2. Caratheodory's Extension

In the following, Ω is a set. Whenever a union of sets is denoted $\uplus$ as opposed to $\cup$, it indicates that the sets involved are pairwise disjoint.

Definition 6 *A* **semi-ring** *on* Ω *is a subset* $\mathcal{S}$ *of the power set* $\mathcal{P}(\Omega)$ *with the following properties:*

$$(i) \qquad \emptyset \in \mathcal{S}$$

$$(ii) \qquad A, B \in \mathcal{S} \ \Rightarrow \ A \cap B \in \mathcal{S}$$

$$(iii) \qquad A, B \in \mathcal{S} \ \Rightarrow \ \exists n \geq 0,\ \exists A_i \in \mathcal{S} : \ A \setminus B = \biguplus_{i=1}^{n} A_i$$

The last property (iii) says that whenever $A, B \in \mathcal{S}$, there is $n \geq 0$ and $A_1, \ldots, A_n$ in $\mathcal{S}$ which are pairwise disjoint, such that $A \setminus B = A_1 \uplus \ldots \uplus A_n$. If $n = 0$, it is understood that the corresponding union is equal to $\emptyset$, (in which case $A \subseteq B$).

Definition 7 *A* **ring** *on* Ω *is a subset* $\mathcal{R}$ *of the power set* $\mathcal{P}(\Omega)$ *with the following properties:*

$$\begin{array}{rl} (i) & \emptyset \in \mathcal{R} \\ (ii) & A, B \in \mathcal{R} \ \Rightarrow \ A \cup B \in \mathcal{R} \\ (iii) & A, B \in \mathcal{R} \ \Rightarrow \ A \setminus B \in \mathcal{R} \end{array}$$

Exercise 1. Show that $A \cap B = A \setminus (A \setminus B)$ and therefore that a ring is closed under pairwise intersection.

Exercise 2. Show that a ring on Ω is also a semi-ring on Ω.

Exercise 3. Suppose that a set Ω can be decomposed as $\Omega = A_1 \uplus A_2 \uplus A_3$ where A_1, A_2 and A_3 are distinct from $\emptyset$ and Ω. Define $\mathcal{S}_1 \stackrel{\triangle}{=} \{\emptyset, A_1, A_2, A_3, \Omega\}$ and $\mathcal{S}_2 \stackrel{\triangle}{=} \{\emptyset, A_1, A_2 \uplus A_3, \Omega\}$. Show that $\mathcal{S}_1$ and $\mathcal{S}_2$ are semi-rings on Ω, but that $\mathcal{S}_1 \cap \mathcal{S}_2$ fails to be a semi-ring on Ω.

Exercise 4. Let $(\mathcal{R}_i)_{i \in I}$ be an arbitrary family of rings on Ω, with $I \neq \emptyset$. Show that $\mathcal{R} \stackrel{\triangle}{=} \cap_{i \in I} \mathcal{R}_i$ is also a ring on Ω.

EXERCISE 5. Let $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. Define:

$$R(\mathcal{A}) \stackrel{\triangle}{=} \{\mathcal{R} \text{ ring on } \Omega : \ \mathcal{A} \subseteq \mathcal{R}\}$$

Show that $\mathcal{P}(\Omega)$ is a ring on Ω, and that $R(\mathcal{A})$ is not empty. Define:

$$\mathcal{R}(\mathcal{A}) \stackrel{\triangle}{=} \bigcap_{\mathcal{R} \in R(\mathcal{A})} \mathcal{R}$$

Show that $\mathcal{R}(\mathcal{A})$ is a ring on Ω such that $\mathcal{A} \subseteq \mathcal{R}(\mathcal{A})$, and that it is the smallest ring on Ω with such property, (i.e. if $\mathcal{R}$ is a ring on Ω and $\mathcal{A} \subseteq \mathcal{R}$ then $\mathcal{R}(\mathcal{A}) \subseteq \mathcal{R}$).

Definition 8 *Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. We call* **ring generated** *by $\mathcal{A}$, the ring on Ω, denoted $\mathcal{R}(\mathcal{A})$, equal to the intersection of all rings on Ω, which contain $\mathcal{A}$.*

EXERCISE 6.Let $\mathcal{S}$ be a semi-ring on Ω. Define the set $\mathcal{R}$ of all finite unions of pairwise disjoint elements of $\mathcal{S}$, i.e.

$$\mathcal{R} \stackrel{\triangle}{=} \{A : \ A = \uplus_{i=1}^{n} A_i \text{ for some } n \geq 0, A_i \in \mathcal{S}\}$$

(where if $n = 0$, the corresponding union is empty, i.e. $\emptyset \in \mathcal{R}$). Let $A = \uplus_{i=1}^{n} A_i$ and $B = \uplus_{j=1}^{p} B_j \in \mathcal{R}$:

1. Show that $A \cap B = \uplus_{i,j}(A_i \cap B_j)$ and that $\mathcal{R}$ is closed under pairwise intersection.

2. Show that if $p \geq 1$ then $A \setminus B = \cap_{j=1}^{p}(\uplus_{i=1}^{n}(A_i \setminus B_j))$.

3. Show that $\mathcal{R}$ is closed under pairwise difference.

4. Show that $A \cup B = (A \setminus B) \uplus B$ and conclude that $\mathcal{R}$ is a ring on Ω.

5. Show that $\mathcal{R}(\mathcal{S}) = \mathcal{R}$.

Exercise 7. Everything being as before, define:

$$\mathcal{R}' \stackrel{\triangle}{=} \{A : \; A = \cup_{i=1}^{n} A_i \text{ for some } n \geq 0, A_i \in \mathcal{S}\}$$

(We do not require the sets involved in the union to be pairwise disjoint). Using the fact that $\mathcal{R}$ is closed under finite union, show that $\mathcal{R}' \subseteq \mathcal{R}$, and conclude that $\mathcal{R}' = \mathcal{R} = \mathcal{R}(\mathcal{S})$.

Definition 9 *Let* $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ *with* $\emptyset \in \mathcal{A}$. *We call* **measure** *on* $\mathcal{A}$, *any map* $\mu : \mathcal{A} \to [0, +\infty]$ *with the following properties:*

$$(i) \qquad \mu(\emptyset) = 0$$

$$(ii) \qquad A \in \mathcal{A}, A_n \in \mathcal{A} \text{ and } A = \biguplus_{n=1}^{+\infty} A_n \ \Rightarrow\ \mu(A) = \sum_{n=1}^{+\infty} \mu(A_n)$$

The $\uplus$ indicates that we assume the A_n's to be pairwise disjoint in the l.h.s. of (ii). It is customary to say in view of condition (ii) that a measure is *countably additive*.

EXERCISE 8.If $\mathcal{A}$ is a σ-algebra on Ω explain why property (ii) can be replaced by:

$$(ii)' \ A_n \in \mathcal{A} \text{ and } A = \biguplus_{n=1}^{+\infty} A_n \ \Rightarrow\ \mu(A) = \sum_{n=1}^{+\infty} \mu(A_n)$$

Exercise 9. Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ with $\emptyset \in \mathcal{A}$ and $\mu : \mathcal{A} \to [0, +\infty]$ be a measure on $\mathcal{A}$.

1. Show that if $A_1, \ldots, A_n \in \mathcal{A}$ are pairwise disjoint and the union $A = \uplus_{i=1}^n A_i$ lies in $\mathcal{A}$, then $\mu(A) = \mu(A_1) + \ldots + \mu(A_n)$.
2. Show that if $A, B \in \mathcal{A}, A \subseteq B$ and $B \setminus A \in \mathcal{A}$ then $\mu(A) \leq \mu(B)$.

Exercise 10. Let $\mathcal{S}$ be a semi-ring on Ω, and $\mu : \mathcal{S} \to [0, +\infty]$ be a measure on $\mathcal{S}$. Suppose that there exists an extension of μ on $\mathcal{R}(\mathcal{S})$, i.e. a measure $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$ such that $\bar{\mu}_{|\mathcal{S}} = \mu$.

1. Let A be an element of $\mathcal{R}(\mathcal{S})$ with representation $A = \uplus_{i=1}^n A_i$ as a finite union of pairwise disjoint elements of $\mathcal{S}$. Show that $\bar{\mu}(A) = \sum_{i=1}^n \mu(A_i)$
2. Show that if $\bar{\mu}' : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$ is another measure with $\bar{\mu}'_{|\mathcal{S}} = \mu$, i.e. another extension of μ on $\mathcal{R}(\mathcal{S})$, then $\bar{\mu}' = \bar{\mu}$.

Exercise 11. Let $\mathcal{S}$ be a semi-ring on Ω and $\mu : \mathcal{S} \to [0, +\infty]$ be a measure. Let A be an element of $\mathcal{R}(\mathcal{S})$ with two representations:

$$A = \biguplus_{i=1}^{n} A_i = \biguplus_{j=1}^{p} B_j$$

as a finite union of pairwise disjoint elements of $\mathcal{S}$.

1. For $i = 1, \ldots, n$, show that $\mu(A_i) = \sum_{j=1}^{p} \mu(A_i \cap B_j)$
2. Show that $\sum_{i=1}^{n} \mu(A_i) = \sum_{j=1}^{p} \mu(B_j)$
3. Explain why we can define a map $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$ as:

$$\bar{\mu}(A) \stackrel{\triangle}{=} \sum_{i=1}^{n} \mu(A_i)$$

4. Show that $\bar{\mu}(\emptyset) = 0$.

Exercise 12. Everything being as before, suppose that $(A_n)_{n\geq 1}$ is a sequence of pairwise disjoint elements of $\mathcal{R}(\mathcal{S})$, each A_n having the representation:

$$A_n = \biguplus_{k=1}^{p_n} A_n^k \ , \ n \geq 1$$

as a finite union of disjoint elements of $\mathcal{S}$. Suppose moreover that $A = \uplus_{n=1}^{+\infty} A_n$ is an element of $\mathcal{R}(\mathcal{S})$ with representation $A = \uplus_{j=1}^{p} B_j$, as a finite union of pairwise disjoint elements of $\mathcal{S}$.

1. Show that for $j = 1, \ldots, p$, $B_j = \cup_{n=1}^{+\infty} \cup_{k=1}^{p_n} (A_n^k \cap B_j)$ and explain why B_j is of the form $B_j = \uplus_{m=1}^{+\infty} C_m$ for some sequence $(C_m)_{m\geq 1}$ of pairwise disjoint elements of $\mathcal{S}$.

2. Show that $\mu(B_j) = \sum_{n=1}^{+\infty} \sum_{k=1}^{p_n} \mu(A_n^k \cap B_j)$

3. Show that for $n \geq 1$ and $k = 1, \ldots, p_n$, $A_n^k = \uplus_{j=1}^{p} (A_n^k \cap B_j)$

4. Show that $\mu(A_n^k) = \sum_{j=1}^{p} \mu(A_n^k \cap B_j)$

5. Recall the definition of $\bar{\mu}$ of exercise (11) and show that it is a measure on $\mathcal{R}(\mathcal{S})$.

EXERCISE 13. Prove the following theorem:

Theorem 2 *Let $\mathcal{S}$ be a semi-ring on Ω. Let $\mu : \mathcal{S} \to [0, +\infty]$ be a measure on $\mathcal{S}$. There exists a unique measure $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$ such that $\bar{\mu}_{|\mathcal{S}} = \mu$.*

Definition 10 *We define an* **outer-measure** *on* Ω *as being any map* $\mu^* : \mathcal{P}(\Omega) \to [0, +\infty]$ *with the following properties:*

$$\begin{array}{ll} (i) & \mu^*(\emptyset) = 0 \\ (ii) & A \subseteq B \ \Rightarrow \ \mu^*(A) \leq \mu^*(B) \\ (iii) & \mu^*\left(\bigcup_{n=1}^{+\infty} A_n\right) \leq \sum_{n=1}^{+\infty} \mu^*(A_n) \end{array}$$

EXERCISE 14. Show that $\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$, where μ^* is an outer-measure on Ω and $A, B \subseteq \Omega$.

Definition 11 *Let* μ^* *be an outer-measure on* Ω*. We define:*

$$\Sigma(\mu^*) \stackrel{\triangle}{=} \{A \subseteq \Omega : \ \mu^*(T) = \mu^*(T \cap A) + \mu^*(T \cap A^c) \ , \ \forall T \subseteq \Omega\}$$

We call $\Sigma(\mu^*)$ *the* **σ-algebra associated** *with the outer-measure* μ^**.*

Note that the fact that $\Sigma(\mu^*)$ is indeed a σ-algebra on Ω, remains to be proved. This will be your task in the following exercises.

Exercise 15. Let μ^* be an outer-measure on Ω. Let $\Sigma = \Sigma(\mu^*)$ be the σ-algebra associated with μ^*. Let $A, B \in \Sigma$ and $T \subseteq \Omega$

1. Show that $\Omega \in \Sigma$ and $A^c \in \Sigma$.
2. Show that $\mu^*(T \cap A) = \mu^*(T \cap A \cap B) + \mu^*(T \cap A \cap B^c)$
3. Show that $T \cap A^c = T \cap (A \cap B)^c \cap A^c$
4. Show that $T \cap A \cap B^c = T \cap (A \cap B)^c \cap A$
5. Show that $\mu^*(T \cap A^c) + \mu^*(T \cap A \cap B^c) = \mu^*(T \cap (A \cap B)^c)$
6. Adding $\mu^*(T \cap (A \cap B))$ on both sides 5., conclude that $A \cap B \in \Sigma$.
7. Show that $A \cup B$ and $A \setminus B$ belong to Σ.

Exercise 16. Everything being as before, let $A_n \in \Sigma, n \geq 1$. Define $B_1 = A_1$ and $B_{n+1} = A_{n+1} \setminus (A_1 \cup \ldots \cup A_n)$. Show that the B_n's are pairwise disjoint elements of Σ and that $\cup_{n=1}^{+\infty} A_n = \uplus_{n=1}^{+\infty} B_n$.

EXERCISE 17. Everything being as before, show that if $B, C \in \Sigma$ and $B \cap C = \emptyset$, then $\mu^*(T \cap (B \uplus C)) = \mu^*(T \cap B) + \mu^*(T \cap C)$ for any $T \subseteq \Omega$.

EXERCISE 18.Everything being as before, let $(B_n)_{n\geq 1}$ be a sequence of pairwise disjoint elements of Σ, and let $B \stackrel{\triangle}{=} \uplus_{n=1}^{+\infty} B_n$. Let $N \geq 1$.

1. Explain why $\uplus_{n=1}^{N} B_n \in \Sigma$
2. Show that $\mu^*(T \cap (\uplus_{n=1}^{N} B_n)) = \sum_{n=1}^{N} \mu^*(T \cap B_n)$
3. Show that $\mu^*(T \cap B^c) \leq \mu^*(T \cap (\uplus_{n=1}^{N} B_n)^c)$
4. Show that $\mu^*(T \cap B^c) + \sum_{n=1}^{+\infty} \mu^*(T \cap B_n) \leq \mu^*(T)$, and:
5. $\mu^*(T) \leq \mu^*(T \cap B^c) + \mu^*(T \cap B) \leq \mu^*(T \cap B^c) + \sum_{n=1}^{+\infty} \mu^*(T \cap B_n)$
6. Show that $B \in \Sigma$ and $\mu^*(B) = \sum_{n=1}^{+\infty} \mu^*(B_n)$.
7. Show that Σ is a σ-algebra on Ω, and $\mu^*_{|\Sigma}$ is a measure on Σ.

Theorem 3 *Let $\mu^* : \mathcal{P}(\Omega) \to [0, +\infty]$ be an outer-measure on Ω. Then $\Sigma(\mu^*)$, the so-called σ-*algebra associated *with μ^*, is indeed a σ-algebra on Ω and $\mu^*_{|\Sigma(\mu^*)}$, is a measure on $\Sigma(\mu^*)$.*

Exercise 19. Let $\mathcal{R}$ be a ring on Ω and $\mu : \mathcal{R} \to [0, +\infty]$ be a measure on $\mathcal{R}$. For all $T \subseteq \Omega$, define:

$$\mu^*(T) \stackrel{\triangle}{=} \inf \left\{ \sum_{n=1}^{+\infty} \mu(A_n) \ , \ (A_n) \text{ is an } \mathcal{R}\text{-cover of } T \right\}$$

where an $\mathcal{R}$-cover of T is defined as any sequence $(A_n)_{n\geq 1}$ of elements of $\mathcal{R}$ such that $T \subseteq \cup_{n=1}^{+\infty} A_n$. By convention $\inf \emptyset \stackrel{\triangle}{=} +\infty$.

1. Show that $\mu^*(\emptyset) = 0$.
2. Show that if $A \subseteq B$ then $\mu^*(A) \leq \mu^*(B)$.

3. Let $(A_n)_{n\geq 1}$ be a sequence of subsets of Ω, with $\mu^*(A_n) < +\infty$ for all $n \geq 1$. Given $\epsilon > 0$, show that for all $n \geq 1$, there exists an $\mathcal{R}$-cover $(A_n^p)^{p\geq 1}$ of A_n such that:

$$\sum_{p=1}^{+\infty} \mu(A_n^p) < \mu^*(A_n) + \epsilon/2^n$$

   Why is it important to assume $\mu^*(A_n) < +\infty$.

4. Show that there exists an $\mathcal{R}$-cover (R_k) of $\cup_{n=1}^{+\infty} A_n$ such that:

$$\sum_{k=1}^{+\infty} \mu(R_k) = \sum_{n=1}^{+\infty}\sum_{p=1}^{+\infty} \mu(A_n^p)$$

5. Show that $\mu^*(\cup_{n=1}^{+\infty} A_n) \leq \epsilon + \sum_{n=1}^{+\infty} \mu^*(A_n)$

6. Show that μ^* is an outer-measure on Ω.

Exercise 20. Everything being as before, Let $A \in \mathcal{R}$. Let $(A_n)_{n\geq 1}$ be an $\mathcal{R}$-cover of A and put $B_1 = A_1 \cap A$, and:

$$B_{n+1} \stackrel{\triangle}{=} (A_{n+1} \cap A) \setminus ((A_1 \cap A) \cup \ldots \cup (A_n \cap A))$$

1. Show that $\mu^*(A) \leq \mu(A)$.
2. Show that $(B_n)_{n\geq 1}$ is a sequence of pairwise disjoint elements of $\mathcal{R}$ such that $A = \uplus_{n=1}^{+\infty} B_n$.
3. Show that $\mu(A) \leq \mu^*(A)$ and conclude that $\mu^*_{|\mathcal{R}} = \mu$.

Exercise 21. Everything being as before, Let $A \in \mathcal{R}$ and $T \subseteq \Omega$.

1. Show that $\mu^*(T) \leq \mu^*(T \cap A) + \mu^*(T \cap A^c)$.
2. Let (T_n) be an $\mathcal{R}$-cover of T. Show that $(T_n \cap A)$ and $(T_n \cap A^c)$ are $\mathcal{R}$-covers of $T \cap A$ and $T \cap A^c$ respectively.
3. Show that $\mu^*(T \cap A) + \mu^*(T \cap A^c) \leq \mu^*(T)$.

4. Show that $\mathcal{R} \subseteq \Sigma(\mu^*)$.

5. Conclude that $\sigma(\mathcal{R}) \subseteq \Sigma(\mu^*)$.

Exercise 22. Prove the following theorem:

Theorem 4 (Caratheodory's extension) *Let $\mathcal{R}$ be a ring on Ω and $\mu : \mathcal{R} \to [0, +\infty]$ be a measure on $\mathcal{R}$. There exists a measure $\mu' : \sigma(\mathcal{R}) \to [0, +\infty]$ such that $\mu'_{|\mathcal{R}} = \mu$.*

Exercise 23. Let $\mathcal{S}$ be a semi-ring on Ω. Show that $\sigma(\mathcal{R}(\mathcal{S})) = \sigma(\mathcal{S})$.

Exercise 24. Prove the following theorem:

Theorem 5 *Let $\mathcal{S}$ be a semi-ring on Ω and $\mu : \mathcal{S} \to [0, +\infty]$ be a measure on $\mathcal{S}$. There exists a measure $\mu' : \sigma(\mathcal{S}) \to [0, +\infty]$ such that $\mu'_{|\mathcal{S}} = \mu$.*