\infty$. if $\alpha \neq 0$, we have:

$$\|f\|_\infty = \|\frac{1}{\alpha}.(\alpha f)\|_\infty \leq \frac{1}{|\alpha|}\|\alpha f\|_\infty$$

It follows that $\|\alpha f\|_\infty = |\alpha|.\|f\|_\infty$, (also true if $\alpha = 0$).

8. $\|f - g\|_\infty = 0$ implies $f = g$ μ-a.s., but not $f = g$. It follows that $(f, g) \rightarrow \|f - g\|_\infty$ may not be a metric on $L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

Exercise 4

Exercise 5.

1. Since $N \neq \emptyset$, $1_N \neq 0$, so $f \neq g$. Since $N \in \mathcal{F}$, the map $f = 1_N$ is measurable, and being $\mathbf{R}$-valued, it is also $\mathbf{C}$-valued. Moreover, since $\mu(N) = 0$, $\|f\|_p = 0 < +\infty$ (whether $p = +\infty$ or lies in $[1, +\infty[$), and we see that $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Since $g = 0$, it is $\mathbf{C}$-valued, measurable and $\|g\|_p = 0 < +\infty$, so $g \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

2. Let U be open in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, such that $f \in U$. By definition (75), there exists $\epsilon > 0$, such that $B(f, \epsilon) \subseteq U$. However, $\|f - g\|_p = \|f\|_p = 0$ ($p = +\infty$ or $p \in [1, +\infty[$). So in particular $\|f - g\|_p < \epsilon$. So $g \in B(f, \epsilon)$ and finally $g \in U$.

3. If $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ was Hausdorff, since $f \neq g$, there would exist U, V open sets in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f \in U$, $g \in V$ and $U \cap V = \emptyset$. However from 2., this is impossible, as g would always be an element of U as well as V. We conclude similarly that $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ is not Hausdorff.

Exercise 5

Exercise 6. Let $L^p_{\mathbf{R}}$ and $L^p_{\mathbf{C}}$ denote $L^p_{\mathbf{R}}(\Omega,\mathcal{F},\mu)$ and $L^p_{\mathbf{C}}(\Omega,\mathcal{F},\mu)$ respectively. Let $\mathcal{T}$ be the usual topology on $L^p_{\mathbf{C}}$ and $\mathcal{T}'$ be the usual topology on $L^p_{\mathbf{R}}$. We want to prove that $\mathcal{T}'=\mathcal{T}_{|L^p_{\mathbf{R}}}$, i.e. that $\mathcal{T}'$ is the topology on $L^p_{\mathbf{R}}$ induced by $\mathcal{T}$. Given $f\in L^p_{\mathbf{R}}$ and $\epsilon>0$, let $B(f,\epsilon)$ denote the open ball in $L^p_{\mathbf{C}}$ and $B'(f,\epsilon)$ denote the open ball the $L^p_{\mathbf{R}}$. Then $B'(f,\epsilon)=B(f,\epsilon)\cap L^p_{\mathbf{R}}$. It is a simple exercise to show that any open ball in $L^p_{\mathbf{R}}$ or $L^p_{\mathbf{C}}$, is in fact open with respect to their usual topology. Let $U'\in\mathcal{T}'$. For all $f\in U'$, there exists $\epsilon_f>0$ such that $f\in B'(f,\epsilon_f)\subseteq U'$. It follows that:

$$U'=\cup_{f\in U'}B'(f,\epsilon_f)=(\cup_{f\in U'}B(f,\epsilon_f))\cap L^p_{\mathbf{R}}$$

and we see that $U'\in\mathcal{T}_{|L^p_{\mathbf{R}}}$. So $\mathcal{T}'\subseteq\mathcal{T}_{|L^p_{\mathbf{R}}}$. Conversely, let $U'\in\mathcal{T}_{|L^p_{\mathbf{R}}}$. There exists $U\in\mathcal{T}$ such that $U'=U\cap L^p_{\mathbf{R}}$. Let $f\in U'$. Then $f\in U$. There exists $\epsilon>0$ such that $B(f,\epsilon)\subseteq U$. It follows that $B'(f,\epsilon)=B(f,\epsilon)\cap L^p_{\mathbf{R}}\subseteq U'$. So U' is open with respect to the usual topology in $L^p_{\mathbf{R}}$, i.e. $U'\in\mathcal{T}'$. We have proved that $\mathcal{T}_{|L^p_{\mathbf{R}}}\subseteq\mathcal{T}'$, and finally $\mathcal{T}'=\mathcal{T}_{|L^p_{\mathbf{R}}}$.

Exercise 6

Exercise 7. let $(E, \mathcal{T})$ be a topological space and $E' \subseteq E$. Let $\mathcal{T}' = \mathcal{T}_{|E'}$ be the induced topology on E'. We assume that $(x_n)_{n\geq 1}$ is a sequence in E', and that $x \in E'$. Suppose that $x_n \xrightarrow{\mathcal{T}} x$. Let $U' \in \mathcal{T}'$ be such that $x \in U'$. There exists $U \in \mathcal{T}$ such that $U' = U \cap E'$. Since $x \in U$ and $x_n \xrightarrow{\mathcal{T}} x$, there exists $n_0 \geq 1$ such that $x_n \in U$ for all $n \geq n_0$. But $x_n \in E'$ for all $n \geq 1$. So $x_n \in U \cap E' = U'$ for all $n \geq n_0$, and we see that $x_n \xrightarrow{\mathcal{T}'} x$. Conversely, suppose that $x_n \xrightarrow{\mathcal{T}'} x$. Let $U \in \mathcal{T}$ be such that $x \in U$. Then $U \cap E' \in \mathcal{T}'$ and $x \in U \cap E'$. There exists $n_0 \geq 1$, such that $x_n \in U \cap E'$ for all $n \geq n_0$. In particular, $x_n \in U$ for all $n \geq n_0$, and we see that $x_n \xrightarrow{\mathcal{T}} x$. We have proved that $x_n \xrightarrow{\mathcal{T}'} x$ and $x_n \xrightarrow{\mathcal{T}} x$ are equivalent.

Exercise 7

Exercise 8.

1. The notation $f_n \to f$ has been used throughout these tutorials, to refer to a *simple* convergence, i.e. $f_n(\omega) \to f(\omega)$ as $n \to +\infty$, for all $\omega \in \Omega$.

2. Suppose $f_n \stackrel{L^p}{\to} f$. Let $\epsilon > 0$. The open ball $B(f, \epsilon)$ being open with respect to the usual topology in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, there exists $n_0 \geq 1$, such that $f_n \in B(f, \epsilon)$ for all $n \geq n_0$, i.e.:

$$n \geq n_0 \;\Rightarrow\; \|f_n - f\|_p < \epsilon$$

So $\|f_n - f\|_p \to 0$. Conversely, suppose $\|f_n - f\|_p \to 0$. Let U be open in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, such that $f \in U$. From definition (75), there exists $\epsilon > 0$ such that $B(f, \epsilon) \subseteq U$. By assumption, there exists $n_0 \geq 0$, such that $\|f_n - f\|_p < \epsilon$ for all $n \geq n_0$. So $f_n \in B(f, \epsilon)$ for all $n \geq n_0$. Hence, we see that $f_n \in U$ for all $n \geq n_0$, and we have proved that $f_n \stackrel{L^p}{\to} f$. We conclude that $f_n \stackrel{L^p}{\to} f$ and $\|f_n - f\|_p \to 0$ are equivalent.

3. Suppose $f_n \stackrel{L^p}{\rightarrow} f$ and $f_n \stackrel{L^p}{\rightarrow} g$. From 2., we have $\|f_n - f\|_p \to 0$ and $\|f_n - g\|_p \to 0$. Using the triangle inequality (ex. (3) for $p \in [1, +\infty[$ and ex. (4) for $p = +\infty$):

$$\|f - g\|_p \leq \|f_n - f\|_p + \|f_n - g\|_p$$

for all $n \geq 1$. It follows that we have $\|f - g\|_p < \epsilon$ for all $\epsilon > 0$, and consequently $\|f - g\|_p = 0$. From ex. (3) and ex. (4) we conclude that $f = g$ μ-a.s.

Exercise 8

Exercise 9. Take $f_n = 1_N = f$ for all $n \geq 1$. Take $g = 0$. Then f_n, f and g are all elements of $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, and $f \neq g$. Moreover, for all $n \geq 1$, we have $\|f_n - f\|_p = \|f_n - g\|_p = 0$. So $f_n \stackrel{L^p}{\to} f$ and $f_n \stackrel{L^p}{\to} g$. The purpose of this exercise is to show that a limit in L^p may not be unique ($f \neq g$). However, it is unique, up to μ-almost sure equality (See exercise (8)).

Exercise 9

Exercise 10. Suppose $f_n \stackrel{L^p}{\to} f$. Let $\epsilon > 0$. There exists $n_0 \geq 1$, with:

$$n \geq n_0 \;\Rightarrow\; \|f_n - f\|_p \leq \epsilon/2$$

From the triangle inequality, for all $n, m \geq 1$:

$$\|f_n - f_m\|_p \leq \|f_n - f\|_p + \|f_m - f\|_p$$

It follows that we have:

$$n, m \geq n_0 \;\Rightarrow\; \|f_n - f_m\|_p \leq \epsilon$$

We conclude that $(f_n)_{n\geq 1}$ is a Cauchy sequence in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

Exercise 10

Exercise 11.

1. Take $\epsilon = 1/2$. There exists $n_1 \geq 1$, such that:

$$n, m \geq n_1 \ \Rightarrow\ \|f_n - f_m\|_p \leq \frac{1}{2}$$

   In particular, we have:

$$n \geq n_1 \ \Rightarrow\ \|f_n - f_{n_1}\|_p \leq \frac{1}{2}$$

2. Let $k \geq 1$. We have $n_1 < \ldots < n_k$, such that for all $j = 1, \ldots, k$:

$$n \geq n_j \ \Rightarrow\ \|f_n - f_{n_j}\|_p \leq \frac{1}{2^j}$$

   Take $\epsilon = 1/2^{k+1}$. There exists $n'_{k+1} \geq 1$, such that:

$$n, m \geq n'_{k+1} \ \Rightarrow\ \|f_n - f_m\|_p \leq \frac{1}{2^{k+1}}$$

Take $n_{k+1} = \max(n_k + 1, n'_{k+1})$. Then $n_k < n_{k+1}$, and:

$$n \geq n_{k+1} \;\Rightarrow\; \|f_n - f_{n_{k+1}}\|_p \leq \frac{1}{2^{k+1}}$$

3. By induction from 2., we can construct a strictly increasing sequence of integers $(n_k)_{k\geq 1}$, such that for all $k \geq 1$:

$$n \geq n_k \;\Rightarrow\; \|f_n - f_{n_k}\|_p \leq \frac{1}{2^k}$$

In particular, $\|f_{n_{k+1}} - f_{n_k}\|_p \leq 1/2^k$ for all $k \geq 1$. It follows that we have found a subsequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$, such that:

$$\sum_{k=1}^{+\infty} \|f_{n_{k+1}} - f_{n_k}\|_p < +\infty$$

Exercise 11

Exercise 12.

1. Each finite sum $g_N = \sum_{n=1}^{N} |f_{n+1} - f_n|$ is well-defined and measurable. It follows that $g = \sup_{N \geq 1} g_N$ is itself measurable. It is obviously non-negative.

2. Suppose $p = +\infty$. From exercise (4), for all $n \geq 1$, we have:

$$|f_{n+1} - f_n| \leq \|f_{n+1} - f_n\|_\infty \ , \ \mu\text{-a.s.}$$

The set $N_n = \{|f_{n+1} - f_n| > \|f_{n+1} - f_n\|_\infty\}$ which lies in $\mathcal{F}$, is such that $\mu(N_n) = 0$. It follows that if $N = \cup_{n \geq 1} N_n$, then $\mu(N) = 0$. However, for all $\omega \in N^c$, we have:

$$g(\omega) = \sum_{n=1}^{+\infty} |f_{n+1}(\omega) - f_n(\omega)| \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_\infty$$

We conclude that $g \leq \sum_{n=1}^{\infty} \|f_{n+1} - f_n\|_\infty$ μ-a.s.

3. Let $p \in [1, +\infty[$ and $N \geq 1$. By the triangle inequality (ex. (3)):

$$\left\| \sum_{n=1}^{N} |f_{n+1} - f_n| \right\|_p \leq \sum_{n=1}^{N} \|f_{n+1} - f_n\|_p \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p$$

4. Let $p \in [1, +\infty[$. Given $N \geq 1$, let $g_N = \sum_{n=1}^{N} |f_{n+1} - f_n|$. Then $g_N \to g$ as $N \to +\infty$. The map $x \to x^p$ being continuous on $[0, +\infty]$, we have $g_N^p \to g^p$, and in particular $g^p = \liminf g_N^p$ as $N \to +\infty$. Using Fatou's lemma (20), we see that:

$$\int g^p d\mu \leq \liminf_{N \geq 1} \int g_N^p d\mu \tag{5}$$

However, from 3., we have $\|g_N\|_p \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p$, for all $N \geq 1$. Since $p \geq 0$, the map $x \to x^p$ is non-decreasing on $[0, +\infty]$, and therefore:

$$\int g_N^p d\mu \leq \left(\sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p \right)^p \tag{6}$$

From inequalities (5) and (6), we conclude that:

$$\int g^p d\mu \leq \left(\sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p \right)^p$$

and finally:

$$\left(\int g^p d\mu \right)^{\frac{1}{p}} \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p$$

5. Let $p \in [1, +\infty]$. If $p = +\infty$, from 2. we have:

$$g \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p \ , \ \mu\text{-a.s.} \tag{7}$$

By assumption, the series in (7) is finite. So $g < +\infty$ μ-a.s. If $p \in [1, +\infty[$, from 4. we have:

$$\left(\int g^p d\mu \right)^{\frac{1}{p}} \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p$$

So $\int g^p d\mu < +\infty$. Since $(+\infty)\mu(\{g^p = +\infty\}) \leq \int g^p d\mu$, we see that $\mu(\{g^p = +\infty\}) = 0$ and finally $g < +\infty$ μ-a.s.

6. Let $A = \{g < +\infty\}$. Let $\omega \in A$. Then $g(\omega) < +\infty$. The series $\sum_{n=1}^{+\infty} |f_{n+1}(\omega) - f_n(\omega)|$ is therefore finite. Let $\epsilon > 0$. There exists $n_0 \geq 1$, such that:

$$n \geq n_0 \;\Rightarrow\; \sum_{k=n}^{+\infty} |f_{k+1}(\omega) - f_k(\omega)| \leq \epsilon$$

Given $m > n \geq n_0$, we have:

$$|f_m(\omega) - f_n(\omega)| \leq \sum_{k=n}^{m-1} |f_{k+1}(\omega) - f_k(\omega)| \leq \epsilon$$

We conclude that the sequence $(f_n(\omega))_{n\geq 1}$ is Cauchy in **C**. It therefore has a limit[1], denoted $z(\omega)$.

[1] The completeness of **C** is proved in the next Tutorial.

7. From 6., $f_n(\omega) \to z(\omega) = f(\omega)$ for all $\omega \in A$. Since by definition, $f(\omega) = 0$ for all $\omega \in A^c$, we see that $f_n(\omega)1_A(\omega) \to f(\omega)$ for all $\omega \in \Omega$. Hence, we have $f_n 1_A \to f$, and since $f_n 1_A$ is measurable for all $n \geq 1$, we see from theorem (17) that $f = \lim f_n 1_A$ is itself measurable. Since $\mu(A^c) = 0$ and $f_n(\omega) \to f(\omega)$ on A, we have $f_n \to f$ μ-a.s.

8. Suppose $p = +\infty$. For all $n \geq 1$, we have:

$$|f_n - f_1| \leq \sum_{k=1}^{n-1} |f_{k+1} - f_k| \leq g$$

So $|f_n| \leq |f_1| + g$. Taking the limit as $n \to +\infty$, we obtain $|f| \leq |f_1| + g$ μ-a.s. Let $M = \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_\infty$. Then by assumption, $M < +\infty$ and from 2. we have $g \leq M$ μ-a.s. Moreover, since $f_1 \in L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, using exercise (4), we have $|f_1| \leq \|f_1\|_\infty$ μ-a.s. with $\|f_1\|_\infty < +\infty$. Hence, we see that

$|f| \leq \|f_1\|_\infty + M$ μ-a.s., and consequently:

$$\|f\|_\infty \leq \|f_1\|_\infty + \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_\infty < +\infty$$

f is therefore **C**-valued, measurable and with $\|f\|_\infty < +\infty$. We have proved that $f \in L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

9. Let $p \in [1, +\infty[$. The series $\sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p$ being finite, there exists $n_0 \geq 1$, such that:

$$n \geq n_0 \ \Rightarrow \ \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_p \leq 1$$

Let $n \geq n_0$. By the triangle inequality:

$$\|f_n - f_{n_0}\|_p \leq \sum_{k=n_0}^{n-1} \|f_{k+1} - f_k\|_p \leq 1$$

Hence, we see that:

$$n \geq n_0 \ \Rightarrow \ \int |f_n - f_{n_0}|^p d\mu \leq 1^p = 1 \tag{8}$$

From 6., $f_n(\omega) \to f(\omega)$ as $n \to +\infty$, for all $\omega \in A$, where $\mu(A^c) = 0$. In particular:

$$1_A|f - f_{n_0}|^p = \liminf_{n\geq 1} 1_A|f_n - f_{n_0}|^p$$

Using inequality (8) and Fatou's lemma (20), we obtain:[2]

$$\int |f - f_{n_0}|^p d\mu \leq \liminf_{n\geq 1} \int |f_n - f_{n_0}|^p d\mu \leq 1$$

In particular, $\int |f - f_{n_0}|^p d\mu < +\infty$. Since $f - f_{n_0}$ is **C**-valued and measurable, we conclude that $f - f_{n_0} \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

10. Let $p \in [1, +\infty]$. If $p = +\infty$, then $f \in L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ was proved in 8. If $p \in [1, +\infty[$, we saw in 9. that $f - f_{n_0} \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ for

[2] Note that $n \geq n_0 \ \Rightarrow \ u_n \leq 1$ is enough to ensure $\liminf_{n\geq 1} u_n \leq 1$.

some $n_0 \geq 1$. Since f_{n_0} is itself an element of $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we conclude from exercise (3) that $f = (f - f_{n_0}) + f_{n_0}$ is also an element of $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

11. The purpose of this exercise is to prove that given a sequence $(f_n)_{n\geq 1}$ in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $\sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p < +\infty$, there exists $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, such that $f_n \to f$ μ-a.s. We now assume that all f_n's are in fact $\mathbf{R}$-valued, i.e. $f_n \in L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. There exists $f^* \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f_n \to f^*$ μ-a.s. However, $f^*(\omega)$ may not be $\mathbf{R}$-valued for all $\omega \in \Omega$. Yet, if $N \in \mathcal{F}$ is such that $\mu(N) = 0$ and $f_n(\omega) \to f^*(\omega)$ for all $\omega \in N^c$, then f^* is $\mathbf{R}$-valued on N^c (as a limit of an $\mathbf{R}$-valued sequence). If we define $f = f^* 1_{N^c}$, then f is $\mathbf{R}$-valued and measurable, with $\|f\|_p = \|f^*\|_p < +\infty$. So $f \in L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ and furthermore since $f = f^*$ μ-a.s., $f_n \to f$ μ-a.s.

Exercise 12

Exercise 13.

1. Yes, there does exist $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f_n \to f$ μ-a.s. This was precisely the object of the previous exercise.

2. Suppose $p = +\infty$, and let $n < m$. From exercise (4), we have $|f_{m+1} - f_n| \leq \|f_{m+1} - f_n\|_\infty$ μ-a.s. Furthermore, from the triangle inequality, $\|f_{m+1} - f_n\|_\infty \leq \sum_{k=n}^m \|f_{k+1} - f_k\|_\infty$. It follows that:

$$|f_{m+1} - f_n| \leq \sum_{k=n}^{m} \|f_{k+1} - f_k\|_\infty \ , \ \mu\text{-a.s.} \tag{9}$$

3. Suppose $p = +\infty$ and let $n \geq 1$. For all $m > n$, let $N_m \in \mathcal{F}$ be such that $\mu(N_m) = 0$, and inequality (9) holds for all $\omega \in N_m^c$. Furthermore, since $f_{m+1} \to f$ μ-a.s., let $M \in \mathcal{F}$ be such that $\mu(M) = 0$, and $f_{m+1}(\omega) \to f(\omega)$ for all $\omega \in M^c$. Then, if $N = M \cup (\cup_{m>n} N_m)$, we have $N \in \mathcal{F}$, $\mu(N) = 0$ and for all

$\omega \in N^c$, $f_{m+1}(\omega) \to f(\omega)$, together with, for all $m > n$:

$$|f_{m+1}(\omega) - f_n(\omega)| \leq \sum_{k=n}^{m} \|f_{k+1} - f_k\|_\infty$$

Taking the limit as $m \to +\infty$, we obtain:

$$|f(\omega) - f_n(\omega)| \leq \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_\infty$$

This being true for all $\omega \in N^c$, we have proved that:

$$|f - f_n| \leq \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_\infty \ , \ \mu\text{-a.s.}$$

From definition (74), we conclude that:

$$\|f - f_n\|_\infty \leq \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_\infty$$

4. Let $p \in [1, +\infty[$ and $n < m$. From exercise (3), we have:

$$\left(\int |f_{m+1} - f_n|^p d\mu \right)^{\frac{1}{p}} = \|f_{m+1} - f_n\|_p \leq \sum_{k=n}^{m} \|f_{k+1} - f_k\|_p$$

5. Let $p \in [1, +\infty[$ and $n \geq 1$. Let $N \in \mathcal{F}$ be such that $\mu(N) = 0$, and $f_{m+1}(\omega) \to f(\omega)$ for all $\omega \in N^c$. Then, we have:

$$|f - f_n|^p 1_{N^c} = \liminf_{m>n} |f_{m+1} - f_n|^p 1_{N^c}$$

Using Fatou's lemma (20), we obtain:

$$\int |f - f_n|^p d\mu \leq \liminf_{m>n} \int |f_{m+1} - f_n|^p d\mu$$

Hence, from 4. we see that:

$$\int |f - f_n|^p d\mu \leq \left(\sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_p \right)^p$$

and consequently:

$$\|f - f_n\|_p \leq \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_p$$

6. Let $p \in [1, +\infty]$. whether $p = +\infty$ or $p \in [1, +\infty[$, from 3. and 5., for all $n \geq 1$, we have $\|f - f_n\|_p \leq \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_p$. Since by assumption, the series $\sum_{k=1}^{+\infty} \|f_{k+1} - f_k\|_p$ is finite, we conclude that $\|f - f_n\|_p \to 0$, as $n \to +\infty$. It follows that not only $f_n \to f$ μ-a.s., but also $f_n \stackrel{L^p}{\to} f$.

7. Suppose $g \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is such that $f_n \stackrel{L^p}{\to} g$. Then $f_n \stackrel{L^p}{\to} f$ together with $f_n \stackrel{L^p}{\to} g$. From ex. (8), $f = g$ μ-a.s. Furthermore, since $f_n \to f$ μ-a.s., we see that $f_n \to g$ μ-a.s. The purpose of this exercise (and the previous) is to prove theorem (44).

Exercise 13

Exercise 14.

1. Since $f_n \stackrel{L^p}{\to} f$, from exercise (10), $(f_n)_{n\geq 1}$ is a Cauchy sequence in $L^p_{\mathbf{C}}(\Omega,\mathcal{F},\mu)$. Using exercise (11), there exists a sub-sequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$, such that $\sum_{k=1}^{+\infty}\|f_{n_{k+1}}-f_{n_k}\|_p<+\infty$.

2. Applying theorem (44) to the sequence $(f_{n_k})_{k\geq 1}$, there exists $g\in L^p_{\mathbf{C}}(\Omega,\mathcal{F},\mu)$, such that $f_{n_k}\to g$ μ-a.s.

3. Also from theorem (44), the convergence $f_{n_k}\to g$ μ-a.s. and $f_{n_k}\stackrel{L^p}{\to} g$ are equivalent. Hence, we also have $f_{n_k}\stackrel{L^p}{\to} g$. However, since by assumption $f_n\stackrel{L^p}{\to} f$, we see that $f_{n_k}\stackrel{L^p}{\to} f$, and consequently from exercise (8), $f=g$ μ-a.s.

4. From 2., $f_{n_k}\to g$ μ-a.s., and from 3., $f=g$ μ-a.s. It follows that $f_{n_k}\to f$ μ-a.s. Given a sequence $(f_n)_{n\geq 1}$ and f in $L^p_{\mathbf{C}}(\Omega,\mathcal{F},\mu)$, such that $f_n\stackrel{L^p}{\to} f$, we have been able to extract a sub-sequence $(f_{n_k})_{k\geq 1}$ such that $f_{n_k}\to f$ μ-a.s. This proves theorem (45).

Exercise 14

Exercise 15. Suppose $(f_n)_{n\geq 1}$ is a sequence in $L^p_{\mathbf{R}}(\Omega,\mathcal{F},\mu)$, and $f\in L^p_{\mathbf{R}}(\Omega,\mathcal{F},\mu)$ such that $f_n\stackrel{L^p}{\to} f$. Then in particular, all f_n's and f are elements of $L^p_{\mathbf{C}}(\Omega,\mathcal{F},\mu)$ with $\|f-f_n\|_p\to 0$ as $n\to+\infty$. From theorem (45), we can extract a sub-sequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$, such that $f_{n_k}\to f$ μ-a.s. This proves theorem (45), where $L^p_{\mathbf{C}}(\Omega,\mathcal{F},\mu)$ is replaced by $L^p_{\mathbf{R}}(\Omega,\mathcal{F},\mu)$. Anyone who feels there was very little to prove in this exercise, could make a very good point.

Exercise 15

Exercise 16.

1. Since $(f_n)_{n\geq 1}$ is Cauchy in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, from exercise (11), we can extract a sub-sequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$, such that:

$$\sum_{k=1}^{+\infty} \|f_{n_{k+1}} - f_{n_k}\|_p < +\infty$$

   From theorem (44), there exists $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, such that $f_{n_k} \to f$ μ-a.s., as well as $f_{n_k} \stackrel{L^p}{\to} f$.

2. Let $\epsilon > 0$. $(f_n)_{n\geq 1}$ being Cauchy, there exists $n_0 \geq 1$, such that:

$$n, m \geq n_0 \;\Rightarrow\; \|f_m - f_n\|_p \leq \frac{\epsilon}{2}$$

   Furthermore, since $f_{n_k} \stackrel{L^p}{\to} f$, there exists $k_0 \geq 1$, such that:

$$k \geq k_0 \;\Rightarrow\; \|f - f_{n_k}\|_p \leq \frac{\epsilon}{2}$$

However, $n_k \uparrow +\infty$ as $k \rightarrow +\infty$. There exists $k'_0 \geq 1$, such that $k \geq k'_0 \ \Rightarrow \ n_k \geq n_0$. Choose an arbitrary $k \geq \max(k_0, k'_0)$. Then $\|f - f_{n_k}\|_p \leq \epsilon/2$ together with $n_k \geq n_0$. Hence, for all $n \geq n_0$, we have:

$$\|f - f_n\|_p \leq \|f - f_{n_k}\|_p + \|f_{n_k} - f_n\|_p \leq \epsilon$$

We have found $n_0 \geq 1$ such that:

$$n \geq n_0 \ \Rightarrow \ \|f - f_n\|_p \leq \epsilon$$

This shows that $f_n \stackrel{L^p}{\rightarrow} f$. The purpose of this exercise, is to prove theorem (46). It is customary to say in light of this theorem, that $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is *complete*, even though as defined in these tutorials, $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is not strictly speaking a metric space.

Exercise 16

Exercise 17. Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. Then in particular, it is a Cauchy sequence in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. From theorem (46), there exists $f^* \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f_n \stackrel{L^p}{\rightarrow} f^*$. Furthermore, from theorem (45), there exists a sub-sequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$, such that $f_{n_k} \rightarrow f^*$ μ-a.s. It follows that f^* is in fact $\mathbf{R}$-valued μ-almost surely. There exists $N \in \mathcal{F}$, $\mu(N) = 0$, such that $f^*(\omega) \in \mathbf{R}$ for all $\omega \in N^c$. Take $f = f^* 1_{N^c}$. Then f is $\mathbf{R}$-valued, measurable and $\|f\|_p = \|f^*\|_p < +\infty$. So $f \in L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. Furthermore, $\|f - f_n\|_p = \|f^* - f_n\|_p \rightarrow 0$, which shows that $f_n \stackrel{L^p}{\rightarrow} f$. This proves theorem (46), where $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is replaced by $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

Exercise 17