10. Bounded Linear Functionals in L^2

In the following, $(\Omega, \mathcal{F}, \mu)$ is a measure space.

Definition 78 *We call* **subsequence** *of a sequence* $(x_n)_{n\geq 1}$*, any sequence of the form* $(x_{\phi(n)})_{n\geq 1}$ *where* $\phi : \mathbf{N}^* \to \mathbf{N}^*$ *is a strictly increasing map.*

Exercise 1. Let (E, d) be a metric space, with metric topology $\mathcal{T}$. Let $(x_n)_{n\geq 1}$ be a sequence in E. For all $n \geq 1$, let F_n be the closure of the set $\{x_k : k \geq n\}$.

1. Show that for all $x \in E$, $x_n \xrightarrow{\mathcal{T}} x$ is equivalent to:

$$\forall \epsilon > 0 \ , \ \exists n_0 \geq 1 \ , \ n \geq n_0 \ \Rightarrow \ d(x_n, x) \leq \epsilon$$

2. Show that $(F_n)_{n\geq 1}$ is a decreasing sequence of closed sets in E.

3. Show that if $F_n \downarrow \emptyset$, then $(F_n^c)_{n\geq 1}$ is an open covering of E.

4. Show that if $(E, \mathcal{T})$ is compact then $\cap_{n=1}^{+\infty} F_n \neq \emptyset$.

5. Show that if $(E, \mathcal{T})$ is compact, there exists $x \in E$ such that for all $n \geq 1$ and $\epsilon > 0$, we have $B(x, \epsilon) \cap \{x_k \ , \ k \geq n\} \neq \emptyset$.

6. By induction, construct a subsequence $(x_{n_p})_{p\geq 1}$ of $(x_n)_{n\geq 1}$ such that $x_{n_p} \in B(x, 1/p)$ for all $p \geq 1$.

7. Conclude that if $(E, \mathcal{T})$ is compact, any sequence $(x_n)_{n\geq 1}$ in E has a convergent subsequence.

Exercise 2. Let (E, d) be a metric space, with metric topology $\mathcal{T}$. We assume that any sequence $(x_n)_{n\geq 1}$ in E has a convergent subsequence. Let $(V_i)_{i\in I}$ be an open covering of E. For $x \in E$, let:

$$r(x) \stackrel{\triangle}{=} \sup\{r > 0 \ : \ B(x, r) \subseteq V_i \ , \ \text{for some } i \in I\}$$

1. Show that $\forall x \in E$, $\exists i \in I$, $\exists r > 0$, such that $B(x, r) \subseteq V_i$.

2. Show that $\forall x \in E$, $r(x) > 0$.

Exercise 3. Further to ex. (2), suppose $\inf_{x\in E} r(x) = 0$.

1. Show that for all $n \geq 1$, there is $x_n \in E$ such that $r(x_n) < 1/n$.
2. Extract a subsequence $(x_{n_k})_{k\geq 1}$ of $(x_n)_{n\geq 1}$ converging to some $x^* \in E$. Let $r^* > 0$ and $i \in I$ be such that $B(x^*, r^*) \subseteq V_i$. Show that we can find some $k_0 \geq 1$, such that $d(x^*, x_{n_{k_0}}) < r^*/2$ and $r(x_{n_{k_0}}) \leq r^*/4$.
3. Show that $d(x^*, x_{n_{k_0}}) < r^*/2$ implies that $B(x_{n_{k_0}}, r^*/2) \subseteq V_i$. Show that this contradicts $r(x_{n_{k_0}}) \leq r^*/4$, and conclude that $\inf_{x\in E} r(x) > 0$.

Exercise 4. Further to ex. (3), Let r_0 with $0 < r_0 < \inf_{x\in E} r(x)$. Suppose that E cannot be covered by a finite number of open balls with radius r_0.

1. Show the existence of a sequence $(x_n)_{n\geq 1}$ in E, such that for all $n \geq 1$, $x_{n+1} \notin B(x_1, r_0) \cup \ldots \cup B(x_n, r_0)$.

2. Show that for all $n > m$ we have $d(x_n, x_m) \geq r_0$.

3. Show that $(x_n)_{n\geq 1}$ cannot have a convergent subsequence.

4. Conclude that there exists a finite subset $\{x_1, \ldots, x_n\}$ of E such that $E = B(x_1, r_0) \cup \ldots \cup B(x_n, r_0)$.

5. Show that for all $x \in E$, we have $B(x, r_0) \subseteq V_i$ for some $i \in I$.

6. Conclude that $(E, \mathcal{T})$ is compact.

7. Prove the following:

Theorem 47 *A metrizable topological space $(E, \mathcal{T})$ is compact, if and only if for every sequence $(x_n)_{n\geq 1}$ in E, there exists a subsequence $(x_{n_k})_{k\geq 1}$ of $(x_n)_{n\geq 1}$ and some $x \in E$, such that $x_{n_k} \xrightarrow{\mathcal{T}} x$.*

Exercise 5. Let $a,b\in\mathbf{R}$, $a<b$ and $(x_n)_{n\geq 1}$ be a sequence in $]a,b[$.

1. Show that $(x_n)_{n\geq 1}$ has a convergent subsequence.
2. Can we conclude that $]a,b[$ is a compact subset of $\mathbf{R}$?

Exercise 6. Let $E=[-M,M]\times\ldots\times[-M,M]\subseteq\mathbf{R}^n$, where $n\geq 1$ and $M\in\mathbf{R}^+$. Let $\mathcal{T}_{\mathbf{R}^n}$ be the usual product topology on $\mathbf{R}^n$, and $\mathcal{T}_E=(\mathcal{T}_{\mathbf{R}^n})_{|E}$ be the induced topology on E.

1. Let $(x_p)_{p\geq 1}$ be a sequence in E. Let $x\in E$. Show that $x_p\stackrel{\mathcal{T}_E}{\to}x$ is equivalent to $x_p\stackrel{\mathcal{T}_{\mathbf{R}^n}}{\to}x$.
2. Propose a metric on $\mathbf{R}^n$, inducing the topology $\mathcal{T}_{\mathbf{R}^n}$.
3. Let $(x_p)_{p\geq 1}$ be a sequence in $\mathbf{R}^n$. Let $x\in\mathbf{R}^n$. Show that $x_p\stackrel{\mathcal{T}_{\mathbf{R}^n}}{\to}x$ if and only if, $x_p^i\stackrel{\mathcal{T}_{\mathbf{R}}}{\to}x^i$ for all $i\in\mathbf{N}_n$.

Exercise 7. Further to ex. (6), suppose $(x_p)_{p\geq 1}$ is a sequence in E.

1. Show the existence of a subsequence $(x_{\phi(p)})_{p\geq 1}$ of $(x_p)_{p\geq 1}$, such that $x^1_{\phi(p)} \stackrel{\mathcal{T}_{[-M,M]}}{\rightarrow} x^1$ for some $x^1 \in [-M,M]$.

2. Explain why the above convergence is equivalent to $x^1_{\phi(p)} \stackrel{\mathcal{T}_{\mathbf{R}}}{\rightarrow} x^1$.

3. Suppose that $1 \leq k \leq n-1$ and $(y_p)_{p\geq 1} = (x_{\phi(p)})_{p\geq 1}$ is a subsequence of $(x_p)_{p\geq 1}$ such that:

$$\forall j = 1,\ldots,k \ ,\ x^j_{\phi(p)} \stackrel{\mathcal{T}_{\mathbf{R}}}{\rightarrow} x^j \text{ for some } x^j \in [-M,M]$$

   Show the existence of a subsequence $(y_{\psi(p)})_{p\geq 1}$ of $(y_p)_{p\geq 1}$ such that $y^{k+1}_{\psi(p)} \stackrel{\mathcal{T}_{\mathbf{R}}}{\rightarrow} x^{k+1}$ for some $x^{k+1} \in [-M,M]$.

4. Show that $\phi\circ\psi : \mathbf{N}^* \rightarrow \mathbf{N}^*$ is strictly increasing.

5. Show that $(x_{\phi\circ\psi(p)})_{p\geq 1}$ is a subsequence of $(x_p)_{p\geq 1}$ such that:

$$\forall j = 1,\ldots,k+1\ ,\ x^j_{\phi\circ\psi(p)} \stackrel{\mathcal{T}_{\mathbf{R}}}{\to} x^j \in [-M,M]$$

6. Show the existence of a subsequence $(x_{\phi(p)})_{p\geq 1}$ of $(x_p)_{p\geq 1}$, and $x\in E$, such that $x_{\phi(p)} \stackrel{\mathcal{T}_E}{\to} x$

7. Show that $(E,\mathcal{T}_E)$ is a compact topological space.

EXERCISE 8. Let A be a closed subset of $\mathbf{R}^n$, $n\geq 1$, which is bounded with respect to the usual metric of $\mathbf{R}^n$.

1. Show that $A\subseteq E = [-M,M]\times\ldots\times[-M,M]$, for some $M\in\mathbf{R}^+$.

2. Show from $E\setminus A = E\cap A^c$ that A is closed in E.

3. Show $(A,(\mathcal{T}_{\mathbf{R}^n})_{|A})$ is a compact topological space.

4. Conversely, let A is a compact subset of $\mathbf{R}^n$. Show that A is closed and bounded.

Theorem 48 *A subset of $\mathbf{R}^n$ is compact if and only if it is closed and bounded with respect to its usual metric.*

Exercise 9. Let $n \geq 1$. Consider the map:

$$\phi : \left\{ \begin{array}{ccc} \mathbf{C}^n & \to & \mathbf{R}^{2n} \\ (a_1 + ib_1, \ldots, a_n + ib_n) & \to & (a_1, b_1, \ldots, a_n, b_n) \end{array} \right.$$

1. Recall the expressions of the usual metrics $d_{\mathbf{C}^n}$ and $d_{\mathbf{R}^{2n}}$ of $\mathbf{C}^n$ and $\mathbf{R}^{2n}$ respectively.
2. Show that for all $z, z' \in \mathbf{C}^n$, $d_{\mathbf{C}^n}(z, z') = d_{\mathbf{R}^{2n}}(\phi(z), \phi(z'))$.
3. Show that ϕ is a homeomorphism from $\mathbf{C}^n$ to $\mathbf{R}^{2n}$.

4. Show that a subset K of $\mathbf{C}^n$ is compact, if and only if $\phi(K)$ is a compact subset of $\mathbf{R}^{2n}$.

5. Show that K is closed, if and only if $\phi(K)$ is closed.

6. Show that K is bounded, if and only if $\phi(K)$ is bounded.

7. Show that a subset K of $\mathbf{C}^n$ is compact, if and only if it is closed and bounded with respect to its usual metric.

Definition 79 *Let (E, d) be a metric space. A sequence $(x_n)_{n\geq 1}$ in E is said to be a* **Cauchy sequence** *with respect to the metric d, if and only if for all $\epsilon > 0$, there exists $n_0 \geq 1$ such that:*

$$n, m \geq n_0 \;\Rightarrow\; d(x_n, x_m) \leq \epsilon$$

Definition 80 *We say that a metric space (E, d) is* **complete**, *if and only if for any Cauchy sequence $(x_n)_{n\geq 1}$ in E, there exists $x \in E$ such that $(x_n)_{n\geq 1}$ converges to x.*

Exercise 10.

1. Explain why strictly speaking, given $p \in [1, +\infty]$, definition (77) of Cauchy sequences in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is not a covered by definition (79).

2. Explain why $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is not a complete metric space, despite theorem (46) and definition (80).

Exercise 11. Let $(z_k)_{k\geq 1}$ be a Cauchy sequence in $\mathbf{C}^n$, $n \geq 1$, with respect to the usual metric $d(z, z') = \|z - z'\|$, where:

$$\|z\| \stackrel{\triangle}{=} \sqrt{\sum_{i=1}^{n} |z_i|^2}$$

1. Show that the sequence $(z_k)_{k\geq 1}$ is bounded, i.e. that there exists $M \in \mathbf{R}^+$ such that $\|z_k\| \leq M$, for all $k \geq 1$.

2. Define $B = \{z \in \mathbf{C}^n \ , \ \|z\| \leq M\}$. Show that $\delta(B) < +\infty$, and that B is closed in $\mathbf{C}^n$.

3. Show the existence of a subsequence $(z_{k_p})_{p\geq 1}$ of $(z_k)_{k\geq 1}$ such that $z_{k_p} \stackrel{\mathcal{T}_{\mathbf{C}^n}}{\rightarrow} z$ for some $z \in B$.

4. Show that for all $\epsilon > 0$, there exists $p_0 \geq 1$ and $n_0 \geq 1$ such that $d(z, z_{k_{p_0}}) \leq \epsilon/2$ and:
$$k \geq n_0 \ \Rightarrow \ d(z_k, z_{k_{p_0}}) \leq \epsilon/2$$

5. Show that $z_k \stackrel{\mathcal{T}_{\mathbf{C}^n}}{\rightarrow} z$.

6. Conclude that $\mathbf{C}^n$ is complete with respect to its usual metric.

7. For which theorem of Tutorial 9 was the completeness of $\mathbf{C}$ used?

Exercise 12. Let $(x_k)_{k\geq 1}$ be a sequence in $\mathbf{R}^n$ such that $x_k \stackrel{\mathcal{T}_{\mathbf{C}^n}}{\rightarrow} z$, for some $z \in \mathbf{C}^n$.

1. Show that $z \in \mathbf{R}^n$.
2. Show that $\mathbf{R}^n$ is complete with respect to its usual metric.

Theorem 49 *$\mathbf{C}^n$ and $\mathbf{R}^n$ are complete w.r. to their usual metrics.*

Exercise 13. Let (E, d) be a metric space, with metric topology $\mathcal{T}$. Let $F \subseteq E$, and $\bar{F}$ denote the closure of F.

1. Explain why, for all $x \in \bar{F}$ and $n \geq 1$, we have $F \cap B(x, 1/n) \neq \emptyset$.
2. Show that for all $x \in \bar{F}$, there exists a sequence $(x_n)_{n\geq 1}$ in F, such that $x_n \xrightarrow{\mathcal{T}} x$.
3. Show conversely that if there is a sequence $(x_n)_{n\geq 1}$ in F with $x_n \xrightarrow{\mathcal{T}} x$, then $x \in \bar{F}$.

4. Show that F is closed if and only if for all sequence $(x_n)_{n\geq 1}$ in F such that $x_n \xrightarrow{\mathcal{T}} x$ for some $x \in E$, we have $x \in F$.

5. Explain why $(F, \mathcal{T}_{|F})$ is metrizable.

6. Show that if F is complete with respect to the metric $d_{|F\times F}$, then F is closed in E.

7. Let $d_{\bar{\mathbf{R}}}$ be a metric on $\bar{\mathbf{R}}$, inducing the usual topology $\mathcal{T}_{\bar{\mathbf{R}}}$. Show that $d' = (d_{\bar{\mathbf{R}}})_{|\mathbf{R}\times\mathbf{R}}$ is a metric on $\mathbf{R}$, inducing the topology $\mathcal{T}_{\mathbf{R}}$.

8. Find a metric on $[-1, 1]$ which induces its usual topology.

9. Show that $\{-1, 1\}$ is not open in $[-1, 1]$.

10. Show that $\{-\infty, +\infty\}$ is not open in $\bar{\mathbf{R}}$.

11. Show that $\mathbf{R}$ is not closed in $\bar{\mathbf{R}}$.

12. Let $d_{\mathbf{R}}$ be the usual metric of $\mathbf{R}$. Show that $d' = (d_{\bar{\mathbf{R}}})_{|\mathbf{R}\times\mathbf{R}}$ and $d_{\mathbf{R}}$ induce the same topology on $\mathbf{R}$, but that however, $\mathbf{R}$

is complete with respect to $d_{\mathbf{R}}$, whereas it cannot be complete with respect to d'.

Definition 81 *Let $\mathcal{H}$ be a $\mathbf{K}$-vector space, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. We call* **inner-product** *on $\mathcal{H}$, any map $\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H}\to\mathbf{K}$ with the following properties:*

$$\begin{array}{rl} (i) & \forall x,y\in\mathcal{H}\ ,\ \langle x,y\rangle = \overline{\langle y,x\rangle} \\ (ii) & \forall x,y,z\in\mathcal{H}\ ,\ \langle x+z,y\rangle = \langle x,y\rangle + \langle z,y\rangle \\ (iii) & \forall x,y\in\mathcal{H},\forall\alpha\in\mathbf{K}\ ,\ \langle\alpha x,y\rangle = \alpha\langle x,y\rangle \\ (iv) & \forall x\in\mathcal{H}\ ,\ \langle x,x\rangle\geq 0 \\ (v) & \forall x\in\mathcal{H}\ ,\ (\langle x,x\rangle = 0\ \Leftrightarrow\ x=0) \end{array}$$

where for all $z\in\mathbf{C}$, $\bar{z}$ denotes the complex conjugate of z. For all $x\in\mathcal{H}$, we call **norm** *of x, denoted $\|x\|$, the number defined by:*

$$\|x\| \stackrel{\triangle}{=} \sqrt{\langle x,x\rangle}$$

Exercise 14. Let $\langle\cdot,\cdot\rangle$ be an inner-product on a $\mathbf{K}$-vector space $\mathcal{H}$.

1. Show that for all $y \in \mathcal{H}$, the map $x \to \langle x, y\rangle$ is linear.
2. Show that for all $x \in \mathcal{H}$, the map $y \to \langle x, y\rangle$ is linear if $\mathbf{K} = \mathbf{R}$, and conjugate-linear if $\mathbf{K} = \mathbf{C}$.

Exercise 15. Let $\langle\cdot,\cdot\rangle$ be an inner-product on a $\mathbf{K}$-vector space $\mathcal{H}$. Let $x, y \in \mathcal{H}$. Let $A = \|x\|^2$, $B = |\langle x, y\rangle|$ and $C = \|y\|^2$. let $\alpha \in \mathbf{K}$ be such that $|\alpha| = 1$ and:

$$B = \alpha\overline{\langle x, y\rangle}$$

1. Show that $A, B, C \in \mathbf{R}^+$.
2. For all $t \in \mathbf{R}$, show that $\langle x - t\alpha y, x - t\alpha y\rangle = A - 2tB + t^2C$.
3. Show that if $C = 0$ then $B^2 \leq AC$.

4. Suppose that $C \neq 0$. Show that $P(t) = A - 2tB + t^2C$ has a minimal value which is in $\mathbf{R}^+$, and conclude that $B^2 \leq AC$.

5. Conclude with the following:

Theorem 50 (Cauchy-Schwarz's inequality:second) *Let $\mathcal{H}$ be a $\mathbf{K}$-vector space, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, and $\langle \cdot, \cdot \rangle$ be an inner-product on $\mathcal{H}$. Then, for all $x, y \in \mathcal{H}$, we have:*

$$|\langle x, y \rangle| \leq \|x\|.\|y\|$$

EXERCISE 16. For all $f, g \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we define:

$$\langle f, g \rangle \stackrel{\triangle}{=} \int_\Omega f\bar{g}d\mu$$

1. Use the first Cauchy-Schwarz inequality (42) to prove that for all $f, g \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we have $f\bar{g} \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Conclude that $\langle f, g \rangle$ is a well-defined complex number.

2. Show that for all $f \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we have $\|f\|_2 = \sqrt{\langle f, f \rangle}$.

3. Make another use of the first Cauchy-Schwarz inequality to show that for all $f, g \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we have:

$$|\langle f, g \rangle| \leq \|f\|_2 . \|g\|_2 \tag{1}$$

4. Go through definition (81), and indicate which of the properties $(i) - (v)$ fails to be satisfied by $\langle \cdot, \cdot \rangle$. Conclude that $\langle \cdot, \cdot \rangle$ is not an inner-product on $L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, and therefore that inequality (*) is not a particular case of the second Cauchy-Schwarz inequality (50).

5. Let $f, g \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. By considering $\int (|f| + t|g|)^2 d\mu$ for $t \in \mathbf{R}$, imitate the proof of the second Cauchy-Schwarz inequality to show that:

$$\int_\Omega |fg| d\mu \leq \left(\int_\Omega |f|^2 d\mu \right)^{\frac{1}{2}} \left(\int_\Omega |g|^2 d\mu \right)^{\frac{1}{2}}$$

6. Let $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ non-negative and measurable. Show that if $\int f^2 d\mu$ and $\int g^2 d\mu$ are finite, then f and g are μ-almost surely equal to elements of $L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Deduce from 5. a new proof of the first Cauchy-Schwarz inequality:

$$\int_\Omega fg d\mu \leq \left(\int_\Omega f^2 d\mu\right)^{\frac{1}{2}} \left(\int_\Omega g^2 d\mu\right)^{\frac{1}{2}}$$

Exercise 17. Let $\langle \cdot, \cdot \rangle$ be an inner-product on a $\mathbf{K}$-vector space $\mathcal{H}$.

1. Show that for all $x, y \in \mathcal{H}$, we have:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + \langle x, y \rangle + \overline{\langle x, y \rangle}$$

2. Using the second Cauchy-Schwarz inequality (50), show that:

$$\|x+y\| \leq \|x\| + \|y\|$$

3. Show that $d_{\langle \cdot, \cdot \rangle}(x, y) = \|x - y\|$ defines a metric on $\mathcal{H}$.

Definition 82 *Let $\mathcal{H}$ be a $\mathbf{K}$-vector space, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, and $\langle \cdot, \cdot \rangle$ be an inner-product on $\mathcal{H}$. We call* **norm topology** *on $\mathcal{H}$, denoted $\mathcal{T}_{\langle \cdot, \cdot \rangle}$, the metric topology associated with $d_{\langle \cdot, \cdot \rangle}(x, y) = \|x - y\|$.*

Definition 83 *We call* **Hilbert space** *over $\mathbf{K}$ where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, any ordered pair $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ where $\langle \cdot, \cdot \rangle$ is an inner-product on a $\mathbf{K}$-vector space $\mathcal{H}$, which is complete w.r. to $d_{\langle \cdot, \cdot \rangle}(x, y) = \|x - y\|$.*

Exercise 18. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbf{K}$ and let $\mathcal{M}$ be a closed linear subspace of $\mathcal{H}$, (closed with respect to the norm topology $\mathcal{T}_{\langle \cdot, \cdot \rangle}$). Define $[\cdot, \cdot] = \langle \cdot, \cdot \rangle_{|\mathcal{M} \times \mathcal{M}}$.

1. Show that $[\cdot, \cdot]$ is an inner-product on the $\mathbf{K}$-vector space $\mathcal{M}$.
2. With obvious notations, show that $d_{[\cdot, \cdot]} = (d_{\langle \cdot, \cdot \rangle})_{|\mathcal{M} \times \mathcal{M}}$.
3. Deduce that $\mathcal{T}_{[\cdot, \cdot]} = (\mathcal{T}_{\langle \cdot, \cdot \rangle})_{|\mathcal{M}}$.

Exercise 19. Further to ex. (18), Let $(x_n)_{n\geq 1}$ be a Cauchy sequence in $\mathcal{M}$, with respect to the metric $d_{[\cdot,\cdot]}$.

1. Show that $(x_n)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{H}$.
2. Explain why there exists $x \in \mathcal{H}$ such that $x_n \stackrel{\mathcal{T}_{\langle\cdot,\cdot\rangle}}{\to} x$.
3. Explain why $x \in \mathcal{M}$.
4. Explain why we also have $x_n \stackrel{\mathcal{T}_{[\cdot,\cdot]}}{\to} x$.
5. Explain why $(\mathcal{M}, \langle\cdot,\cdot\rangle_{|\mathcal{M}\times\mathcal{M}})$ is a Hilbert space over $\mathbf{K}$.

Exercise 20. For all $z, z' \in \mathbf{C}^n$, $n \geq 1$, we define:

$$\langle z, z'\rangle \stackrel{\triangle}{=} \sum_{i=1}^{n} z_i \bar{z}_i'$$

1. Show that $\langle\cdot,\cdot\rangle$ is an inner-product on $\mathbf{C}^n$.

2. Show that the metric $d_{\langle\cdot,\cdot\rangle}$ is equal to the usual metric of $\mathbf{C}^n$.

3. Conclude that $(\mathbf{C}^n, \langle\cdot,\cdot\rangle)$ is a Hilbert space over $\mathbf{C}$.

4. Show that $\mathbf{R}^n$ is a closed subset of $\mathbf{C}^n$.

5. Show however that $\mathbf{R}^n$ is not a linear subspace of $\mathbf{C}^n$.

6. Show that $(\mathbf{R}^n, \langle\cdot,\cdot\rangle_{|\mathbf{R}^n\times\mathbf{R}^n})$ is a Hilbert space over $\mathbf{R}$.

Definition 84 *We call* **usual inner-product** *in* $\mathbf{K}^n$*, where* $\mathbf{K} = \mathbf{R}$ *or* $\mathbf{C}$*, the inner-product denoted* $\langle\cdot,\cdot\rangle$ *and defined by:*

$$\forall x, y \in \mathbf{K}^n \ , \ \langle x, y\rangle = \sum_{i=1}^n x_i \bar{y}_i$$

Theorem 51 *$\mathbf{C}^n$ and $\mathbf{R}^n$ together with their usual inner-products, are Hilbert spaces over $\mathbf{C}$ and $\mathbf{R}$ respectively.*

Definition 85 *Let $\mathcal{H}$ be a $\mathbf{K}$-vector space, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let $\mathcal{C} \subseteq \mathcal{H}$. We say that $\mathcal{C}$ is a* **convex subset** *or $\mathcal{H}$, if and only if, for all $x, y \in \mathcal{C}$ and $t \in [0,1]$, we have $tx + (1-t)y \in \mathcal{C}$.*

EXERCISE 21. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbf{K}$. Let $\mathcal{C} \subseteq \mathcal{H}$ be a non-empty closed convex subset of $\mathcal{H}$. Let $x_0 \in \mathcal{H}$. Define:

$$\delta_{\min} \stackrel{\triangle}{=} \inf\{\|x - x_0\| \ : \ x \in \mathcal{C}\}$$

1. Show the existence of a sequence $(x_n)_{n\geq 1}$ in $\mathcal{C}$ such that $\|x_n - x_0\| \to \delta_{\min}$.

2. Show that for all $x, y \in \mathcal{H}$, we have:

$$\|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 - 4\left\|\frac{x+y}{2}\right\|^2$$

3. Explain why for all $n, m \geq 1$, we have:

$$\delta_{\min} \leq \left\| \frac{x_n + x_m}{2} - x_0 \right\|$$

4. Show that for all $n, m \geq 1$, we have:

$$\|x_n - x_m\|^2 \leq 2\|x_n - x_0\|^2 + 2\|x_m - x_0\|^2 - 4\delta_{\min}^2$$

5. Show the existence of some $x^* \in \mathcal{H}$, such that $x_n \stackrel{\mathcal{T}_{\langle \cdot, \cdot \rangle}}{\rightarrow} x^*$.

6. Explain why $x^* \in \mathcal{C}$

7. Show that for all $x, y \in \mathcal{H}$, we have $|\ \|x\| - \|y\|\ | \leq \|x - y\|$.

8. Show that $\|x_n - x_0\| \rightarrow \|x^* - x_0\|$.

9. Conclude that we have found $x^* \in \mathcal{C}$ such that:

$$\|x^* - x_0\| = \inf\{\|x - x_0\| \ : \ x \in \mathcal{C}\}$$

10. Let y^* be another element of $\mathcal{C}$ with such property. Show that:

$$\|x^* - y^*\|^2 \leq 2\|x^* - x_0\|^2 + 2\|y^* - x_0\|^2 - 4\delta_{\min}^2$$

11. Conclude that $x^* = y^*$.

Theorem 52 *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let $\mathcal{C}$ be a non-empty, closed and convex subset of $\mathcal{H}$. For all $x_0 \in \mathcal{H}$, there exists a unique $x^* \in \mathcal{C}$ such that:*

$$\|x^* - x_0\| = \inf\{\|x - x_0\| \ : \ x \in \mathcal{C}\}$$

Definition 86 *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let $\mathcal{G} \subseteq \mathcal{H}$. We call* **orthogonal** *of $\mathcal{G}$, the subset of $\mathcal{H}$ denoted $\mathcal{G}^\perp$ and defined by:*

$$\mathcal{G}^\perp \stackrel{\triangle}{=} \{\ x \in \mathcal{H} \ : \ \langle x, y\rangle = 0\ ,\ \forall y \in \mathcal{G}\ \}$$

Exercise 22. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbf{K}$ and $\mathcal{G} \subseteq \mathcal{H}$.

1. Show that $\mathcal{G}^\perp$ is a linear subspace of $\mathcal{H}$, even if $\mathcal{G}$ isn't.
2. Show that $\phi_y : \mathcal{H} \to K$ defined by $\phi_y(x) = \langle x, y \rangle$ is continuous.
3. Show that $\mathcal{G}^\perp = \cap_{y \in \mathcal{G}} \phi_y^{-1}(\{0\})$.
4. Show that $\mathcal{G}^\perp$ is a closed subset of $\mathcal{H}$, even if $\mathcal{G}$ isn't.
5. Show that $\emptyset^\perp = \{0\}^\perp = \mathcal{H}$.
6. Show that $\mathcal{H}^\perp = \{0\}$.

Exercise 23. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbf{K}$. Let $\mathcal{M}$ be a closed linear subspace of $\mathcal{H}$, and $x_0 \in \mathcal{H}$.

1. Explain why there exists $x^* \in \mathcal{M}$ such that:

$$\|x^* - x_0\| = \inf\{ \ \|x - x_0\| \ : \ x \in \mathcal{M} \ \}$$

2. Define $y^* = x_0 - x^* \in \mathcal{H}$. Show that for all $y \in \mathcal{M}$ and $\alpha \in \mathbf{K}$:
$$\|y^*\|^2 \leq \|y^* - \alpha y\|^2$$

3. Show that for all $y \in \mathcal{M}$ and $\alpha \in \mathbf{K}$, we have:
$$0 \leq -\alpha\langle y, y^*\rangle - \overline{\alpha\langle y, y^*\rangle} + |\alpha|^2.\|y\|^2$$

4. For all $y \in \mathcal{M} \setminus \{0\}$, taking $\alpha = \overline{\langle y, y^*\rangle}/\|y\|^2$, show that:
$$0 \leq -\frac{|\langle y, y^*\rangle|^2}{\|y\|^2}$$

5. Conclude that $x^* \in \mathcal{M}$, $y^* \in \mathcal{M}^\perp$ and $x_0 = x^* + y^*$.

6. Show that $\mathcal{M} \cap \mathcal{M}^\perp = \{0\}$

7. Show that $x^* \in \mathcal{M}$ and $y^* \in \mathcal{M}^\perp$ with $x_0 = x^* + y^*$, are unique.

Theorem 53 *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let $\mathcal{M}$ be a closed linear subspace of $\mathcal{H}$. Then, for all $x_0 \in \mathcal{H}$, there is a unique decomposition:*

$$x_0 = x^* + y^*$$

where $x^* \in \mathcal{M}$ and $y^* \in \mathcal{M}^\perp$.

Definition 87 *Let $\mathcal{H}$ be a $\mathbf{K}$-vector space, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. We call* **linear functional**, *any map $\lambda : \mathcal{H} \to \mathbf{K}$, such that for all $x, y \in \mathcal{H}$ and $\alpha \in \mathbf{K}$:*

$$\lambda(x + \alpha y) = \lambda(x) + \alpha\lambda(y)$$

EXERCISE 24. Let λ be a linear functional on a $\mathbf{K}$-Hilbert[1] space $\mathcal{H}$.

1. Suppose that λ is continuous at some point $x_0 \in \mathcal{H}$. Show the existence of $\eta > 0$ such that:

$$\forall x \in \mathcal{H}\ ,\ \|x - x_0\| \leq \eta \ \Rightarrow\ |\lambda(x) - \lambda(x_0)| \leq 1$$

[1]Norm vector spaces are introduced later in these tutorials.

Show that for all $x \in \mathcal{H}$ with $x \neq 0$, we have $|\lambda(\eta x/\|x\|)| \leq 1$.

2. Show that if λ is continuous at x_0, there exits $M \in \mathbf{R}^+$, with:

$$\forall x \in \mathcal{H}\ ,\ |\lambda(x)| \leq M\|x\| \tag{2}$$

3. Show conversely that if (2) holds, λ is continuous everywhere.

Definition 88 *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert*[2] *space over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let λ be a linear functional on $\mathcal{H}$. Then, the following are equivalent:*

$$(i) \qquad \lambda : (\mathcal{H}, \mathcal{T}_{\langle \cdot, \cdot \rangle}) \to (K, \mathcal{T}_{\mathbf{K}}) \ \text{ is continuous}$$

$$(ii) \qquad \exists M \in \mathbf{R}^+\ ,\ \forall x \in \mathcal{H}\ ,\ |\lambda(x)| \leq M.\|x\|$$

In which case, we say that λ is a **bounded linear functional**.

[2]Norm vector spaces are introduced later in these tutorials.

EXERCISE 25. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbf{K}$. Let λ be a bounded linear functional on $\mathcal{H}$, such that $\lambda(x) \neq 0$ for some $x \in \mathcal{H}$, and define $\mathcal{M} = \lambda^{-1}(\{0\})$.

1. Show the existence of $x_0 \in \mathcal{H}$, such that $x_0 \notin \mathcal{M}$.

2. Show the existence of $x^* \in \mathcal{M}$ and $y^* \in \mathcal{M}^\perp$ with $x_0 = x^* + y^*$.

3. Deduce the existence of some $z \in \mathcal{M}^\perp$ such that $\|z\| = 1$.

4. Show that for all $\alpha \in \mathbf{K} \setminus \{0\}$ and $x \in \mathcal{H}$, we have:

$$\frac{\lambda(x)}{\bar{\alpha}} \langle z, \alpha z \rangle = \lambda(x)$$

5. Show that in order to have:

$$\forall x \in \mathcal{H} \ , \ \lambda(x) = \langle x, \alpha z \rangle$$

it is sufficient to choose $\alpha \in \mathbf{K} \setminus \{0\}$ such that:

$$\forall x \in \mathcal{H} \ , \ \frac{\lambda(x) z}{\bar{\alpha}} - x \in \mathcal{M}$$

6. Show the existence of $y \in \mathcal{H}$ such that:

$$\forall x \in \mathcal{H}\ ,\ \lambda(x) = \langle x, y \rangle$$

7. Show the uniqueness of such $y \in \mathcal{H}$.

Theorem 54 *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let λ be a bounded linear functional on $\mathcal{H}$. Then, there exists a unique $y \in \mathcal{H}$ such that: $\forall x \in \mathcal{H}\ ,\ \lambda(x) = \langle x, y \rangle$.*

Definition 89 *Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. We call* **K-vector space**, *any set $\mathcal{H}$, together with operators $\oplus$ and $\otimes$ for which there exits an element $0_{\mathcal{H}} \in \mathcal{H}$ such that for all $x, y, z \in \mathcal{H}$ and $\alpha, \beta \in \mathbf{K}$, we have:*

$$\begin{array}{rl} (i) & 0_{\mathcal{H}} \oplus x = x \\ (ii) & \exists (-x) \in \mathcal{H}\ ,\ (-x) \oplus x = 0_{\mathcal{H}} \\ (iii) & x \oplus (y \oplus z) = (x \oplus y) \oplus z \end{array}$$

$$
\begin{array}{rl}
(iv) & x \oplus y = y \oplus x \\
(v) & 1 \otimes x = x \\
(vi) & \alpha \otimes (\beta \otimes x) = (\alpha\beta) \otimes x \\
(vii) & (\alpha + \beta) \otimes x = (\alpha \otimes x) \oplus (\beta \otimes x) \\
(viii) & \alpha \otimes (x \oplus y) = (\alpha \otimes x) \oplus (\alpha \otimes y)
\end{array}
$$

EXERCISE 26. For all $f \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$, define:

$$\mathcal{H} \stackrel{\triangle}{=} \{ \ [f] \ : \ f \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \ \}$$

where $[f] = \{g \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) : g = f, \mu\text{-a.s.}\}$. Let $0_{\mathcal{H}} = [0]$, and for all $[f], [g] \in \mathcal{H}$, and $\alpha \in \mathbf{K}$, we define:

$$
\begin{array}{rcl}
[f] \oplus [g] & \stackrel{\triangle}{=} & [f + g] \\
\alpha \otimes [f] & \stackrel{\triangle}{=} & [\alpha f]
\end{array}
$$

We assume f, f', g and g' are elements of $L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$.

1. Show that for $f = g$ μ-a.s. is equivalent to $[f] = [g]$.
2. Show that if $[f] = [f']$ and $[g] = [g']$, then $[f + g] = [f' + g']$.
3. Conclude that $\oplus$ is well-defined.
4. Show that $\otimes$ is also well-defined.
5. Show that $(\mathcal{H}, \oplus, \otimes)$ is a $\mathbf{K}$-vector space.

Exercise 27. Further to ex. (26), we define for all $[f], [g] \in \mathcal{H}$:

$$\langle [f], [g] \rangle_{\mathcal{H}} \stackrel{\triangle}{=} \int_\Omega f \bar{g} d\mu$$

1. Show that $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is well-defined.
2. Show that $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is an inner-product on $\mathcal{H}$.
3. Show that $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ is a Hilbert space over $\mathbf{K}$.

4. Why is $\langle f,g\rangle \stackrel{\triangle}{=} \int_\Omega f\bar{g}d\mu$ not an inner-product on $L^2_{\mathbf{K}}(\Omega,\mathcal{F},\mu)$?

Exercise 28. Further to ex. (27), Let $\lambda : L^2_{\mathbf{K}}(\Omega,\mathcal{F},\mu) \to \mathbf{K}$ be a continuous linear functional[3]. Define $\Lambda : \mathcal{H} \to \mathbf{K}$ by $\Lambda([f]) = \lambda(f)$.

1. Show the existence of $M \in \mathbf{R}^+$ such that:

$$\forall f \in L^2_{\mathbf{K}}(\Omega,\mathcal{F},\mu)\ ,\ |\lambda(f)| \leq M.\|f\|_2$$

2. Show that if $[f]=[g]$ then $\lambda(f)=\lambda(g)$.

3. Show that Λ is a well defined bounded linear functional on $\mathcal{H}$.

4. Conclude with the following:

[3] As defined in these tutorials, $L^2_{\mathbf{K}}(\Omega,\mathcal{F},\mu)$ is not a Hilbert space (not even a norm vector space). However, both $L^2_{\mathbf{K}}(\Omega,\mathcal{F},\mu)$ and $\mathbf{K}$ have natural topologies and it is therefore meaningful to speak of *continuous linear functional*. Note however that we are slightly outside the framework of definition (88).

Theorem 55 *Let* $\lambda : L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \to \mathbf{K}$ *be a continuous linear functional, where* $\mathbf{K} = \mathbf{R}$ *or* $\mathbf{C}$. *There exists* $g \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$ *such that:*

$$\forall f \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \ , \ \lambda(f) = \int_\Omega f\bar{g}d\mu$$