11. Complex Measures

In the following, $(\Omega, \mathcal{F})$ denotes an arbitrary measurable space.

Definition 90 *Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers. We say that $(a_n)_{n\geq 1}$ has the* **permutation property** *if and only if, for all bijections $\sigma : \mathbf{N}^* \to \mathbf{N}^*$, the series $\sum_{k=1}^{+\infty} a_{\sigma(k)}$ converges in $\mathbf{C}$*[1]

Exercise 1. Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers.

1. Show that if $(a_n)_{n\geq 1}$ has the permutation property, then the same is true of $(Re(a_n))_{n\geq 1}$ and $(Im(a_n))_{n\geq 1}$.

2. Suppose $a_n \in \mathbf{R}$ for all $n \geq 1$. Show that if $\sum_{k=1}^{+\infty} a_k$ converges:

$$\sum_{k=1}^{+\infty} |a_k| = +\infty \Rightarrow \sum_{k=1}^{+\infty} a_k^+ = \sum_{k=1}^{+\infty} a_k^- = +\infty$$

[1] which excludes $\pm\infty$ as limit.

Exercise 2. Let $(a_n)_{n\geq 1}$ be a sequence in $\mathbf{R}$, such that the series $\sum_{k=1}^{+\infty} a_k$ converges, and $\sum_{k=1}^{+\infty} |a_k| = +\infty$. Let $A > 0$. We define:

$$N^+ \stackrel{\triangle}{=} \{k \geq 1 \ : \ a_k \geq 0\} \ \ , \ \ N^- \stackrel{\triangle}{=} \{k \geq 1 \ : \ a_k < 0\}$$

1. Show that N^+ and N^- are infinite.

2. Let $\phi^+ : \mathbf{N}^* \to N^+$ and $\phi^- : \mathbf{N}^* \to N^-$ be two bijections. Show the existence of $k_1 \geq 1$ such that:

$$\sum_{k=1}^{k_1} a_{\phi^+(k)} \geq A$$

3. Show the existence of an increasing sequence $(k_p)_{p\geq 1}$ such that:

$$\sum_{k=k_{p-1}+1}^{k_p} a_{\phi^+(k)} \geq A$$

for all $p \geq 1$, where $k_0 = 0$.

4. Consider the permutation $\sigma : \mathbf{N}^* \to \mathbf{N}^*$ defined informally by:

$$(\phi^-(1), \underbrace{\phi^+(1), \ldots, \phi^+(k_1)}, \phi^-(2), \underbrace{\phi^+(k_1+1), \ldots, \phi^+(k_2)}, \ldots)$$

representing $(\sigma(1), \sigma(2), \ldots)$. More specifically, define $k_0^* = 0$ and $k_p^* = k_p + p$ for all $p \geq 1$. For all $n \in \mathbf{N}^*$ and $p \geq 1$ with: [2]

$$k_{p-1}^* < n \leq k_p^* \tag{1}$$

we define:

$$\sigma(n) = \begin{cases} \phi^-(p) & \text{if} \quad n = k_{p-1}^* + 1 \\ \phi^+(n-p) & \text{if} \quad n > k_{p-1}^* + 1 \end{cases} \tag{2}$$

Show that $\sigma : \mathbf{N}^* \to \mathbf{N}^*$ is indeed a bijection.

[2]Given an integer $n \geq 1$, there exists a unique $p \geq 1$ such that (1) holds.

5. Show that if $\sum_{k=1}^{+\infty} a_{\sigma(k)}$ converges, there is $N \geq 1$, such that:

$$n \geq N \ , \ p \geq 1 \ \Rightarrow \ \left| \sum_{k=n+1}^{n+p} a_{\sigma(k)} \right| < A$$

6. Explain why $(a_n)_{n\geq 1}$ cannot have the permutation property.

7. Prove the following theorem:

Theorem 56 *Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers such that for all bijections $\sigma : \mathbf{N}^* \to \mathbf{N}^*$, the series $\sum_{k=1}^{+\infty} a_{\sigma(k)}$ converges. Then, the series $\sum_{k=1}^{+\infty} a_k$ converges absolutely, i.e.*

$$\sum_{k=1}^{+\infty} |a_k| < +\infty$$

Definition 91 *Let $(\Omega, \mathcal{F})$ be a measurable space and $E \in \mathcal{F}$. We call* **measurable partition** *of E, any sequence $(E_n)_{n\geq 1}$ of pairwise disjoint elements of $\mathcal{F}$, such that $E = \uplus_{n\geq 1} E_n$.*

Definition 92 *We call* **complex measure** *on a measurable space $(\Omega, \mathcal{F})$ any map $\mu : \mathcal{F} \to \mathbf{C}$, such that for all $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ measurable partition of E, the series $\sum_{n=1}^{+\infty} \mu(E_n)$ converges to $\mu(E)$. The set of all complex measures on $(\Omega, \mathcal{F})$ is denoted $M^1(\Omega, \mathcal{F})$.*

Definition 93 *We call* **signed measure** *on a measurable space $(\Omega, \mathcal{F})$, any complex measure on $(\Omega, \mathcal{F})$ with values in $\mathbf{R}$.*[3]

Exercise 3.

1. Show that a measure on $(\Omega, \mathcal{F})$ may not be a complex measure.
2. Show that for all $\mu \in M^1(\Omega, \mathcal{F})$, $\mu(\emptyset) = 0$.

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[3] In these tutorials, signed measure may not have values in $\{-\infty, +\infty\}$.

3. Show that a finite measure on $(\Omega, \mathcal{F})$ is a complex measure with values in $\mathbf{R}^+$, and conversely.

4. Let $\mu \in M^1(\Omega, \mathcal{F})$. Let $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E. Show that:

$$\sum_{n=1}^{+\infty} |\mu(E_n)| < +\infty$$

5. Let μ be a measure on $(\Omega, \mathcal{F})$ and $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Define:

$$\forall E \in \mathcal{F} \ , \ \nu(E) \stackrel{\triangle}{=} \int_E f d\mu$$

Show that ν is a complex measure on $(\Omega, \mathcal{F})$.

Definition 94 *Let μ be a complex measure on a measurable space $(\Omega, \mathcal{F})$. We call* **total variation** *of μ, the map $|\mu| : \mathcal{F} \to [0, +\infty]$, defined by:*

$$\forall E \in \mathcal{F} \ , \ |\mu|(E) \stackrel{\triangle}{=} \sup \sum_{n=1}^{+\infty} |\mu(E_n)|$$

where the 'sup' is taken over all measurable partitions $(E_n)_{n\geq 1}$ of E.

EXERCISE 4. Let μ be a complex measure on $(\Omega, \mathcal{F})$.

1. Show that for all $E \in \mathcal{F}$, $|\mu(E)| \leq |\mu|(E)$.
2. Show that $|\mu|(\emptyset) = 0$.

EXERCISE 5. Let μ be a complex measure on $(\Omega, \mathcal{F})$. Let $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E.

1. Show that there exists $(t_n)_{n\geq 1}$ in $\mathbf{R}$, with $t_n < |\mu|(E_n)$ for all n.

2. Show that for all $n \geq 1$, there exists a measurable partition $(E_n^p)_{p\geq 1}$ of E_n such that:

$$t_n < \sum_{p=1}^{+\infty} |\mu(E_n^p)|$$

3. Show that $(E_n^p)_{n,p\geq 1}$ is a measurable partition of E.

4. Show that for all $N \geq 1$, we have $\sum_{n=1}^{N} t_n \leq |\mu|(E)$.

5. Show that for all $N \geq 1$, we have:

$$\sum_{n=1}^{N} |\mu|(E_n) \leq |\mu|(E)$$

6. Suppose that $(A_p)_{p\geq 1}$ is another arbitrary measurable partition

of E. Show that for all $p \geq 1$:

$$|\mu(A_p)| \leq \sum_{n=1}^{+\infty} |\mu(A_p \cap E_n)|$$

7. Show that for all $n \geq 1$:

$$\sum_{p=1}^{+\infty} |\mu(A_p \cap E_n)| \leq |\mu|(E_n)$$

8. Show that:

$$\sum_{p=1}^{+\infty} |\mu(A_p)| \leq \sum_{n=1}^{+\infty} |\mu|(E_n)$$

9. Show that $|\mu| : \mathcal{F} \to [0, +\infty]$ is a measure on $(\Omega, \mathcal{F})$.

Exercise 6. Let $a, b \in \mathbf{R}, a < b$. Let $F \in C^1([a,b];\mathbf{R})$, and define:

$$\forall x \in [a,b]\ ,\ H(x) \stackrel{\triangle}{=} \int_a^x F'(t)dt$$

1. Show that $H \in C^1([a,b];\mathbf{R})$ and $H' = F'$.

2. Show that:

$$F(b) - F(a) = \int_a^b F'(t)dt$$

3. Show that:

$$\frac{1}{2\pi}\int_{-\pi/2}^{+\pi/2} \cos\theta d\theta = \frac{1}{\pi}$$

4. Let $u \in \mathbf{R}^n$ and $\tau_u : \mathbf{R}^n \to \mathbf{R}^n$ be the translation $\tau_u(x) = x + u$. Show that the Lebesgue measure dx on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ is invariant by translation τ_u, i.e. $dx(\{\tau_u \in B\}) = dx(B)$ for all $B \in \mathcal{B}(\mathbf{R}^n)$.

5. Show that for all $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, and $u \in \mathbf{R}^n$:

$$\int_{\mathbf{R}^n} f(x+u)dx = \int_{\mathbf{R}^n} f(x)dx$$

6. Show that for all $\alpha \in \mathbf{R}$, we have:

$$\int_{-\pi}^{+\pi} \cos^+(\alpha - \theta)d\theta = \int_{-\pi-\alpha}^{+\pi-\alpha} \cos^+ \theta d\theta$$

7. Let $\alpha \in \mathbf{R}$ and $k \in \mathbf{Z}$ such that $k \leq \alpha/2\pi < k+1$. Show:

$$-\pi - \alpha \leq -2k\pi - \pi < \pi - \alpha \leq -2k\pi + \pi$$

8. Show that:

$$\int_{-\pi-\alpha}^{-2k\pi-\pi} \cos^+ \theta d\theta = \int_{\pi-\alpha}^{-2k\pi+\pi} \cos^+ \theta d\theta$$

9. Show that:

$$\int_{-\pi-\alpha}^{+\pi-\alpha} \cos^+ \theta d\theta = \int_{-2k\pi-\pi}^{-2k\pi+\pi} \cos^+ \theta d\theta = \int_{-\pi}^{+\pi} \cos^+ \theta d\theta$$

10. Show that for all $\alpha \in \mathbf{R}$:

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} \cos^+(\alpha - \theta) d\theta = \frac{1}{\pi}$$

Exercise 7. Let $z_1, \ldots, z_N$ be N complex numbers. Let $\alpha_k \in \mathbf{R}$ be such that $z_k = |z_k| e^{i\alpha_k}$, for all $k = 1, \ldots, N$. For all $\theta \in [-\pi, +\pi]$, we define $S(\theta) = \{k = 1, \ldots, N \ : \ \cos(\alpha_k - \theta) > 0\}$.

1. Show that for all $\theta \in [-\pi, +\pi]$, we have:

$$\left| \sum_{k \in S(\theta)} z_k \right| = \left| \sum_{k \in S(\theta)} z_k e^{-i\theta} \right| \geq \sum_{k \in S(\theta)} |z_k| \cos(\alpha_k - \theta)$$

2. Define $\phi : [-\pi, +\pi] \to \mathbf{R}$ by $\phi(\theta) = \sum_{k=1}^{N} |z_k| \cos^+(\alpha_k - \theta)$. Show the existence of $\theta_0 \in [-\pi, +\pi]$ such that:

$$\phi(\theta_0) = \sup_{\theta \in [-\pi, +\pi]} \phi(\theta)$$

3. Show that:

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} \phi(\theta) d\theta = \frac{1}{\pi} \sum_{k=1}^{N} |z_k|$$

4. Conclude that:

$$\frac{1}{\pi} \sum_{k=1}^{N} |z_k| \leq \left| \sum_{k \in S(\theta_0)} z_k \right|$$

Exercise 8. Let $\mu \in M^1(\Omega, \mathcal{F})$. Suppose that $|\mu|(E) = +\infty$ for some $E \in \mathcal{F}$. Define $t = \pi(1 + |\mu(E)|) \in \mathbf{R}^+$.

1. Show that there is a measurable partition $(E_n)_{n\geq 1}$ of E, with:

$$t < \sum_{n=1}^{+\infty} |\mu(E_n)|$$

2. Show the existence of $N \geq 1$ such that:

$$t < \sum_{n=1}^{N} |\mu(E_n)|$$

3. Show the existence of $S \subseteq \{1,\ldots,N\}$ such that:

$$\sum_{n=1}^{N} |\mu(E_n)| \leq \pi \left|\sum_{n\in S} \mu(E_n)\right|$$

4. Show that $|\mu(A)| > t/\pi$, where $A = \uplus_{n\in S} E_n$.

5. Let $B = E \setminus A$. Show that $|\mu(B)| \geq |\mu(A)| - |\mu(E)|$.

6. Show that $E = A \uplus B$ with $|\mu(A)| > 1$ and $|\mu(B)| > 1$.

7. Show that $|\mu|(A) = +\infty$ or $|\mu|(B) = +\infty$.

EXERCISE 9. Let $\mu \in M^1(\Omega, \mathcal{F})$. Suppose that $|\mu|(\Omega) = +\infty$.

1. Show the existence of $A_1, B_1 \in \mathcal{F}$, such that $\Omega = A_1 \uplus B_1$, $|\mu(A_1)| > 1$ and $|\mu|(B_1) = +\infty$.

2. Show the existence of a sequence $(A_n)_{n\geq 1}$ of pairwise disjoint elements of $\mathcal{F}$, such that $|\mu(A_n)| > 1$ for all $n \geq 1$.

3. Show that the series $\sum_{n=1}^{+\infty} \mu(A_n)$ does not converge to $\mu(A)$ where $A = \uplus_{n=1}^{+\infty} A_n$.

4. Conclude that $|\mu|(\Omega) < +\infty$.

Theorem 57 *Let μ be a complex measure on a measurable space $(\Omega, \mathcal{F})$. Then, its total variation $|\mu|$ is a finite measure on $(\Omega, \mathcal{F})$.*

EXERCISE 10. Show that $M^1(\Omega, \mathcal{F})$ is a $\mathbf{C}$-vector space, with:

$$\begin{aligned} (\lambda + \mu)(E) &\stackrel{\triangle}{=} \lambda(E) + \mu(E) \\ (\alpha\lambda)(E) &\stackrel{\triangle}{=} \alpha.\lambda(E) \end{aligned}$$

where $\lambda, \mu \in M^1(\Omega, \mathcal{F})$, $\alpha \in \mathbf{C}$, and $E \in \mathcal{F}$.

Definition 95 *Let $\mathcal{H}$ be a $\mathbf{K}$-vector space, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. We call* **norm** *on $\mathcal{H}$, any map $N : \mathcal{H} \to \mathbf{R}^+$, with the following properties:*

$$\begin{array}{rl} (i) & \forall x \in \mathcal{H}\ ,\ (N(x) = 0 \ \Leftrightarrow\ x = 0) \\ (ii) & \forall x \in \mathcal{H}, \forall \alpha \in \mathbf{K}\ ,\ N(\alpha x) = |\alpha| N(x) \\ (iii) & \forall x, y \in \mathcal{H}\ ,\ N(x + y) \leq N(x) + N(y) \end{array}$$

Exercise 11.

1. Explain why $\|.\|_p$ may not be a norm on $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$.
2. Show that $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ is a norm, when $\langle\cdot,\cdot\rangle$ is an inner-product.
3. Show that $\|\mu\| \stackrel{\triangle}{=} |\mu|(\Omega)$ defines a norm on $M^1(\Omega, \mathcal{F})$.

Exercise 12. Let $\mu \in M^1(\Omega, \mathcal{F})$ be a signed measure. Show that:

$$\begin{aligned} \mu^+ &\stackrel{\triangle}{=} \frac{1}{2}(|\mu| + \mu) \\ \mu^- &\stackrel{\triangle}{=} \frac{1}{2}(|\mu| - \mu) \end{aligned}$$

are finite measures such that:

$$\mu = \mu^+ - \mu^- \quad , \quad |\mu| = \mu^+ + \mu^-$$

Exercise 13. Let $\mu \in M^1(\Omega, \mathcal{F})$ and $l : \mathbf{R}^2 \to \mathbf{R}$ be a linear map.

1. Show that l is continuous.
2. Show that $l \circ \mu$ is a signed measure on $(\Omega, \mathcal{F})$. [4]
3. Show that all $\mu \in M^1(\Omega, \mathcal{F})$ can be decomposed as:

$$\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$$

where $\mu_1, \mu_2, \mu_3, \mu_4$ are finite measures.

[4] $l \circ \mu$ refers strictly speaking to $l(Re(\mu), Im(\mu))$.