3. Stieltjes-Lebesgue Measure

Definition 12 *Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ and $\mu : \mathcal{A} \to [0, +\infty]$ be a map. We say that μ is* **finitely additive** *if and only if, given $n \geq 1$:*

$$A \in \mathcal{A}, A_i \in \mathcal{A}, A = \biguplus_{i=1}^{n} A_i \ \Rightarrow \ \mu(A) = \sum_{i=1}^{n} \mu(A_i)$$

We say that μ is **finitely sub-additive** *if and only if, given $n \geq 1$:*

$$A \in \mathcal{A}, A_i \in \mathcal{A}, A \subseteq \bigcup_{i=1}^{n} A_i \ \Rightarrow \ \mu(A) \leq \sum_{i=1}^{n} \mu(A_i)$$

Exercise 1. Let $\mathcal{S} \stackrel{\triangle}{=} \{]a,b] \ , \ a,b \in \mathbf{R}\}$ be the set of all intervals $]a,b]$, defined as $]a,b] = \{x \in \mathbf{R}, a < x \leq b\}$.

1. Show that $]a,b]\cap]c,d] =]a \vee c, b \wedge d]$
2. Show that $]a,b]\setminus]c,d] =]a, b \wedge c]\cup]a \vee d, b]$

3. Show that $c \leq d \ \Rightarrow \ b \wedge c \leq a \vee d$.

4. Show that $\mathcal{S}$ is a semi-ring on $\mathbf{R}$.

Exercise 2. Suppose $\mathcal{S}$ is a semi-ring in Ω and $\mu : \mathcal{S} \to [0, +\infty]$ is finitely additive. Show that μ can be extended to a finitely additive map $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$, with $\bar{\mu}_{|\mathcal{S}} = \mu$.

Exercise 3. Everything being as before, Let $A \in \mathcal{R}(\mathcal{S})$, $A_i \in \mathcal{R}(\mathcal{S})$, $A \subseteq \cup_{i=1}^n A_i$ where $n \geq 1$. Define $B_1 = A_1 \cap A$ and for $i = 1, \ldots, n-1$:

$$B_{i+1} \stackrel{\triangle}{=} (A_{i+1} \cap A) \setminus ((A_1 \cap A) \cup \ldots \cup (A_i \cap A))$$

1. Show that $B_1, \ldots, B_n$ are pairwise disjoint elements of $\mathcal{R}(\mathcal{S})$ such that $A = \uplus_{i=1}^n B_i$.

2. Show that for all $i = 1, \ldots, n$, we have $\bar{\mu}(B_i) \leq \bar{\mu}(A_i)$.

3. Show that $\bar{\mu}$ is finitely sub-additive.

4. Show that μ is finitely sub-additive.

Exercise 4. Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. Let $\mathcal{S}$ be the semi-ring on $\mathbf{R}$, $\mathcal{S} = \{]a,b]\ ,\ a,b \in \mathbf{R}\}$. Define the map $\mu : \mathcal{S} \to [0,+\infty]$ by $\mu(\emptyset) = 0$, and:

$$\forall a \leq b\ ,\ \mu(]a,b]) \stackrel{\triangle}{=} F(b) - F(a) \tag{1}$$

Let $a < b$ and $a_i < b_i$ for $i = 1, \ldots, n$ and $n \geq 1$, with :

$$]a,b] = \biguplus_{i=1}^{n}]a_i, b_i]$$

1. Show that there is $i_1 \in \{1, \ldots, n\}$ such that $a_{i_1} = a$.

2. Show that $]b_{i_1}, b] = \uplus_{i \in \{1,\ldots,n\} \setminus \{i_1\}}]a_i, b_i]$

3. Show the existence of a permutation $(i_1, \ldots, i_n)$ of $\{1, \ldots, n\}$ such that $a = a_{i_1} < b_{i_1} = a_{i_2} < \ldots < b_{i_n} = b$.

4. Show that μ is finitely additive and finitely sub-additive.

EXERCISE 5. μ being defined as before, suppose $a < b$ and $a_n < b_n$ for $n \geq 1$ with:

$$]a, b] = \biguplus_{n=1}^{+\infty}]a_n, b_n]$$

Given $N \geq 1$, let $(i_1, \ldots, i_N)$ be a permutation of $\{1, \ldots, N\}$ with:

$$a \leq a_{i_1} < b_{i_1} \leq a_{i_2} < \ldots < b_{i_N} \leq b$$

1. Show that $\sum_{k=1}^{N} F(b_{i_k}) - F(a_{i_k}) \leq F(b) - F(a)$.

2. Show that $\sum_{n=1}^{+\infty} \mu(]a_n, b_n]) \leq \mu(]a, b])$

3. Given $\epsilon > 0$, show that there is $\eta \in]0, b-a[$ such that:

$$0 \leq F(a + \eta) - F(a) \leq \epsilon$$

4. For $n \geq 1$, show that there is $\eta_n > 0$ such that:
$$0 \leq F(b_n + \eta_n) - F(b_n) \leq \frac{\epsilon}{2^n}$$

5. Show that $[a + \eta, b] \subseteq \cup_{n=1}^{+\infty}]a_n, b_n + \eta_n[$.

6. Explain why there exist $p \geq 1$ and integers $n_1, \ldots, n_p$ such that:
$$]a + \eta, b] \subseteq \cup_{k=1}^{p}]a_{n_k}, b_{n_k} + \eta_{n_k}]$$

7. Show that $F(b) - F(a) \leq 2\epsilon + \sum_{n=1}^{+\infty} F(b_n) - F(a_n)$

8. Show that $\mu : \mathcal{S} \to [0, +\infty]$ is a measure.

Definition 13 *A* **topology** *on* Ω *is a subset* $\mathcal{T}$ *of the power set* $\mathcal{P}(\Omega)$*, with the following properties:*

$$\begin{array}{ll} (i) & \Omega, \emptyset \in \mathcal{T} \\ (ii) & A, B \in \mathcal{T} \ \Rightarrow \ A \cap B \in \mathcal{T} \\ (iii) & A_i \in \mathcal{T}, \forall i \in I \ \Rightarrow \ \bigcup_{i \in I} A_i \in \mathcal{T} \end{array}$$

Property (iii) of definition (13) can be translated as: *for any family* $(A_i)_{i \in I}$ *of elements of* $\mathcal{T}$*, the union* $\cup_{i \in I} A_i$ *is still an element of* $\mathcal{T}$*.* Hence, a topology on Ω, is a set of subsets of Ω containing Ω and the empty set, which is closed under finite intersection and arbitrary union.

Definition 14 *A* **topological space** *is an ordered pair* $(\Omega, \mathcal{T})$*, where* Ω *is a set and* $\mathcal{T}$ *is a topology on* Ω*.*

Definition 15 *Let $(\Omega, \mathcal{T})$ be a topological space. We say that $A \subseteq \Omega$ is an* **open set** *in Ω, if and only if it is an element of the topology $\mathcal{T}$. We say that $A \subseteq \Omega$ is a* **closed set** *in Ω, if and only if its complement A^c is an open set in Ω.*

Definition 16 *Let $(\Omega, \mathcal{T})$ be a topological space. We define the* **Borel σ-algebra** *on Ω, denoted $\mathcal{B}(\Omega)$, as the σ-algebra on Ω, generated by the topology $\mathcal{T}$. In other words, $\mathcal{B}(\Omega) = \sigma(\mathcal{T})$*

Definition 17 *We define the* **usual topology** *on $\mathbf{R}$, denoted $\mathcal{T}_{\mathbf{R}}$, as the set of all $U \subseteq \mathbf{R}$ such that:*

$$\forall x \in U \ , \ \exists \epsilon > 0 \ , \ ]x - \epsilon, x + \epsilon[\subseteq U$$

Exercise 6. Show that $\mathcal{T}_{\mathbf{R}}$ is indeed a topology on $\mathbf{R}$.

Exercise 7. Consider the semi-ring $\mathcal{S} \stackrel{\triangle}{=} \{]a, b] \ , \ a, b \in \mathbf{R}\}$. Let $\mathcal{T}_{\mathbf{R}}$ be the usual topology on $\mathbf{R}$, and $\mathcal{B}(\mathbf{R})$ be the Borel σ-algebra on $\mathbf{R}$.

1. Let $a \leq b$. Show that $]a, b] = \cap_{n=1}^{+\infty}]a, b + 1/n[$.

2. Show that $\sigma(\mathcal{S}) \subseteq \mathcal{B}(\mathbf{R})$.

3. Let U be an open subset of $\mathbf{R}$. Show that for all $x \in U$, there exist $a_x, b_x \in \mathbf{Q}$ such that $x \in]a_x, b_x] \subseteq U$.

4. Show that $U = \cup_{x \in U}]a_x, b_x]$.

5. Show that the set $I \stackrel{\triangle}{=} \{]a_x, b_x]\ ,\ x \in U\}$ is countable.

6. Show that U can be written $U = \cup_{i \in I} A_i$ with $A_i \in \mathcal{S}$.

7. Show that $\sigma(\mathcal{S}) = \mathcal{B}(\mathbf{R})$.

Theorem 6 *Let $\mathcal{S}$ be the semi-ring $\mathcal{S} = \{]a, b]\ , a, b \in \mathbf{R}\}$. Then, the Borel σ-algebra $\mathcal{B}(\mathbf{R})$ on $\mathbf{R}$, is generated by $\mathcal{S}$, i.e. $\mathcal{B}(\mathbf{R}) = \sigma(\mathcal{S})$.*

Definition 18 *A* **measurable space** *is an ordered pair $(\Omega, \mathcal{F})$ where Ω is a set and $\mathcal{F}$ is a σ-algebra on Ω.*

Definition 19 *A* **measure space** *is a triple* $(\Omega, \mathcal{F}, \mu)$ *where* $(\Omega, \mathcal{F})$ *is a measurable space and* $\mu : \mathcal{F} \to [0, +\infty]$ *is a measure on* $\mathcal{F}$*.*

EXERCISE 8. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $(A_n)_{n\geq 1}$ be a sequence of elements of $\mathcal{F}$ such that $A_n \subseteq A_{n+1}$ for all $n \geq 1$, and let $A = \cup_{n=1}^{+\infty} A_n$ (we write $A_n \uparrow A$). Define $B_1 = A_1$ and for all $n \geq 1$, $B_{n+1} = A_{n+1} \setminus A_n$.

1. Show that (B_n) is a sequence of pairwise disjoint elements of $\mathcal{F}$ such that $A = \uplus_{n=1}^{+\infty} B_n$.

2. Given $N \geq 1$ show that $A_N = \uplus_{n=1}^{N} B_n$.

3. Show that $\mu(A_N) \to \mu(A)$ as $N \to +\infty$

4. Show that $\mu(A_n) \leq \mu(A_{n+1})$ for all $n \geq 1$.

Theorem 7 *Let* $(\Omega, \mathcal{F}, \mu)$ *be a measure space. Then if* $(A_n)_{n\geq 1}$ *is a sequence of elements of* $\mathcal{F}$*, such that* $A_n \uparrow A$*, we have* $\mu(A_n) \uparrow \mu(A)$[1]*.*

Exercise 9.Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $(A_n)_{n\geq 1}$ be a sequence of elements of $\mathcal{F}$ such that $A_{n+1} \subseteq A_n$ for all $n \geq 1$, and let $A = \cap_{n=1}^{+\infty} A_n$ (we write $A_n \downarrow A$). We assume that $\mu(A_1) < +\infty$.

1. Define $B_n \stackrel{\triangle}{=} A_1 \setminus A_n$ and show that $B_n \in \mathcal{F}, B_n \uparrow A_1 \setminus A$.
2. Show that $\mu(B_n) \uparrow \mu(A_1 \setminus A)$
3. Show that $\mu(A_n) = \mu(A_1) - \mu(A_1 \setminus A_n)$
4. Show that $\mu(A) = \mu(A_1) - \mu(A_1 \setminus A)$
5. Why is $\mu(A_1) < +\infty$ important in deriving those equalities.
6. Show that $\mu(A_n) \to \mu(A)$ as $n \to +\infty$

[1] i.e. the sequence $(\mu(A_n))_{n\geq 1}$ is non-decreasing and converges to $\mu(A)$.

7. Show that $\mu(A_{n+1}) \leq \mu(A_n)$ for all $n \geq 1$.

Theorem 8 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Then if $(A_n)_{n\geq 1}$ is a sequence of elements of $\mathcal{F}$, such that $A_n \downarrow A$ and $\mu(A_1) < +\infty$, we have $\mu(A_n) \downarrow \mu(A)$.*

Exercise 10. Take $\Omega = \mathbf{R}$ and $\mathcal{F} = \mathcal{B}(\mathbf{R})$. Suppose μ is a measure on $\mathcal{B}(\mathbf{R})$ such that $\mu(]a, b]) = b - a$, for $a < b$. Take $A_n =]n, +\infty[$.

1. Show that $A_n \downarrow \emptyset$.
2. Show that $\mu(A_n) = +\infty$, for all $n \geq 1$.
3. Conclude that $\mu(A_n) \downarrow \mu(\emptyset)$ fails to be true.

Exercise 11. Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. Show the existence of a measure $\mu : \mathcal{B}(\mathbf{R}) \to [0, +\infty]$ such that:

$$\forall a, b \in \mathbf{R}\ ,\ a \leq b\ ,\ \mu(]a, b]) = F(b) - F(a) \qquad (2)$$

Exercise 12. Let μ_1, μ_2 be two measures on $\mathcal{B}(\mathbf{R})$ with property (2). For $n \geq 1$, we define:

$$\mathcal{D}_n \stackrel{\triangle}{=} \{B \in \mathcal{B}(\mathbf{R}) \ , \ \mu_1(B \cap]-n,n]) = \mu_2(B \cap]-n,n])\}$$

1. Show that $\mathcal{D}_n$ is a Dynkin system on $\mathbf{R}$.
2. Explain why $\mu_1(]-n,n]) < +\infty$ and $\mu_2(]-n,n]) < +\infty$ is needed when proving 1.
3. Show that $\mathcal{S} \stackrel{\triangle}{=} \{]a,b] \ , \ a,b \in \mathbf{R}\} \subseteq \mathcal{D}_n$.
4. Show that $\mathcal{B}(\mathbf{R}) \subseteq \mathcal{D}_n$.
5. Show that $\mu_1 = \mu_2$.
6. Prove the following theorem.

Theorem 9 *Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. There exists a unique measure $\mu : \mathcal{B}(\mathbf{R}) \to [0, +\infty]$ such that:*

$$\forall a, b \in \mathbf{R} \ , \ a \leq b \ , \ \mu(]a, b]) = F(b) - F(a)$$

Definition 20 *Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. We call* **Stieltjes measure** *on $\mathbf{R}$ associated with F, the unique measure on $\mathcal{B}(\mathbf{R})$, denoted dF, such that:*

$$\forall a, b \in \mathbf{R} \ , \ a \leq b \ , \ dF(]a, b]) = F(b) - F(a)$$

Definition 21 *We call* **Lebesgue measure** *on $\mathbf{R}$, the unique measure on $\mathcal{B}(\mathbf{R})$, denoted dx, such that:*

$$\forall a, b \in \mathbf{R} \ , \ a \leq b \ , \ dx(]a, b]) = b - a$$

Exercise 13. Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. Let $x_0 \in \mathbf{R}$.

1. Show that the limit $F(x_0-) = \lim_{x<x_0, x \to x_0} F(x)$ exists and is an element of $\mathbf{R}$.

2. Show that $\{x_0\} = \cap_{n=1}^{+\infty}]x_0 - 1/n, x_0]$.

3. Show that $\{x_0\} \in \mathcal{B}(\mathbf{R})$

4. Show that $dF(\{x_0\}) = F(x_0) - F(x_0-)$

Exercise 14. Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. Let $a \leq b$.

1. Show that $]a, b] \in \mathcal{B}(\mathbf{R})$ and $dF(]a, b]) = F(b) - F(a)$

2. Show that $[a, b] \in \mathcal{B}(\mathbf{R})$ and $dF([a, b]) = F(b) - F(a-)$

3. Show that $]a, b[\in \mathcal{B}(\mathbf{R})$ and $dF(]a, b[) = F(b-) - F(a)$

4. Show that $[a, b[\in \mathcal{B}(\mathbf{R})$ and $dF([a, b[) = F(b-) - F(a-)$

EXERCISE 15. Let $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. Let $\Omega' \subseteq \Omega$. Define:

$$\mathcal{A}_{|\Omega'} \stackrel{\triangle}{=} \{A \cap \Omega' \ , \ A \in \mathcal{A}\}$$

1. Show that if $\mathcal{A}$ is a topology on Ω, $\mathcal{A}_{|\Omega'}$ is a topology on Ω'.

2. Show that if $\mathcal{A}$ is a σ-algebra on Ω, $\mathcal{A}_{|\Omega'}$ is a σ-algebra on Ω'.

Definition 22 *Let Ω be a set, and $\Omega' \subseteq \Omega$. Let $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. We call* **trace** *of $\mathcal{A}$ on Ω', the subset $\mathcal{A}_{|\Omega'}$ of the power set $\mathcal{P}(\Omega')$ defined by:*

$$\mathcal{A}_{|\Omega'} \stackrel{\triangle}{=} \{A \cap \Omega' \, , \, A \in \mathcal{A}\}$$

Definition 23 *Let $(\Omega, \mathcal{T})$ be a topological space and $\Omega' \subseteq \Omega$. We call* **induced topology** *on Ω', denoted $\mathcal{T}_{|\Omega'}$, the topology on Ω' defined by:*

$$\mathcal{T}_{|\Omega'} \stackrel{\triangle}{=} \{A \cap \Omega' \ , \ A \in \mathcal{T}\}$$

In other words, the induced topology $\mathcal{T}_{|\Omega'}$ is the trace of $\mathcal{T}$ on Ω'.

Exercise 16. Let $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. Let $\Omega' \subseteq \Omega$, and $\mathcal{A}_{|\Omega'}$ be the trace of $\mathcal{A}$ on Ω'. Define:

$$\Gamma \stackrel{\triangle}{=} \{A \in \sigma(\mathcal{A}) \ , \ A \cap \Omega' \in \sigma(\mathcal{A}_{|\Omega'})\}$$

where $\sigma(\mathcal{A}_{|\Omega'})$ refers to the σ-algebra generated by $\mathcal{A}_{|\Omega'}$ on Ω'.

1. Explain why the notation $\sigma(\mathcal{A}_{|\Omega'})$ by itself is ambiguous.

2. Show that $\mathcal{A} \subseteq \Gamma$.

3. Show that Γ is a σ-algebra on Ω.

4. Show that $\sigma(\mathcal{A}_{|\Omega'}) = \sigma(\mathcal{A})_{|\Omega'}$

Theorem 10 *Let $\Omega' \subseteq \Omega$ and $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. Then, the trace on Ω' of the σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$, is equal to the σ-algebra on Ω' generated by the trace of $\mathcal{A}$ on Ω'. In other words, $\sigma(\mathcal{A})_{|\Omega'} = \sigma(\mathcal{A}_{|\Omega'})$.*

Exercise 17.Let $(\Omega, \mathcal{T})$ be a topological space and $\Omega' \subseteq \Omega$ with its induced topology $\mathcal{T}_{|\Omega'}$.

1. Show that $\mathcal{B}(\Omega)_{|\Omega'} = \mathcal{B}(\Omega')$.
2. Show that if $\Omega' \in \mathcal{B}(\Omega)$ then $\mathcal{B}(\Omega') \subseteq \mathcal{B}(\Omega)$.
3. Show that $\mathcal{B}(\mathbf{R}^+) = \{A \cap \mathbf{R}^+ \ , \ A \in \mathcal{B}(\mathbf{R})\}$.
4. Show that $\mathcal{B}(\mathbf{R}^+) \subseteq \mathcal{B}(\mathbf{R})$.

Exercise 18.Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\Omega' \subseteq \Omega$

1. Show that $(\Omega', \mathcal{F}_{|\Omega'})$ is a measurable space.

2. If $\Omega' \in \mathcal{F}$, show that $\mathcal{F}_{|\Omega'} \subseteq \mathcal{F}$.

3. If $\Omega' \in \mathcal{F}$, show that $(\Omega', \mathcal{F}_{|\Omega'}, \mu_{|\Omega'})$ is a measure space, where $\mu_{|\Omega'}$ is defined as $\mu_{|\Omega'} = \mu_{|(\mathcal{F}_{|\Omega'})}$.

Exercise 19. Let $F : \mathbf{R}^+ \to \mathbf{R}$ be a right-continuous, non-decreasing map with $F(0) \geq 0$. Define:

$$\bar{F}(x) \stackrel{\triangle}{=} \begin{cases} 0 & \text{if} \quad x < 0 \\ F(x) & \text{if} \quad x \geq 0 \end{cases}$$

1. Show that $\bar{F} : \mathbf{R} \to \mathbf{R}$ is right-continuous and non-decreasing.

2. Show that $\mu : \mathcal{B}(\mathbf{R}^+) \to [0, +\infty]$ defined by $\mu = d\bar{F}_{|\mathcal{B}(\mathbf{R}^+)}$, is a measure on $\mathcal{B}(\mathbf{R}^+)$ with the properties:

$$\begin{array}{ll} (i) & \mu(\{0\}) = F(0) \\ (ii) & \forall 0 \leq a \leq b \ , \ \mu(]a, b]) = F(b) - F(a) \end{array}$$

Exercise 20. Define: $\mathcal{C} = \{\{0\}\} \cup \{]a, b] \ , \ 0 \leq a \leq b\}$

1. Show that $\mathcal{C} \subseteq \mathcal{B}(\mathbf{R}^+)$
2. Let U be open in $\mathbf{R}^+$. Show that U is of the form:

   $$U = \bigcup_{i \in I} (\mathbf{R}^+ \cap]a_i, b_i])$$

   where I is a countable set and $a_i, b_i \in \mathbf{R}$ with $a_i \leq b_i$.
3. For all $i \in I$, show that $\mathbf{R}^+ \cap]a_i, b_i] \in \sigma(\mathcal{C})$.
4. Show that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbf{R}^+)$

Exercise 21. Let μ_1 and μ_2 be two measures on $\mathcal{B}(\mathbf{R}^+)$ with:

$$\begin{array}{rl} (i) & \mu_1(\{0\}) = \mu_2(\{0\}) = F(0) \\ (ii) & \mu_1(]a, b]) = \mu_2(]a, b]) = F(b) - F(a) \end{array}$$

for all $0 \leq a \leq b$. For $n \geq 1$, we define:

$$\mathcal{D}_n = \{B \in \mathcal{B}(\mathbf{R}^+)\ ,\ \mu_1(B \cap [0,n]) = \mu_2(B \cap [0,n])\}$$

1. Show that $\mathcal{D}_n$ is a Dynkin system on $\mathbf{R}^+$ with $\mathcal{C} \subseteq \mathcal{D}_n$, where the set $\mathcal{C}$ is defined as in exercise (20).
2. Explain why $\mu_1([0,n]) < +\infty$ and $\mu_2([0,n]) < +\infty$ is important when proving 1.
3. Show that $\mu_1 = \mu_2$.
4. Prove the following theorem.

Theorem 11 *Let $F : \mathbf{R}^+ \to \mathbf{R}$ be a right-continuous, non-decreasing map with $F(0) \geq 0$. There exists a unique $\mu : \mathcal{B}(\mathbf{R}^+) \to [0,+\infty]$ measure on $\mathcal{B}(\mathbf{R}^+)$ such that:*

$$\begin{array}{ll} (i) & \mu(\{0\}) = F(0) \\ (ii) & \forall 0 \leq a \leq b\ ,\ \mu(]a,b]) = F(b) - F(a) \end{array}$$

Definition 24 *Let* $F : \mathbf{R}^+ \to \mathbf{R}$ *be a right-continuous, non-decreasing map with* $F(0) \geq 0$*. We call* **Stieltjes measure** *on* $\mathbf{R}^+$ *associated with* F*, the unique measure on* $\mathcal{B}(\mathbf{R}^+)$*, denoted* dF*, such that:*

$$(i) \qquad dF(\{0\}) = F(0)$$

$$(ii) \qquad \forall 0 \leq a \leq b \ , \ dF(]a,b]) = F(b) - F(a)$$

Solutions to Exercises

Exercise 1.

1. $x \in]a,b] \cap]c,d]$ is equivalent to $a < x \leq b$ and $c < x \leq d$. This is in turn equivalent to:

$$a \vee c \stackrel{\triangle}{=} \max(a,c) < x \leq \min(b,d) \stackrel{\triangle}{=} b \wedge d$$

We have proved that:

$$]a,b] \cap]c,d] =]a \vee c, b \wedge d]$$

2. Suppose $x \in]a,b] \setminus]c,d]$. Then, either $x \leq c$ or $d < x$. In the first case, $x \in]a, b \wedge c]$. In the second, $x \in]a \vee d, b]$. Conversely, if $x \in]a, b \wedge c] \cup]a \vee d, b]$, then $a < x \leq b$ is true. Moreover, $x \leq c$ or $d < x$. In any case, $x \notin]c,d]$. So $x \in]a,b] \setminus]c,d]$. We have proved that:

$$]a,b] \setminus]c,d] =]a, b \wedge c] \cup]a \vee d, b]$$

3. If $c \leq d$, then in particular:

$$b \wedge c \leq c \leq d \leq a \vee d$$

4. $\mathcal{S}$ is a set of subsets of $\mathbf{R}$ which obviously contains the empty set. From 1., it is also closed under finite intersection. Let $]a, b]$ and $]c, d]$ be two elements of $\mathcal{S}$. If $c > d$, then $]c, d] = \emptyset$ and we have $]a, b]\setminus]c, d] =]a, b]$. If $c \leq d$, then it follows from 3. that $b \wedge c \leq a \vee d$. We conclude from 2. that:

$$]a, b]\setminus]c, d] =]a, b \wedge c] \uplus]a \vee d, b]$$

In any case, $]a, b]\setminus]c, d]$ can be written as a finite union of pairwise disjoint elements of $\mathcal{S}$. We have proved that $\mathcal{S}$ is indeed a semi-ring on $\mathbf{R}$, as defined in definition (6).

Exercise 1

Exercise 2. The solution to this exercise is very similar to the proof of theorem (2) : a measure defined on a semi-ring can be extended to a measure defined on the ring generated by this semi-ring. In this case, we are dealing with a finitely additive map which is not exactly a measure, but the techniques involved are almost the same. We know from the previous tutorial that the ring $\mathcal{R}(\mathcal{S})$ generated by the semi-ring $\mathcal{S}$, is the set of all finite unions of pairwise disjoint elements of $\mathcal{S}$. It is tempting to define $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$, by:

$$\forall A = \uplus_{i=1}^{n} A_i \in \mathcal{R}(\mathcal{S}) \quad , \quad \bar{\mu}(A) \stackrel{\triangle}{=} \sum_{i=1}^{n} \mu(A_i) \tag{3}$$

However, such definition may not be valid, unless the sum involved in equation (3), is independent of the particular representation of $A \in \mathcal{R}(\mathcal{S})$ as a finite union of pairwise disjoint elements of $\mathcal{S}$. Suppose that $A = \uplus_{j=1}^{p} B_j$ is another such representation of A. Then, for all $i = 1, \dots, n$, we have:

$$A_i = A_i \cap A = \uplus_{j=1}^{p} A_i \cap B_j$$

Since each $A_i \cap B_j$ is an element of $\mathcal{S}$, and μ is finitely additive, for all $i = 1, \ldots, n$, we have:

$$\mu(A_i) = \sum_{j=1}^{p} \mu(A_i \cap B_j)$$

and similarly for all $j = 1, \ldots, p$:

$$\mu(B_j) = \sum_{i=1}^{n} \mu(A_i \cap B_j)$$

from which we conclude that:

$$\sum_{i=1}^{n} \mu(A_i) = \sum_{i=1}^{n} \sum_{j=1}^{p} \mu(A_i \cap B_j) = \sum_{j=1}^{p} \mu(B_j)$$

It follows that the map $\bar{\mu}$ as defined by equation (3), is perfectly well defined. Let $A_1, \ldots, A_n$ be n pairwise disjoint elements of $\mathcal{R}(\mathcal{S})$, $n \geq 1$, each A_i having the representation:

$$A_i = \uplus_{k=1}^{p_i} A_i^k \quad , \quad i = 1, \ldots, n$$

as a finite union of pairwise disjoint elements of $\mathcal{S}$. Suppose moreover that $A = \uplus_{i=1}^n A_i$ (which is an element of $\mathcal{R}(\mathcal{S})$ since a ring is closed under finite union). Then A has a representation:

$$A = \bigcup_{i=1}^{n} \bigcup_{k=1}^{p_i} A_i^k$$

where the A_i^k's are pairwise disjoint. From the very definition of $\bar{\mu}$:

$$\bar{\mu}(A) = \sum_{i=1}^{n} \sum_{k=1}^{p_i} \mu(A_i^k)$$

and furthermore for all $i = 1, \ldots, n$:

$$\bar{\mu}(A_i) = \sum_{k=1}^{p_i} \mu(A_i^k)$$

So we conclude that:

$$\bar{\mu}(A) = \sum_{i=1}^{n} \bar{\mu}(A_i)$$

We have proved that $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$ is a finitely additive map. Finally, if $A \in \mathcal{S}$, taking $n = 1$ and $A_1 = A$, $A = \uplus_{i=1}^{n} A_i$ is a representation of A as a finite union of pairwise disjoint elements of $\mathcal{S}$. By definition of $\bar{\mu}$, $\bar{\mu}(A) = \sum_{i=1}^{n} \mu(A_i) = \mu(A)$. Hence, we see that $\bar{\mu}_{|\mathcal{S}} = \mu$. We have proved the existence of a finitely additive map $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$, such that $\bar{\mu}_{|\mathcal{S}} = \mu$.

Exercise 2

Exercise 3.

1. A ring being closed under finite union, intersection and difference, each B_i is an element of $\mathcal{R}(\mathcal{S})$. Suppose $B_i \cap B_j \neq \emptyset$ for some $i, j = 1, \ldots, n$. Without loss of generality we can assume that $i \leq j$. Suppose that $i < j$ and let $x \in B_i \cap B_j$. From $x \in B_i$ we have $x \in A_i \cap A$. From $x \in B_j$, we have $x \notin (A_1 \cap A) \cup \ldots \cup (A_{j-1} \cap A)$. In particular $x \notin A_i \cap A$. This is a contradiction, and it follows that $i = j$. The B_i's are therefore pairwise disjoint. For all $i = 1, \ldots, n$ we have $B_i \subseteq A_i \cap A \subseteq A$. hence $\uplus_{i=1}^n B_i \subseteq A$. Conversely, suppose $x \in A \subseteq \cup_{i=1}^n A_i$. There exists $i \in \{1, \ldots, n\}$ such that $x \in A_i$. Let i be the smallest of such integer. If $i = 1$, then $x \in A_1 \cap A = B_1$. If $i > 1$, then $x \in A_i \cap A$ and $x \notin A_j \cap A$ for all $j < i$. So $x \in B_i$. In any case, $x \in B_i$. It follows that $A \subseteq \uplus_{i=1}^n B_i$. We have proved that $B_1, \ldots, B_n$ are pairwise disjoint elements of $\mathcal{R}(\mathcal{S})$ with $A = \uplus_{i=1}^n B_i$.

2. $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$ being defined as in exercise (2), it is a

finitely additive map. We have $B_i \subseteq A_i \cap A \subseteq A_i$, for all $i = 1, \ldots, n$. It follows that $A_i = B_i \uplus (A_i \setminus B_i)$, from which we conclude that :

$$\bar{\mu}(A_i) = \bar{\mu}(B_i) + \bar{\mu}(A_i \setminus B_i) \geq \bar{\mu}(B_i)$$

3. From $A = \uplus_{i=1}^n B_i$ and $\bar{\mu}$ being finitely additive, we have:

$$\bar{\mu}(A) = \sum_{i=1}^n \bar{\mu}(B_i)$$

Using 2., we obtain:

$$\bar{\mu}(A) \leq \sum_{i=1}^n \bar{\mu}(A_i)$$

This is true for all $A \in \mathcal{R}(\mathcal{S})$ and $A_1, \ldots, A_n$ in $\mathcal{R}(\mathcal{S})$ such that $A \subseteq \cup_{i=1}^n A_i$. It follows from definition (12) that $\bar{\mu}$ is indeed a finitely sub-additive map.

4. Suppose $A \in \mathcal{S}$ and $A_1, \ldots, A_n \in \mathcal{S}$, $(n \geq 1)$, with $A \subseteq \cup_{i=1}^n A_i$. Since $\bar{\mu}_{|\mathcal{S}} = \mu$, and $\bar{\mu}$ is finitely sub-additive (from 3.), we have:

$$\mu(A) = \bar{\mu}(A) \leq \sum_{i=1}^n \bar{\mu}(A_i) = \sum_{i=1}^n \mu(A_i)$$

It follows from definition (12) that μ is indeed finitely sub-additive. The purpose of this exercise is to show that any finitely additive map defined on a semi-ring $\mathcal{S}$, is in fact also finitely sub-additive. Note that proving that $\bar{\mu}$ is finitely sub-additive is pretty straightforward. This is because $\bar{\mu}$ is defined on a ring, which is closed under various finite operations (union, intersection, difference). However, μ being defined on a semi-ring only, it is impossible to apply the same line of argument as the one used for $\bar{\mu}$. It is in fact necessary for us to initially extend μ from $\mathcal{S}$ to $\mathcal{R}(\mathcal{S})$, then prove the finite sub-additivity on $\mathcal{R}(\mathcal{S})$, and conclude with the finite sub-additivity of μ on $\mathcal{S}$.

Exercise 3

Exercise 4.

1. Take i_1 such that $a_{i_1} = \min(a_1, \ldots, a_n)$. From $]a_{i_1}, b_{i_1}] \subseteq]a, b]$ and $a_{i_1} < b_{i_1}$, we see that $a \leq a_{i_1} < b_{i_1} \leq b$. Suppose that $a < a_{i_1}$, and let x be such that $a < x < a_{i_1} \leq b$. Since $x \in]a, b]$, there is $j \in \{1, \ldots, n\}$ such that $x \in]a_j, b_j]$. By definition of i_1, we have $a_{i_1} \leq a_j < x$. This is a contradiction, and it follows that $a_{i_1} = a$. We have proved the existence of $i_1 \in \{1, \ldots, n\}$ such that $a_{i_1} = a$.

2. Suppose $x \in]a_i, b_i]$ for some $i \in \{1, \ldots, n\}$, $i \neq i_1$. Since $]a_i, b_i] \subseteq]a, b]$, $x \in]a, b]$ and $x \leq b$. Also, $a \leq a_i$. From 1., $a_{i_1} = a$. It follows that $a_{i_1} \leq a_i < x$. However, the $]a_i, b_i]$'s being pairwise disjoint and $i \neq i_1$, $x \notin]a_{i_1}, b_{i_1}]$. Therefore $x > b_{i_1}$. We have proved that $x \in]b_{i_1}, b]$ and consequently:

$$\biguplus_{i=1, i \neq i_1}^{n}]a_i, b_i] \subseteq]b_{i_1}, b]$$

Conversely, let $x \in]b_{i_1}, b] \subseteq]a, b]$. There exists $i \in \{1, \ldots, n\}$ such that $x \in]a_i, b_i]$. If $i = i_1$, then $x \in]a_{i_1}, b_{i_1}]$ which contradicts $b_{i_1} < x$. It follows that $i \neq i_1$ and:

$$]b_{i_1}, b] \subseteq \biguplus_{i=1, i\neq i_1}]a_i, b_i]$$

3. Using 1. and 2., starting from:

$$]a, b] = \biguplus_{i=1}^{n}]a_i, b_i]$$

we have $i_1 \in \{1, \ldots, n\}$ such that $a = a_{i_1} < b_{i_1}$ and:

$$]b_{i_1}, b] = \biguplus_{i=1, i\neq i_1}^{n}]a_i, b_i]$$

Going one step further, there exists $i_2 \in \{1, \ldots, n\} \setminus \{i_1\}$ such that $b_{i_1} = a_{i_2} < b_{i_2}$ and:

$$]b_{i_2}, b] = \biguplus_{i=1, i\neq i_1, i_2}^{n}]a_i, b_i]$$

By induction, we define $i_1 \ldots, i_n$ distinct integers in $\{1, \ldots, n\}$, (hence a permutation on $\{1, \ldots, n\}$), such that:

$$a = a_{i_1} < b_{i_1} = a_{i_2} < \ldots < b_{i_n}$$

and $]b_{i_n}, b] = \emptyset$. Since $]a_{i_n}, b_{i_n}] \subseteq]a, b]$ and $a_{i_n} < b_{i_n}$, we have $b_{i_n} \leq b$. From $]b_{i_n}, b] = \emptyset$, we conclude that $b_{i_n} = b$.

4. Let $(i_1, \ldots, i_n)$ be a permutation of $\{1, \ldots, n\}$, such that:

$$a = a_{i_1} < b_{i_1} = a_{i_2} < \ldots < b_{i_n} = b$$

We have:

$$F(b) - F(a) = \sum_{k=1}^{n} F(b_{i_k}) - F(a_{i_k})$$

from which we see that:

$$\mu(]a,b]) = \sum_{k=1}^{n} \mu(]a_{i_k}, b_{i_k}]) = \sum_{i=1}^{n} \mu(]a_i, b_i])$$

This is true for all $a < b$, $n \geq 1$ and $a_i < b_i$ for $i = 1, \ldots, n$, such that:

$$]a,b] = \biguplus_{i=1}^{n}]a_i, b_i]$$

Suppose $A \in \mathcal{S}$, $n \geq 1$ and $A_1, \ldots, A_n \in \mathcal{S}$, with $A = \uplus_{i=1}^n A_i$. If $A = \emptyset$, then for all $i = 1, \ldots, n$, we have $A_i = \emptyset$. In particular, $\mu(A) = \sum_{i=1}^n \mu(A_i)$ is obviously satisfied. If $A \neq \emptyset$, then A is of the form $A =]a, b]$ for some $a < b$ in $\mathbf{R}$. If we consider $J = \{i = 1, \ldots, n, A_i \neq \emptyset\}$, then $J \neq \emptyset$, and for all $i \in J$, A_i is of the form $A_i =]a_i, b_i]$ with $a_i < b_i$. Moreover, $A = \uplus_{i \in J} A_i$ and it follows from our previous developments that $\mu(A) = \sum_{i \in J} \mu(A_i)$. However, for all $i = 1, \ldots, n$, if $i \notin J$, then

$A_i = \emptyset$ and $\mu(A_i) = 0$. Consequently:

$$\mu(A) = \sum_{i \in J} \mu(A_i) + \sum_{i \notin J} \mu(A_i) = \sum_{i=1}^{n} \mu(A_i)$$

We have proved that $\mu : \mathcal{S} \to [0, +\infty]$ as defined by (1) is finitely additive. From exercise (3), it is also finitely sub-additive.

Exercise 4

Exercise 5.

1. The sum $\sum_{k=1}^{N} F(b_{i_k}) - F(a_{i_k})$ can be written as:

$$F(b_{i_N}) - F(a_{i_1}) + \sum_{k=1}^{N-1} F(b_{i_k}) - F(a_{i_{k+1}})$$

F being non-decreasing, with $b_{i_N} \leq b$ and $a \leq a_{i_1}$, we have $F(b_{i_N}) \leq F(b)$ and $F(a) \leq F(a_{i_1})$. Moreover, since $b_{i_k} \leq a_{i_{k+1}}$ for all $k = 1, \ldots, N-1$, we have $F(b_{i_k}) \leq F(a_{i_{k+1}})$. It follows that:

$$\sum_{k=1}^{N} F(b_{i_k}) - F(a_{i_k}) \leq F(b) - F(a)$$

2. Let $N \geq 1$, and $(i_1, \ldots, i_N)$ be a permutation of $\{1, \ldots, N\}$ such that $a_{i_1} \leq a_{i_2} \leq \ldots \leq a_{i_N}$. Since $]a_{i_1}, b_{i_1}] \subseteq]a, b]$ (and the fact that $a_{i_1} < b_{i_1}$), we have $a \leq a_{i_1} < b_{i_1}$. We also have $]a_{i_N}, b_{i_N}] \subseteq]a, b]$ with $a_{i_N} < b_{i_N}$. Hence, $a_{i_N} < b_{i_N} \leq b$. Let $k \in \{1, \ldots, N-1\}$. Since the $]a_n, b_n]$'s are pairwise disjoint,

in particular, $]a_{i_k}, b_{i_k}] \cap]a_{i_{k+1}}, b_{i_{k+1}}] = \emptyset$. Let $\epsilon > 0$ be such that $a_{i_{k+1}} + \epsilon \in]a_{i_{k+1}}, b_{i_{k+1}}]$. Then $a_{i_k} \leq a_{i_{k+1}} < a_{i_{k+1}} + \epsilon$, and $a_{i_{k+1}} + \epsilon$ cannot be an element of $]a_{i_k}, b_{i_k}]$. Hence, $b_{i_k} < a_{i_{k+1}} + \epsilon$. Taking the limit as $\epsilon \to 0$, we have $b_{i_k} \leq a_{i_{k+1}}$. It follows that the permutation $(i_1, \ldots, i_N)$ of $\{1, \ldots, N\}$ is such that:

$$a \leq a_{i_1} < b_{i_1} \leq a_{i_2} < \ldots < b_{i_N} \leq b$$

From 1., we obtain:

$$\sum_{k=1}^{N} F(b_{i_k}) - F(a_{i_k}) \leq F(b) - F(a)$$

and consequently:

$$\sum_{n=1}^{N} \mu(]a_n, b_n]) = \sum_{k=1}^{N} \mu(]a_{i_k}, b_{i_k}]) \leq \mu(]a, b]) \tag{4}$$

Taking the supremum over all $N \geq 1$ (or the limit as $N \to +\infty$)

in the left-hand side of (4), we obtain:

$$\sum_{n=1}^{+\infty} \mu(]a_n, b_n]) \leq \mu(]a, b])$$

3. F being right-continuous, it is right-continuous in $a \in \mathbf{R}$. Given $\epsilon > 0$, there exists $\eta' > 0$ such that:

$$\forall x \in [a, a + \eta'[\quad , \quad |F(x) - F(a)| \leq \epsilon$$

Take $\eta = \min(b - a, \eta')/2$. Then $\eta \in]0, b - a[$, and we have $a + \eta \in [a, a + \eta'[$. Therefore, $|F(a + \eta) - F(a)| \leq \epsilon$, and F being non-decreasing, we finally have:

$$0 \leq F(a + \eta) - F(a) \leq \epsilon$$

4. Given $n \geq 1$, F is right-continuous in $b_n \in \mathbf{R}$. Given $\epsilon > 0$ and $\epsilon' = \epsilon/2^n$, there exists $\eta'_n > 0$ such that:

$$\forall x \in [b_n, b_n + \eta'_n[\quad , \quad |F(x) - F(b_n)| \leq \epsilon'$$

Take $\eta_n = \eta'_n/2$. Then $b_n + \eta_n \in [b_n, b_n + \eta'_n[$, and we have $|F(b_n + \eta_n) - F(b_n)| \leq \epsilon/2^n$. F being non-decreasing, we finally have:

$$0 \leq F(b_n + \eta_n) - F(b_n) \leq \frac{\epsilon}{2^n}$$

5. Let $x \in [a + \eta, b]$. Then $x \in]a, b]$, and there exists $n \geq 1$ such that $x \in]a_n, b_n]$. In particular, $x \in]a_n, b_n + \eta_n[$. It follows that:

$$[a + \eta, b] \subseteq \bigcup_{n=1}^{+\infty}]a_n, b_n + \eta_n[\tag{5}$$

6. We see from (5) that the closed interval $[a+\eta, b]$ of $\mathbf{R}$, is covered by the family of open sets $(]a_n, b_n+\eta_n[)_{n\geq 1}$ in $\mathbf{R}$. Since $[a+\eta, b]$ is a compact subset of $\mathbf{R}$[2], we can extract a finite sub-covering

[2]Note that the notion of *compact* subsets and the fact that any closed interval $[a, b]$ in $\mathbf{R}$ is indeed a compact subset of $\mathbf{R}$, has not been approached so far in these tutorials. This seems to contradict our promise that no results in these tutorials should be used without proof. In fact, Tutorial 8 will give you ample reminders on compactness. Just be a little patient.

of $[a+\eta, b]$. In other words, there exist $p \geq 1$, and integers $n_1, \ldots, n_p$ such that:

$$[a+\eta, b] \subseteq \bigcup_{k=1}^{p}]a_{n_k}, b_{n_k} + \eta_{n_k}[$$

In particular:

$$]a+\eta, b] \subseteq \bigcup_{k=1}^{p}]a_{n_k}, b_{n_k} + \eta_{n_k}] \tag{6}$$

7. From exercise (4), we know that μ as defined in (1), is finitely sub-additive. It follows from (6):

$$\mu(]a+\eta, b]) \leq \sum_{k=1}^{p} \mu(]a_{n_k}, b_{n_k} + \eta_{n_k}]) \tag{7}$$

Since $a+\eta < b$ and $a_n < b_n < b_n+\eta_n$ for all $n \geq 1$, inequality (7) can be written as:

$$F(b) - F(a+\eta) \leq \sum_{k=1}^{p} F(b_{n_k} + \eta_{n_k}) - F(a_{n_k})$$

Using 3. and 4., we obtain:

$$F(b) - F(a) \leq \epsilon + \sum_{k=1}^{p} (F(b_{n_k}) - F(a_{n_k}) + \frac{\epsilon}{2^{n_k}})$$

and since F is non-decreasing, we finally have:

$$F(b) - F(a) \leq 2\epsilon + \sum_{n=1}^{+\infty} F(b_n) - F(a_n) \qquad (8)$$

8. Taking the limit as $\epsilon \to 0$ in (8), we obtain:

$$F(b) - F(a) \leq \sum_{n=1}^{+\infty} F(b_n) - F(a_n)$$

Since $a < b$ and $a_n < b_n$ for all $n \geq 1$, we have:

$$\mu(]a,b]) \leq \sum_{n=1}^{+\infty} \mu(]a_n, b_n])$$

From 2., we conclude that:

$$\mu(]a,b]) = \sum_{n=1}^{+\infty} \mu(]a_n, b_n]) \tag{9}$$

It follows that if $A \in \mathcal{S}$ and $(A_n)_{n\geq 1}$ is a sequence of pairwise disjoint elements of $\mathcal{S}$, such that $A = \uplus_{n=1}^{+\infty} A_n$, we have:

$$\mu(A) = \sum_{n=1}^{+\infty} \mu(A_n) \tag{10}$$

Indeed, if $A = \emptyset$, then all A_n's are empty and (10) is obviously satisfied. If $A \neq \emptyset$, then $A =]a,b]$ for some $a < b$. Moreover, if we define $J = \{n \geq 1, A_n \neq \emptyset\}$, then $A = \uplus_{n\in J} A_n$, and the following holds,

$$\mu(A) = \sum_{n \in J} \mu(A_n) \tag{11}$$

either as a consequence of (9), in the case when J is infinite, or as a consequence of μ being finitely additive (exercise (4)), in the case when J is finite. In any case, (10) follows immediately from (11) and the fact that $\mu(\emptyset) = 0$. Having proved (10), we conclude that $\mu : \mathcal{S} \to [0, +\infty]$ as defined in (1) is indeed a measure on the semi-ring $\mathcal{S}$.

Exercise 5

Exercise 6. Any statement of the form $\forall x \in \emptyset \ldots$[3] is true. So $\emptyset \in \mathcal{T}_{\mathbf{R}}$, and it is clear that $\mathbf{R} \in \mathcal{T}_{\mathbf{R}}$. So (i) of definition (13) is satisfied for $\mathcal{T}_{\mathbf{R}}$. Let $A, B \in \mathcal{T}_{\mathbf{R}}$. Let $x \in A \cap B$. Since $x \in A$, from definition (17), there exists $\epsilon_1 > 0$ such that $]x - \epsilon_1, x + \epsilon_1[\subseteq A$. Since $x \in B$, there exists $\epsilon_2 > 0$ such that $]x - \epsilon_2, x + \epsilon_2[\subseteq B$. It follows that if ϵ is defined as $\epsilon = \min(\epsilon_1, \epsilon_2)$, then $]x - \epsilon, x + \epsilon[\subseteq A \cap B$. Hence $A \cap B \in \mathcal{T}_{\mathbf{R}}$, and (ii) of definition (13) is satisfied for $\mathcal{T}_{\mathbf{R}}$. Let $(A_i)_{i \in I}$ be a family of elements of $\mathcal{T}_{\mathbf{R}}$. Let $x \in \cup_{i \in I} A_i$. There exists $i \in I$ such that $x \in A_i$. Since by assumption $A_i \in \mathcal{T}_{\mathbf{R}}$, there exists $\epsilon > 0$ such that $]x - \epsilon, x + \epsilon[\subseteq A_i$. In particular, $]x - \epsilon, x + \epsilon[\subseteq \cup_{i \in I} A_i$. It follows that $\cup_{i \in I} A_i \in \mathcal{T}_{\mathbf{R}}$, and (iii) of definition (13) is satisfied for $\mathcal{T}_{\mathbf{R}}$. We have proved that $\mathcal{T}_{\mathbf{R}}$ is indeed a topology on $\mathbf{R}$.

Exercise 6

[3] Recall that $\forall x \in \emptyset, H$ is equivalent to $x \in \emptyset \Rightarrow H$, and $G \Rightarrow H$ is equivalent to (G is false) or (both G and H are true).

Exercise 7.

1. For all $n \geq 1$, we have $]a,b] \subseteq]a,b+1/n[$. Hence, we have $]a,b] \subseteq \cap_{n=1}^{+\infty}]a,b+1/n[$. Conversely, if $x \in \cap_{n=1}^{+\infty}]a,b+1/n[$, then for all $n \geq 1$, we have $a < x < b + 1/n$. Taking the limit as $n \to +\infty$, we obtain $a < x \leq b$. It follows that $x \in]a,b]$ and $\cap_{n=1}^{+\infty}]a,b+1/n[\subseteq]a,b]$. Finally, $]a,b] = \cap_{n=1}^{+\infty}]a,b+1/n[$.

2. Let $a, b \in \mathbf{R}$, $a \leq b$. For all $n \geq 1$, the interval $]a,b+1/n[$ is an open set in $\mathbf{R}$, (i.e. an element of $\mathcal{T}_{\mathbf{R}}$). Indeed, if $x \in]a,b+1/n[$, take $\epsilon = \min(b+1/n-x, x-a)$, then $]x-\epsilon, x+\epsilon[\subseteq]a,b+1/n[$. Since $\mathcal{T}_{\mathbf{R}} \subseteq \sigma(\mathcal{T}_{\mathbf{R}}) = \mathcal{B}(\mathbf{R})$, $]a,b+1/n[$ is also a Borel set in $\mathbf{R}$, (i.e. an element of $\mathcal{B}(\mathbf{R})$). From 1., we have:

$$]a,b] = \bigcap_{n=1}^{+\infty}]a,b+1/n[= \left(\bigcup_{n=1}^{+\infty}]a,b+1/n[^c \right)^c$$

$\mathcal{B}(\mathbf{R})$ being a σ-algebra, it is closed under complementation and countable union. It follows that $]a,b] \in \mathcal{B}(\mathbf{R})$. This being true for all $a \leq b$, we have proved that $\mathcal{S} \subseteq \mathcal{B}(\mathbf{R})$. The σ-algebra $\sigma(\mathcal{S})$ generated by $\mathcal{S}$ being the smallest σ-algebra on $\mathbf{R}$ containing $\mathcal{S}$, we finally have $\sigma(\mathcal{S}) \subseteq \mathcal{B}(\mathbf{R})$.

3. Let $U \in \mathcal{T}_{\mathbf{R}}$ and $x \in U$. From definition (17), there exists $\epsilon > 0$ such that $]x-\epsilon, x+\epsilon[\subseteq U$. $\mathbf{Q}$ being the set of all rational numbers, it is dense in $\mathbf{R}$: in other words, for all $a < b$, $\mathbf{Q} \cap]a, b[$ is a non-empty set[4]. In particular, there exist $a_x \in \mathbf{Q} \cap]x-\epsilon, x[$ and $b_x \in \mathbf{Q} \cap]x, x+\epsilon[$. We have $x \in]a_x, b_x] \subseteq U$.

4. Since for all $x \in U$, $]a_x, b_x] \subseteq U$, we have $\cup_{x \in U}]a_x, b_x] \subseteq U$. If $x \in U$, then $x \in]a_x, b_x]$. So $U \subseteq \cup_{x \in U}]a_x, b_x]$. We have proved that $U = \cup_{x \in U}]a_x, b_x]$.

5. Let $I = \{]a_x, b_x], x \in U\}$. Since $\mathbf{Q}$ is a countable set, the product $\mathbf{Q}^2 = \mathbf{Q} \times \mathbf{Q}$ is also countable. There exists a one-to-one map $\phi : \mathbf{Q}^2 \rightarrow \mathbf{N}$. Consider $\psi : I \rightarrow \mathbf{N}$ defined by

[4]This density property of $\mathbf{Q}$ in $\mathbf{R}$ is not proved anywhere in these tutorials. Please refer to any textbook containing a formal construction of the field $\mathbf{R}$.

$\psi(]a_x, b_x]) = \phi(a_x, b_x)$. Then if $\psi(]a_{x'}, b_{x'}]) = \psi(]a_x, b_x])$, we have $\phi(a_{x'}, b_{x'}) = \phi(a_x, b_x)$, and thus, $(a_{x'}, b_{x'}) = (a_x, b_x)$. Hence, $]a_{x'}, b_{x'}] =]a_x, b_x]$. It follows that the map $\psi : I \to \mathbf{N}$ is an injective map. We have proved that I is a countable set.

6. For all $i \in I$, $i =]a_x, b_x]$ for some $x \in U$. So $i \in \mathcal{S}$. Define $A_i = i$. Then $A_i \in \mathcal{S}$ for all $i \in I$, and we have:

$$U = \bigcup_{x \in U}]a_x, b_x] = \bigcup_{i \in I} A_i$$

7. Since I is a countable set, and $A_i \in \mathcal{S}$ for all $i \in I$, we have $U = \cup_{i \in I} A_i \in \sigma(\mathcal{S})$. This being true for all $U \in \mathcal{T}_{\mathbf{R}}$, we have proved that $\mathcal{T}_{\mathbf{R}} \subseteq \sigma(\mathcal{S})$. The Borel σ-algebra $\mathcal{B}(\mathbf{R})$ generated by $\mathcal{T}_{\mathbf{R}}$ being the smallest σ-algebra on $\mathbf{R}$ containing $\mathcal{T}_{\mathbf{R}}$, we have $\mathcal{B}(\mathbf{R}) \subseteq \sigma(\mathcal{S})$. From 2., we conclude that $\mathcal{B}(\mathbf{R}) = \sigma(\mathcal{S})$. The purpose of this exercise is to show theorem (6).

Exercise 7

Exercise 8.

1. A σ-algebra being closed under difference, $(B_n)_{n\geq 1}$ is indeed a sequence of elements of $\mathcal{F}$. Suppose $B_n \cap B_p \neq \emptyset$. Without loss of generality, we can assume that $n \leq p$. Suppose $n < p$ and let $x \in B_n \cap B_p$. From $x \in B_n$, we have $x \in A_n$. From $x \in B_p$, we have $x \notin A_{p-1}$. However, $A_n \subseteq A_{p-1}$. This is a contradiction, and it follows that $n = p$. We have proved that the B_n's are pairwise disjoint. Since $B_n \subseteq A_n$ for all $n \geq 1$, we have $\uplus_{n=1}^{+\infty} B_n \subseteq A$. Conversely, let $x \in A$. There exists $n \geq 1$ such that $x \in A_n$. Let n be the smallest integer such that $x \in A_n$. Then if $n = 1$, $x \in B_1$. If $n > 1$, then $x \in A_n \setminus A_{n-1} = B_n$. In any case $x \in B_n$ and $A \subseteq \uplus_{n=1}^{+\infty} B_n$. We have proved that $(B_n)_{n\geq 1}$ is a sequence of pairwise disjoint elements of $\mathcal{F}$, such that $A = \uplus_{n=1}^{+\infty} B_n$.

2. Let $N \geq 1$. For all $n = 1, \ldots, N$, we have $B_n \subseteq A_n \subseteq A_N$. So $\uplus_{n=1}^{N} B_n \subseteq A_N$. Conversely, let $x \in A_N$. Let n be the smallest integer such that $x \in A_n$. Then $1 \leq n \leq N$. If $n = 1$, then

$x \in B_1$. If $n > 1$, then $x \in A_n \setminus A_{n-1} = B_n$. In any case, $x \in B_n$ and $A_N \subseteq \uplus_{n=1}^{N} B_n$. We have proved that $A_N = \uplus_{n=1}^{N} B_n$.

3. μ being a measure on $\mathcal{F}$, from 1. we obtain:

$$\lim_{N \to +\infty} \sum_{n=1}^{N} \mu(B_n) \stackrel{\triangle}{=} \sum_{n=1}^{+\infty} \mu(B_n) = \mu(A)$$

However, it follows from 2.

$$\sum_{n=1}^{N} \mu(B_n) = \mu(A_N)$$

Hence:

$$\lim_{N \to +\infty} \mu(A_N) = \mu(A)$$

4. Since $A_n \subseteq A_{n+1}$, we have $\mu(A_n) \leq \mu(A_{n+1})$ for all $n \geq 1$. The purpose of this exercise is to prove theorem (7).

Exercise 8

Exercise 9.

1. A σ-algebra being closed under difference, each B_n is an element of $\mathcal{F}$. For all $n \geq 1$, we have:

$$B_n = A_1 \cap A_n^c \subseteq A_1 \cap A_{n+1}^c = B_{n+1}$$

Moreover:

$$\bigcup_{n=1}^{+\infty} B_n = A_1 \cap \left(\bigcup_{n=1}^{+\infty} A_n^c \right) = A_1 \cap \left(\bigcap_{n=1}^{+\infty} A_n \right)^c = A_1 \setminus A$$

We have proved that $B_n \uparrow A_1 \setminus A$.

2. $\mu(B_n) \uparrow \mu(A_1 \setminus A)$ is a direct application of theorem (7).

3. Since $A_n \subseteq A_1$, we have $A_1 = A_n \uplus (A_1 \setminus A_n)$. μ being a measure on $\mathcal{F}$, we obtain $\mu(A_1) = \mu(A_n) + \mu(A_1 \setminus A_n)$. Since $\mu(A_1) < +\infty$, all the terms involved in this equality are finite. Hence, it is legitimate to write:

$$\mu(A_n) = \mu(A_1) - \mu(A_1 \setminus A_n)$$

4. Since $A \subseteq A_1$, we have $A_1 = A \uplus (A_1 \setminus A)$. μ being a measure on $\mathcal{F}$, we obtain $\mu(A_1) = \mu(A) + \mu(A_1 \setminus A)$. Since $\mu(A_1) < +\infty$, all the terms involved in this equality are finite. Hence, it is legitimate to write:

$$\mu(A) = \mu(A_1) - \mu(A_1 \setminus A)$$

5. Since for all $n \geq 1$, $A \subseteq A_n \subseteq A_1$, μ being a measure on $\mathcal{F}$, $\mu(A) \leq \mu(A_n) \leq \mu(A_1)$. Similarly, $A_1 \setminus A \subseteq A_1$ implies that $\mu(A_1 \setminus A) \leq \mu(A_1)$. Having $\mu(A_1) < +\infty$ ensures that all the terms involved in say $\mu(A_1) = \mu(A) + \mu(A_1 \setminus A)$ are finite, allowing to subtract $\mu(A_1 \setminus A)$ on both side of such equality. One common mistake to make is to get involved in algebra with non-finite terms, ending up with meaningless expressions of the form $+\infty - (+\infty)\ldots$

6. Using 2., 3., 4. and the fact that $\mu(A_1) < +\infty$[5]:

$$\lim_{n\to+\infty} \mu(A_n) = \mu(A_1) - \lim_{n\to+\infty} \mu(B_n) = \mu(A_1) - \mu(A_1 \setminus A) = \mu(A)$$

7. For all $n \geq 1$, $A_{n+1} \subseteq A_n$, and therefore $\mu(A_{n+1}) \leq \mu(A_n)$. The purpose of this exercise is to prove theorem (8).

Exercise 9

[5] $\lim_{n\to+\infty}(+\infty - n) = +\infty$, whereas $+\infty - \lim_{n\to+\infty} n$ is meaningless...

Exercise 10.

1. For all $n \geq 1$, we have $A_{n+1} \subseteq A_n$, and:

$$\bigcap_{n=1}^{+\infty} A_n = \bigcap_{n=1}^{+\infty}]n, +\infty[= \emptyset$$

   It follows that $A_n \downarrow \emptyset$.

2. Let $n \geq 1$. Given $p \geq n$, define $A_n^p =]n, p]$. Then $A_n^p \uparrow A_n$ as $p \to +\infty$, and from theorem (7), we have:

$$\mu(A_n) = \lim_{p \to +\infty} \mu(A_n^p) = \lim_{p \to +\infty} p - n = +\infty$$

3. Since $\mu(A_n) = +\infty$ for all $n \geq 1$, $\mu(A_n) \to +\infty$ as $n \to +\infty$. Since $\mu(\emptyset) = 0$, $\mu(A_n) \downarrow \mu(\emptyset)$ fails to be true. The purpose of this exercise is to serve as counter example to theorem (8), if the condition $\mu(A_1) < +\infty$ is relaxed. Indeed, $A_n \downarrow \emptyset$, yet we do not have $\mu(A_n) \downarrow \mu(\emptyset)$. Note however that to apply

theorem (8), $\mu(A_1) < +\infty$ is not strictly speaking necessary: if a slightly weaker assumption is made that $\mu(A_p) < +\infty$ for some $p \geq 1$, one can always apply theorem (8) to the sequence $(A'_n)_{n\geq 1} = (A_{n+p-1})_{n\geq 1} \ldots$

Exercise 10

Exercise 11. Let $\mathcal{S}$ be the semi-ring $\mathcal{S} = \{]a,b], a,b \in \mathbf{R}\}$, and $\mu : \mathcal{S} \to [0,+\infty]$ be the map defined by equation (2). We know from exercise (5) that μ is in fact a measure on $\mathcal{S}$. From theorem (5) , μ can be extended to a measure defined on the σ-algebra $\sigma(\mathcal{S})$ generated by $\mathcal{S}$. In other words, there exists a measure $\bar{\mu} : \sigma(\mathcal{S}) \to [0,+\infty]$, such that $\bar{\mu}_{|\mathcal{S}} = \mu$. From theorem (6), we know that the σ-algebra $\sigma(\mathcal{S})$ is in fact equal to the Borel σ-algebra $\mathcal{B}(\mathbf{R})$ on $\mathbf{R}$. Hence, we have found a measure $\bar{\mu} : \mathcal{B}(\mathbf{R}) \to [0,+\infty]$ such that $\bar{\mu}_{|\mathcal{S}} = \mu$. In particular, we have:

$$\forall a,b \in \mathbf{R}\ ,\ a \leq b\ ,\ \bar{\mu}(]a,b]) = F(b) - F(a)$$

The purpose of this exercise is to prove the existence of the so called *Stieltjes* measure on $\mathbf{R}$, stated in theorem (9). This is a vitally important result, as most other measures ever encountered, are derived one way or another from the Stieltjes measure on $\mathbf{R}$.

Exercise 11

Exercise 12.

1. Since $\mu_1(]-n,n]) = F(n) - F(-n) = \mu_2(]-n,n])$, $\Omega \in \mathcal{D}_n$. Suppose $A, B \in \mathcal{D}_n$, with $A \subseteq B$. We have:

$$\mu_1(B\cap]-n,n]) = \mu_2(B\cap]-n,n]) \tag{12}$$

$$\mu_1(A\cap]-n,n]) = \mu_2(A\cap]-n,n]) \tag{13}$$

Moreover, since $B = A \uplus (B \setminus A)$, for $i = 1, 2$, we have:

$$\mu_i(B\cap]-n,n]) = \mu_i(A\cap]-n,n]) + \mu_i((B \setminus A)\cap]-n,n]) \tag{14}$$

All terms involved in (12), (13) and (14) being finite, subtracting (13) from (12), and using (14), we obtain:

$$\mu_1((B \setminus A)\cap]-n,n]) = \mu_2((B \setminus A)\cap]-n,n])$$

This shows that $B \setminus A \in \mathcal{D}_n$. Let $(A_p)_{p\geq 1}$ be a sequence of elements of $\mathcal{D}_n$ such that $A_p \uparrow A$. Then $A_p\cap]-n,n] \uparrow A\cap]-n,n]$, and from theorem (7), $\mu_i(A_p\cap]-n,n]) \uparrow \mu_i(A\cap]-n,n])$ for all

$i = 1, 2$. However, since $A_p \in \mathcal{D}_n$ for all $p \geq 1$, we have:

$$\mu_1(A_p \cap]-n, n]) = \mu_2(A_p \cap]-n, n])$$

Taking the limit as $p \to +\infty$, we obtain:

$$\mu_1(A \cap]-n, n]) = \mu_2(A \cap]-n, n])$$

So $A \in \mathcal{D}_n$. Having checked (i), (ii) and (iii) of definition (1), we have proved that $\mathcal{D}_n$ is indeed a Dynkin system on $\mathbf{R}$.

2. A crucial step in proving that $\mathcal{D}_n$ is a Dynkin system on $\mathbf{R}$, consists in subtracting (13) from (12). One has to be very careful in avoiding meaningless expressions of the form $+\infty - (+\infty)$. Having $\mu_1(]-n, n]) < +\infty$ and $\mu_2(]-n, n]) < +\infty$ ensures that all terms involved be finite.

3. Since $\mu_1(\emptyset \cap]-n, n]) = 0 = \mu_2(\emptyset \cap]-n, n])$, we have $\emptyset \in \mathcal{D}_n$. Let $a < b$. From exercise (1), $]a, b] \cap]-n, n]$ is an interval of the form $]a', b']$. If $a' < b'$, then:

$$\mu_1(]a', b']) = F(b') - F(a') = \mu_2(]a', b'])$$

Otherwise, $\mu_1(]a',b']) = 0 = \mu_2(]a',b'])$. In any case, we have $\mu_1(]a',b']) = \mu_2(]a',b'])$, and $]a,b] \in \mathcal{D}_n$. We have proved that $\mathcal{S} \subseteq \mathcal{D}_n$.

4. $\mathcal{S}$ being a semi-ring on $\mathbf{R}$, from definition (6), it is closed under finite intersection. Since $\mathcal{S} \subseteq \mathcal{D}_n$, $\mathcal{D}_n$ is a Dynkin system containing a set of subsets of $\mathbf{R}$, which is closed under finite intersection. According to theorem (1), $\mathcal{D}_n$ also contains the σ-algebra generated by $\mathcal{S}$. In other words, $\sigma(\mathcal{S}) \subseteq \mathcal{D}_n$. However, from theorem (6), the σ-algebra generated by $\mathcal{S}$, coincide with the Borel σ-algebra on $\mathbf{R}$, i.e. $\sigma(\mathcal{S}) = \mathcal{B}(\mathbf{R})$. It follows that $\mathcal{B}(\mathbf{R}) \subseteq \mathcal{D}_n$.

5. Let $A \in \mathcal{B}(\mathbf{R})$. from 4., we have $A \in \mathcal{D}_n$. In other words:

$$\mu_1(A \cap]-n,n]) = \mu_2(A \cap]-n,n])$$

This being true for all $n \geq 1$, using theorem (7) and taking the limit as $n \to +\infty$, we obtain: $\mu_1(A) = \mu_2(A)$. This being true for all $A \in \mathcal{B}(\mathbf{R})$, $\mu_1 = \mu_2$.

6. Uniqueness follows from 5. Existence is proved in exercise (11).

Exercise 12

Exercise 13.

1. F being non-decreasing, for all $x < x_0$, $F(x) \leq F(x_0)$. Define:

$$\alpha \stackrel{\triangle}{=} \sup_{x<x_0} F(x)$$

Then $\alpha \leq F(x_0)$ and in particular $\alpha < +\infty$. It follows that given $\epsilon > 0$, $\alpha - \epsilon < \alpha$. Being a supremum, α is the smallest upper-bound of all $F(x)$'s for $x < x_0$. Hence, we see that $\alpha - \epsilon$ cannot be such upper-bound. There exists $x_1 < x_0$ such that $\alpha - \epsilon < F(x_1)$. Since F is non-decreasing, for all $x \in]x_1, x_0[$, we have $\alpha - \epsilon < F(x_1) \leq F(x) \leq \alpha \leq \alpha + \epsilon$. We conclude that for all $\epsilon > 0$, there exists $x_1 < x_0$ such that:

$$\forall x \in]x_1, x_0[\quad , \quad |F(x) - \alpha| \leq \epsilon$$

We have proved the existence of the left limit:

$$F(x_0-) \stackrel{\triangle}{=} \lim_{x<x_0, x\to x_0} F(x) = \alpha \in \mathbf{R}$$

2. It is clear that $\{x_0\} \subseteq \cap_{n=1}^{+\infty}]x_0 - 1/n, x_0]$. Conversely, suppose that $x \in \cap_{n=1}^{+\infty}]x_0 - 1/n, x_0]$. Then for all $n \geq 1$, we have $x_0 - 1/n < x \leq x_0$. Taking the limit as $n \rightarrow +\infty$, we obtain $x_0 \leq x \leq x_0$, i.e. $x = x_0$. So $\cap_{n=1}^{+\infty}]x_0 - 1/n, x_0] \subseteq \{x_0\}$. We have proved that $\{x_0\} = \cap_{n=1}^{+\infty}]x_0 - 1/n, x_0]$.

3. We have $\{x_0\} = (]-\infty, x_0[\cup]x_0, +\infty[)^c$. Open intervals being open sets for the usual topology on $\mathbf{R}$, they are also Borel sets. A σ-algebra being closed under finite union and complementation, we conclude that $\{x_0\} \in \mathcal{B}(\mathbf{R})$.

4. Given $n \geq 1$, let $A_n =]x_0 - 1/n, x_0]$. Since $A_{n+1} \subseteq A_n$, from 2., we have $A_n \downarrow \{x_0\}$. Also, $dF(A_1) = F(x_0) - F(x_0 - 1) \in \mathbf{R}$. In particular, $dF(A_1) < +\infty$. Applying theorem (8), we obtain:

$$dF(\{x_0\}) = \lim_{n \rightarrow +\infty} dF(A_n) = F(x_0) - F(x_0-)$$

Exercise 13

Exercise 14.

1. $]a,b] =]a,+\infty[\cap (]b,+\infty[)^c$. Open intervals being Borel sets, and a σ-algebra being closed under finite intersection and complementation, we have $]a,b] \in \mathcal{B}(\mathbf{R})$. In virtue of definition (20), $dF(]a,b]) = F(b) - F(a)$.

2. $[a,b] = (]-\infty,a[\cup]b,+\infty[)^c$ and is therefore a Borel set. Moreover, using exercise (13):

$$dF([a,b]) = dF(\{a\}) + dF(]a,b]) = F(b) - F(a-)$$

3. $]a,b[$ being open is a Borel set. Moreover, using exercise (13):

$$dF(]a,b[) = dF(]a,b]) - dF(\{b\}) = F(b-) - F(a)$$

4. $[a,b[=]-\infty,b[\cap (]-\infty,a[)^c$ and is therefore a Borel set. Moreover, using exercise (13):

$$dF([a,b[) = dF(\{a\}) + dF(]a,b]) - dF(\{b\}) = F(b-) - F(a-)$$

Exercise 14

Exercise 15.

1. Suppose $\mathcal{A}$ is a topology on Ω. Then $\emptyset$ and Ω are elements of $\mathcal{A}$. It follows that that $\emptyset \cap \Omega' = \emptyset$ and $\Omega \cap \Omega' = \Omega'$ are both elements of $\mathcal{A}_{|\Omega'}$. So (i) of definition (13) is satisfied for $\mathcal{A}_{|\Omega'}$. Let $A', B' \in \mathcal{A}_{|\Omega'}$. There exist $A, B \in \mathcal{A}$ such that $A' = A \cap \Omega'$ and $B' = B \cap \Omega'$. Hence, $A' \cap B' = (A \cap B) \cap \Omega'$. Since $\mathcal{A}$ is a topology, $A \cap B \in \mathcal{A}$. It follows that $A' \cap B' \in \mathcal{A}_{|\Omega'}$, and (ii) of definition (13) is satisfied for $\mathcal{A}_{|\Omega'}$. Let $(A'_i)_{i\in I}$ be a family of elements of $\mathcal{A}_{|\Omega'}$. There exists a family $(A_i)_{i\in I}$ of elements of $\mathcal{A}$, such that $A'_i = A_i \cap \Omega'$, for all $i \in I$. In particular, $\cup_{i\in I} A'_i = (\cup_{i\in I} A_i) \cap \Omega'$. Since $\mathcal{A}$ is a topology, $\cup_{i\in I} A_i \in \mathcal{A}$. It follows that $\cup_{i\in I} A'_i \in \mathcal{A}_{|\Omega'}$ and (iii) of definition (13) is satisfied for $\mathcal{A}_{|\Omega'}$. We have proved that $\mathcal{A}_{|\Omega'}$ is indeed a topology on Ω'.

2. Suppose $\mathcal{A}$ is a σ-algebra on Ω. Then $\Omega \in \mathcal{A}$, and we have $\Omega' = \Omega \cap \Omega' \in \mathcal{A}_{|\Omega'}$. Let $A' \in \mathcal{A}_{|\Omega'}$. There exists $A \in \mathcal{A}$ such that $A' = A \cap \Omega'$. Hence[6], $\Omega' \setminus A' = \Omega' \cap (A')^c = \Omega' \cap A^c$. Since

[6] The notation $(A')^c$ refers to the complement of A' in Ω, i.e. $(A')^c = \Omega \setminus A'$.

$\mathcal{A}$ is a σ-algebra, $A^c \in \mathcal{A}$. It follows that $\Omega' \setminus A' \in \mathcal{A}_{|\Omega'}$, and $\mathcal{A}_{|\Omega'}$ is closed under complementation in Ω'. let $(A'_n)_{n\geq 1}$ be a sequence of elements of $\mathcal{A}_{|\Omega'}$. There exists a sequence $(A_n)_{n\geq 1}$ of elements of $\mathcal{A}$, such that $A'_n = A_n \cap \Omega'$, for all $n \geq 1$. In particular, $\cup_{n=1}^{+\infty} A'_n = (\cup_{n=1}^{+\infty} A_n) \cap \Omega'$. Since $\mathcal{A}$ is a σ-algebra, $\cup_{n=1}^{+\infty} A_n \in \mathcal{A}$. It follows that $\cup_{n=1}^{+\infty} A'_n \in \mathcal{A}_{|\Omega'}$, and $\mathcal{A}_{|\Omega'}$ is closed under countable union. We have proved that $\mathcal{A}_{|\Omega'}$ is indeed a σ-algebra on Ω'.

Exercise 15

The complement of A' in Ω' is denoted $\Omega' \setminus A'$.

Exercise 16.

1. When working in the context of two reference sets Ω' and Ω where $\Omega' \subseteq \Omega$, given $A \subseteq \Omega'$, the notation A^c and the notion of complementation can be confusing: does it refer to the complement of A in Ω, or the complement of A in Ω'... Unless otherwise specified, it is customary to keep the notation A^c for the complement of A *relative to the large set* ($A^c = \Omega \setminus A$). The complement of A relative to the *smaller set* Ω' can still be denoted $\Omega' \setminus A$. Similarly, whenever $\mathcal{A}'$ is a set of subsets of Ω' (like $\mathcal{A}_{|\Omega'}$), then it is also a set of subsets of Ω. Hence, a notation such as $\sigma(\mathcal{A}')$ can be ambiguous and confusing. One the one hand, $\sigma(\mathcal{A}')$ could be referring to the σ-algebra generated by $\mathcal{A}'$ on Ω. One the other hand, $\sigma(\mathcal{A}')$ could be referring to the σ-algebra generated by $\mathcal{A}'$ on Ω'. Hence, it is very important to specify clearly what is meant, when using a notation such as $\sigma(\mathcal{A}')$. In this exercise, $\sigma(\mathcal{A})$ is a σ-algebra on Ω, whereas $\sigma(\mathcal{A}_{|\Omega'})$ is a σ-algebra on Ω'.

2. Let $A \in \mathcal{A}$. Then $A \in \sigma(\mathcal{A})$ and $A \cap \Omega' \in \mathcal{A}_{|\Omega'} \subseteq \sigma(\mathcal{A}_{|\Omega'})$. It follows that $A \in \Gamma$, and $\mathcal{A} \subseteq \Gamma$.

3. $\sigma(\mathcal{A})$ being a σ-algebra on Ω, $\Omega \in \sigma(\mathcal{A})$. $\sigma(\mathcal{A}_{|\Omega'})$ being a σ-algebra on Ω', $\Omega \cap \Omega' = \Omega' \in \sigma(\mathcal{A}_{|\Omega'})$. It follows that $\Omega \in \Gamma$. Let $A \in \Gamma$. Then $A \in \sigma(\mathcal{A})$ and $A \cap \Omega' \in \sigma(\mathcal{A}_{|\Omega'})$. Hence, $A^c \in \sigma(\mathcal{A})$ and $A^c \cap \Omega' = \Omega' \setminus (A \cap \Omega') \in \sigma(\mathcal{A}_{|\Omega'})$. So $A^c \in \Gamma$. It follows that Γ is closed under complementation. Let $(A_n)_{n\geq 1}$ be a sequence of elements of Γ. Then for all $n \geq 1$, $A_n \in \sigma(\mathcal{A})$ and $A_n \cap \Omega' \in \sigma(\mathcal{A}_{|\Omega'})$. It follows that $\cup_{n=1}^{+\infty} A_n \in \sigma(\mathcal{A})$, and $(\cup_{n=1}^{+\infty} A_n) \cap \Omega' = \cup_{n=1}^{+\infty}(A_n \cap \Omega') \in \sigma(\mathcal{A}_{|\Omega'})$. So $\cup_{n=1}^{+\infty} A_n \in \Gamma$. It follows that Γ is closed under countable union. We have proved that Γ is indeed a σ-algebra on Ω.

4. The σ-algebra $\sigma(\mathcal{A})$ on Ω generated by $\mathcal{A}$, being the smallest σ-algebra on Ω containing $\mathcal{A}$, from $\mathcal{A} \subseteq \Gamma$, and the fact that Γ is σ-algebra on Ω, we have $\sigma(\mathcal{A}) \subseteq \Gamma$. In particular, for all $A \in \sigma(\mathcal{A})$, we have $A \cap \Omega' \in \sigma(\mathcal{A}_{|\Omega'})$. Hence, we see that $\sigma(\mathcal{A})_{|\Omega'} \subseteq \sigma(\mathcal{A}_{|\Omega'})$. However, for all $A \in \mathcal{A}$, since $A \in \sigma(\mathcal{A})$,

we have $A \cap \Omega' \in \sigma(\mathcal{A})_{|\Omega'}$. It follows that $\mathcal{A}_{|\Omega'} \subseteq \sigma(\mathcal{A})_{|\Omega'}$. From exercise (15), $\sigma(\mathcal{A})_{|\Omega'}$ is a σ-algebra on Ω'. The σ-algebra $\sigma(\mathcal{A}_{|\Omega'})$ being the smallest σ-algebra on Ω' containing $\mathcal{A}_{|\Omega'}$, we conclude that $\sigma(\mathcal{A}_{|\Omega'}) \subseteq \sigma(\mathcal{A})_{|\Omega'}$. We have proved that $\sigma(\mathcal{A}_{|\Omega'}) = \sigma(\mathcal{A})_{|\Omega'}$. The purpose of this exercise is to prove theorem (10).

Exercise 16

Exercise 17.

1. From theorem (10), $\mathcal{B}(\Omega)_{|\Omega'} = \sigma(\mathcal{T})_{|\Omega'} = \sigma(\mathcal{T}_{|\Omega'}) = \mathcal{B}(\Omega')$.

2. Suppose $\Omega' \in \mathcal{B}(\Omega)$. Let $A' \in \mathcal{B}(\Omega')$. Since $\mathcal{B}(\Omega') = \mathcal{B}(\Omega)_{|\Omega'}$, there exists $A \in \mathcal{B}(\Omega)$ such that $A' = A \cap \Omega'$. A σ-algebra being closed under finite intersection, it follows that $A' \in \mathcal{B}(\Omega)$. We have proved that $\mathcal{B}(\Omega') \subseteq \mathcal{B}(\Omega)$.

3. From 1., we have $\mathcal{B}(\mathbf{R}^+) = \mathcal{B}(\mathbf{R})_{|\mathbf{R}^+} = \{A \cap \mathbf{R}^+ \,,\, A \in \mathcal{B}(\mathbf{R})\}$

4. Since $\mathbf{R}^+ =]-\infty, 0[^c \in \mathcal{B}(\mathbf{R})$, from 2. we have $\mathcal{B}(\mathbf{R}^+) \subseteq \mathcal{B}(\mathbf{R})$.

Exercise 17

Exercise 18.

1. From exercise (15), $\mathcal{F}$ being a σ-algebra on Ω, $\mathcal{F}_{|\Omega'}$ is a σ-algebra on Ω'. from definition (18), it follows that $(\Omega', \mathcal{F}_{|\Omega'})$ is a measurable space.

2. Suppose $\Omega' \in \mathcal{F}$. A σ-algebra being closed under finite intersection, $\mathcal{F}_{|\Omega'} = \{A \cap \Omega', A \in \mathcal{F}\} \subseteq \mathcal{F}$.

3. If $\Omega' \in \mathcal{F}$, from 2., $\mathcal{F}_{|\Omega'} \subseteq \mathcal{F}$. Hence, it is legitimate to consider the restriction $\mu_{|(\mathcal{F}_{|\Omega'})}$ of the map $\mu : \mathcal{F} \to [0, +\infty]$ to the smaller domain $\mathcal{F}_{|\Omega'}$. Denoting such restriction by $\mu_{|\Omega'}$, it is clearly a measure on $\mathcal{F}_{|\Omega'}$ (definition (9)). From definition (19), it follows that $(\Omega', \mathcal{F}_{|\Omega'}, \mu_{|\Omega'})$ is a measure space.

Exercise 18

Exercise 19.

1. Let $x_0 \in \mathbf{R}$. If $x_0 < 0$, then $\bar{F}(x) \to 0 = \bar{F}(x_0)$ as $x \to x_0$. If $x_0 \geq 0$, since F is right-continuous, we have:

$$\lim_{x_0 < x, x \to x_0} \bar{F}(x) = \lim_{x_0 < x, x \to x_0} F(x) = F(x_0) = \bar{F}(x_0)$$

   Hence we see that $\bar{F}$ is itself right-continuous. Let $a \leq b$. If $0 \leq a \leq b$, then $\bar{F}(a) = F(a) \leq F(b) = \bar{F}(b)$. If $a < 0 \leq b$, then $\bar{F}(a) = 0 \leq F(0) \leq F(b) = \bar{F}(b)$. If $a \leq b < 0$, then $\bar{F}(a) = 0 = \bar{F}(b)$. In any case, $\bar{F}(a) \leq \bar{F}(b)$ and $\bar{F}$ is non-decreasing.

2. $\mathcal{B}(\mathbf{R}^+) \subseteq \mathcal{B}(\mathbf{R})$ and μ is well-defined. Using exercise (13):

$$\mu(\{0\}) = d\bar{F}(\{0\}) = \bar{F}(0) - \bar{F}(0-) = F(0)$$

   Moreover, for all $0 \leq a \leq b$:

$$\mu(]a, b]) = d\bar{F}(]a, b]) = \bar{F}(b) - \bar{F}(a) = F(b) - F(a)$$

Exercise 19

Exercise 20.

1. For all $0 \leq a \leq b$, $]a,b] =]a,b] \cap \mathbf{R}^+ \in \mathcal{B}(\mathbf{R})_{|\mathbf{R}^+} = \mathcal{B}(\mathbf{R}^+)$. Moreover, we have $\{0\} =]-1,0] \cap \mathbf{R}^+ \in \mathcal{B}(\mathbf{R}^+)$. we have proved that $\mathcal{C} \subseteq \mathcal{B}(\mathbf{R}^+)$.

2. Let U be open in $\mathbf{R}^+$. By definition (23), there exists V open in $\mathbf{R}$, such that $U = V \cap \mathbf{R}^+$. For all $x \in V$, there exists $\epsilon_x > 0$ such that $]x - \epsilon_x, x + \epsilon_x[\subseteq V$. The set of rational numbers $\mathbf{Q}$ being dense in $\mathbf{R}$, we can choose $p_x \in \mathbf{Q} \cap]x - \epsilon_x, x[$ and $q_x \in \mathbf{Q} \cap]x, x + \epsilon_x[$. We have $x \in]p_x, q_x] \subseteq V$. If we define $I = \{]p_x, q_x], x \in V\}$, then I is a countable set (see exercise (7) for more details). For all $i \in I$, let $a_i = p_x$ and $b_i = q_x$, where $x \in V$ is such that $i =]p_x, q_x]$. From $V = \cup_{x \in V}]p_x, q_x]$, we obtain $V = \cup_{i \in I}]a_i, b_i]$, and finally $U = \cup_{i \in I} (\mathbf{R}^+ \cap]a_i, b_i])$.

3. If $0 \leq a_i \leq b_i$, then $\mathbf{R}^+ \cap]a_i, b_i] =]a_i, b_i] \in \mathcal{C}$. If $a_i < 0 \leq b_i$, then $\mathbf{R}^+ \cap]a_i, b_i] = [0, b_i] = \{0\} \cup]0, b_i] \in \sigma(\mathcal{C})$. If $a_i \leq b_i < 0$, then $\mathbf{R}^+ \cap]a_i, b_i] = \emptyset =]1,1] \in \mathcal{C}$. In any case, $\mathbf{R}^+ \cap]a_i, b_i] \in \sigma(\mathcal{C})$.

4. From 2. and 3., the set I being countable, we have:

$$U = \cup_{i\in I}(\mathbf{R}^+ \cap]a_i, b_i]) \in \sigma(\mathcal{C})$$

This being true for all U open in $\mathbf{R}^+$, we have $\mathcal{T}_{\mathbf{R}^+} \subseteq \sigma(\mathcal{C})$. $\mathcal{B}(\mathbf{R}^+)$ being the smallest σ-algebra on $\mathbf{R}^+$ containing $\mathcal{T}_{\mathbf{R}^+}$, we obtain that $\mathcal{B}(\mathbf{R}^+) \subseteq \sigma(\mathcal{C})$. However from 1., $\mathcal{C} \subseteq \mathcal{B}(\mathbf{R}^+)$. $\sigma(\mathcal{C})$ being the smallest σ-algebra on $\mathbf{R}^+$ containing $\mathcal{C}$, we have $\sigma(\mathcal{C}) \subseteq \mathcal{B}(\mathbf{R}^+)$. We have proved that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbf{R}^+)$.

Exercise 20

Exercise 21.

1. $\mu_1(\{0\} \cap [0, n]) = \mu_1(\{0\}) = \mu_2(\{0\}) = \mu_2(\{0\} \cap [0, n])$. So $\{0\} \in \mathcal{D}_n$. For all $0 \leq a \leq b$, $]a, b] \cap [0, n]$ is either empty, or is an interval of the form $]a', b']$ with $0 \leq a' \leq b'$. In any case, $\mu_1(]a, b] \cap [0, n]) = \mu_2(]a, b] \cap [0, n])$. It follows that $\mathcal{C} \subseteq \mathcal{D}_n$. Since $\mu_1([0, n]) = \mu_1(\{0\}) + \mu_1(]0, n]) = F(n) = \mu_2([0, n])$, we have $\mathbf{R}^+ \in \mathcal{D}_n$ and both $\mu_1([0, n])$ and $\mu_2([0, n])$ are finite. Let $A, B \in \mathcal{D}_n$ with $A \subseteq B$. We have:

$$\mu_1(A \cap [0, n]) = \mu_2(A \cap [0, n])$$

$$\mu_1(B \cap [0, n]) = \mu_2(B \cap [0, n])$$

and for $i = 1, 2$:

$$\mu_i(B \cap [0, n]) = \mu_i(A \cap [0, n]) + \mu_i((B \setminus A) \cap [0, n])$$

All terms being finite, we obtain:

$$\mu_1((B \setminus A) \cap [0, n]) = \mu_2((B \setminus A) \cap [0, n])$$

and it follows that $B \setminus A \in \mathcal{D}_n$. Let $(A_p)_{p\geq 1}$ be a sequence of elements of $\mathcal{D}_n$, with $A_p \uparrow A$. Then $A_p \cap [0,n] \uparrow A \cap [0,n]$. For all $p \geq 1$, we have:

$$\mu_1(A_p \cap [0,n]) = \mu_2(A_p \cap [0,n])$$

Using theorem (7), taking the limit as $p \to +\infty$, we obtain:

$$\mu_1(A \cap [0,n]) = \mu_2(A \cap [0,n])$$

and it follows that $A \in \mathcal{D}_n$. We have proved that $\mathcal{D}_n$ is a Dynkin system on $\mathbf{R}^+$ (definition (1)) with $\mathcal{C} \subseteq \mathcal{D}_n$.

2. $\mu_1([0,n]) < +\infty$ and $\mu_2([0,n]) < +\infty$ is important in ensuring that the algebra required to prove that $B \setminus A \in \mathcal{D}_n$, is indeed meaningful.

3. Let $0 \leq a \leq b$. Then $\{0\} \cap]a,b] = \emptyset =]1,1] \in \mathcal{C}$. If $0 \leq c \leq d$, then $]a,b] \cap]c,d]$ can still be written as $]a',b']$ with $0 \leq a' \leq b'$, and therefore lies in $\mathcal{C}$. It follows that $\mathcal{C}$ is closed under finite intersection. Since $\mathcal{D}_n$ is a Dynkin system on $\mathbf{R}^+$ such that

$\mathcal{C} \subseteq \mathcal{D}_n$, using theorem (1), we see that $\sigma(\mathcal{C}) \subseteq \mathcal{D}_n$. However, from exercise (20), $\sigma(\mathcal{C}) = \mathcal{B}(\mathbf{R}^+)$. It follows that $\mathcal{B}(\mathbf{R}^+) \subseteq \mathcal{D}_n$. Hence, for all $A \in \mathcal{B}(\mathbf{R}^+)$, we have $\mu_1(A\cap[0,n]) = \mu_2(A\cap[0,n])$. Since $A \cap [0,n] \uparrow A$ as $n \to +\infty$, using theorem (7), we obtain $\mu_1(A) = \mu_2(A)$. This being true for all Borel set $A \in \mathcal{B}(\mathbf{R}^+)$, we have proved that $\mu_1 = \mu_2$.

4. Existence follows from exercise (19). Uniqueness is a consequence of 3.

Exercise 21