7. Fubini Theorem

Definition 59 *Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces. Let $E \subseteq \Omega_1 \times \Omega_2$. For all $\omega_1 \in \Omega_1$, we call ω_1-***section** *of E in Ω_2, the set:*

$$E^{\omega_1} \stackrel{\triangle}{=} \{\omega_2 \in \Omega_2 \ : \ (\omega_1, \omega_2) \in E\}$$

Exercise 1. Let $(\Omega_1, \mathcal{F}_1)$, $(\Omega_2, \mathcal{F}_2)$ and (S, Σ) be three measurable spaces, and $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to (S, \Sigma)$ be a measurable map. Given $\omega_1 \in \Omega_1$, define:

$$\Gamma^{\omega_1} \stackrel{\triangle}{=} \{E \subseteq \Omega_1 \times \Omega_2 \ , \ E^{\omega_1} \in \mathcal{F}_2\}$$

1. Show that for all $\omega_1 \in \Omega_1$, Γ^{ω_1} is a σ-algebra on $\Omega_1 \times \Omega_2$.
2. Show that for all $\omega_1 \in \Omega_1$, $\mathcal{F}_1 \amalg \mathcal{F}_2 \subseteq \Gamma^{\omega_1}$.
3. Show that for all $\omega_1 \in \Omega_1$ and $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, we have $E^{\omega_1} \in \mathcal{F}_2$.
4. Given $\omega_1 \in \Omega_1$, show that $\omega \to f(\omega_1, \omega)$ is measurable.

5. Show that $\theta : (\Omega_2 \times \Omega_1, \mathcal{F}_2 \otimes \mathcal{F}_1) \to (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ defined by $\theta(\omega_2, \omega_1) = (\omega_1, \omega_2)$ is a measurable map.

6. Given $\omega_2 \in \Omega_2$, show that $\omega \to f(\omega, \omega_2)$ is measurable.

Theorem 29 *Let (S, Σ), $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be three measurable spaces. Let $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to (S, \Sigma)$ be a measurable map. For all $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$, the map $\omega \to f(\omega_1, \omega)$ is measurable w.r. to $\mathcal{F}_2$ and Σ, and $\omega \to f(\omega, \omega_2)$ is measurable w.r. to $\mathcal{F}_1$ and Σ.*

EXERCISE 2. Let $(\Omega_i, \mathcal{F}_i)_{i \in I}$ be a family of measurable spaces with $\text{card} I \geq 2$. Let $f : (\Pi_{i \in I} \Omega_i, \otimes_{i \in I} \mathcal{F}_i) \to (E, \mathcal{B}(E))$ be a measurable map, where (E, d) is a metric space. Let $i_1 \in I$. Put $E_1 = \Omega_{i_1}$, $\mathcal{E}_1 = \mathcal{F}_{i_1}$, $E_2 = \Pi_{i \in I \setminus \{i_1\}} \Omega_i$, $\mathcal{E}_2 = \otimes_{i \in I \setminus \{i_1\}} \mathcal{F}_i$.

1. Explain why f can be viewed as a map defined on $E_1 \times E_2$.

2. Show that $f : (E_1 \times E_2, \mathcal{E}_1 \otimes \mathcal{E}_2) \to (E, \mathcal{B}(E))$ is measurable.

3. For all $\omega_{i_1} \in \Omega_{i_1}$, show that the map $\omega \to f(\omega_{i_1}, \omega)$ defined on $\Pi_{i \in I \setminus \{i_1\}} \Omega_i$ is measurable w.r. to $\otimes_{i \in I \setminus \{i_1\}} \mathcal{F}_i$ and $\mathcal{B}(E)$.

Definition 60 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. $(\Omega, \mathcal{F}, \mu)$ is said to be a* **finite measure space***, or we say that μ is a* **finite measure***, if and only if $\mu(\Omega) < +\infty$.*

Definition 61 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. $(\Omega, \mathcal{F}, \mu)$ is said to be a* **σ-finite measure space***, or μ a* **σ-finite measure***, if and only if there exists a sequence $(\Omega_n)_{n \geq 1}$ in $\mathcal{F}$ such that $\Omega_n \uparrow \Omega$ and $\mu(\Omega_n) < +\infty$, for all $n \geq 1$.*

EXERCISE 3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

1. Show that $(\Omega, \mathcal{F}, \mu)$ is σ-finite if and only if there exists a sequence $(\Omega_n)_{n \geq 1}$ in $\mathcal{F}$ such that $\Omega = \uplus_{n=1}^{+\infty} \Omega_n$, and $\mu(\Omega_n) < +\infty$ for all $n \geq 1$.

2. Show that if $(\Omega, \mathcal{F}, \mu)$ is finite, then μ has values in $\mathbf{R}^+$.

3. Show that if $(\Omega, \mathcal{F}, \mu)$ is finite, then it is σ-finite.

4. Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. Show that the measure space $(\mathbf{R}, \mathcal{B}(\mathbf{R}), dF)$ is σ-finite, where dF is the Stieltjes measure associated with F.

EXERCISE 4. Let $(\Omega_1, \mathcal{F}_1)$ be a measurable space, and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be a σ-finite measure space. For all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$ and $\omega_1 \in \Omega_1$, define:

$$\Phi_E(\omega_1) \stackrel{\triangle}{=} \int_{\Omega_2} 1_E(\omega_1, x) d\mu_2(x)$$

Let $\mathcal{D}$ be the set of subsets of $\Omega_1 \times \Omega_2$, defined by:

$$\mathcal{D} \stackrel{\triangle}{=} \{E \in \mathcal{F}_1 \otimes \mathcal{F}_2 \ : \ \Phi_E : (\Omega_1, \mathcal{F}_1) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}})) \text{ is measurable}\}$$

1. Explain why for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, the map Φ_E is well defined.

2. Show that $\mathcal{F}_1 \amalg \mathcal{F}_2 \subseteq \mathcal{D}$.

3. Show that if μ_2 is finite, $A, B \in \mathcal{D}$ and $A \subseteq B$, then $B \setminus A \in \mathcal{D}$.

4. Show that if $E_n \in \mathcal{F}_1 \otimes \mathcal{F}_2, n \geq 1$ and $E_n \uparrow E$, then $\Phi_{E_n} \uparrow \Phi_E$.

5. Show that if μ_2 is finite then $\mathcal{D}$ is a Dynkin system on $\Omega_1 \times \Omega_2$.

6. Show that if μ_2 is finite, then the map $\Phi_E : (\Omega_1, \mathcal{F}_1) \rightarrow (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$.

7. Let $(\Omega_2^n)_{n\geq 1}$ in $\mathcal{F}_2$ be such that $\Omega_2^n \uparrow \Omega_2$ and $\mu_2(\Omega_2^n) < +\infty$. Define $\mu_2^n = \mu_2^{\Omega_2^n} = \mu_2(\bullet \cap \Omega_2^n)$. For $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, we put:

$$\Phi_E^n(\omega_1) \stackrel{\triangle}{=} \int_{\Omega_2} 1_E(\omega_1, x) d\mu_2^n(x)$$

Show that $\Phi_E^n : (\Omega_1, \mathcal{F}_1) \rightarrow (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, and:

$$\Phi_E^n(\omega_1) = \int_{\Omega_2} 1_{\Omega_2^n}(x) 1_E(\omega_1, x) d\mu_2(x)$$

Deduce that $\Phi_E^n \uparrow \Phi_E$.

8. Show that the map $\Phi_E : (\Omega_1, \mathcal{F}_1) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$.

9. Let s be a simple function on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$. Show that the map $\omega \to \int_{\Omega_2} s(\omega, x) d\mu_2(x)$ is well defined and measurable with respect to $\mathcal{F}_1$ and $\mathcal{B}(\bar{\mathbf{R}})$.

10. Show the following theorem:

Theorem 30 *Let $(\Omega_1, \mathcal{F}_1)$ be a measurable space, and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be a σ-finite measure space. Then for all non-negative and measurable map $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to [0, +\infty]$, the map:*

$$\omega \to \int_{\Omega_2} f(\omega, x) d\mu_2(x)$$

is measurable with respect to $\mathcal{F}_1$ and $\mathcal{B}(\bar{\mathbf{R}})$.

Exercise 5. Let $(\Omega_i, \mathcal{F}_i)_{i\in I}$ be a family of measurable spaces, with $\mathrm{card} I \geq 2$. Let $i_0 \in I$, and suppose that μ_0 is a σ-finite measure on $(\Omega_{i_0}, \mathcal{F}_{i_0})$. Show that if $f : (\Pi_{i\in I}\Omega_i, \otimes_{i\in I}\mathcal{F}_i) \to [0, +\infty]$ is a non-negative and measurable map, then:

$$\omega \to \int_{\Omega_{i_0}} f(\omega, x) d\mu_0(x)$$

defined on $\Pi_{i\in I\setminus\{i_0\}}\Omega_i$, is measurable w.r. to $\otimes_{i\in I\setminus\{i_0\}}\mathcal{F}_i$ and $\mathcal{B}(\bar{\mathbf{R}})$.

Exercise 6. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two σ-finite measure spaces. For all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, we define:

$$\mu_1 \otimes \mu_2(E) \stackrel{\triangle}{=} \int_{\Omega_1} \left(\int_{\Omega_2} 1_E(x, y) d\mu_2(y) \right) d\mu_1(x)$$

1. Explain why $\mu_1 \otimes \mu_2 : \mathcal{F}_1 \otimes \mathcal{F}_2 \to [0, +\infty]$ is well defined.

2. Show that $\mu_1 \otimes \mu_2$ is a measure on $\mathcal{F}_1 \otimes \mathcal{F}_2$.

3. Show that if $A \times B \in \mathcal{F}_1 \amalg \mathcal{F}_2$, then:

$$\mu_1 \otimes \mu_2(A \times B) = \mu_1(A)\mu_2(B)$$

Exercise 7. Further to ex. (6), suppose that $\mu : \mathcal{F}_1 \otimes \mathcal{F}_2 \to [0, +\infty]$ is another measure on $\mathcal{F}_1 \otimes \mathcal{F}_2$ with $\mu(A \times B) = \mu_1(A)\mu_2(B)$, for all measurable rectangle $A \times B$. Let $(\Omega_1^n)_{n\geq 1}$ and $(\Omega_2^n)_{n\geq 1}$ be sequences in $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively, such that $\Omega_1^n \uparrow \Omega_1$, $\Omega_2^n \uparrow \Omega_2$, $\mu_1(\Omega_1^n) < +\infty$ and $\mu_2(\Omega_2^n) < +\infty$. Define, for all $n \geq 1$:

$$\mathcal{D}_n \stackrel{\triangle}{=} \{E \in \mathcal{F}_1 \otimes \mathcal{F}_2 : \mu(E \cap (\Omega_1^n \times \Omega_2^n)) = \mu_1 \otimes \mu_2(E \cap (\Omega_1^n \times \Omega_2^n))\}$$

1. Show that for all $n \geq 1$, $\mathcal{F}_1 \amalg \mathcal{F}_2 \subseteq \mathcal{D}_n$.

2. Show that for all $n \geq 1$, $\mathcal{D}_n$ is a Dynkin system on $\Omega_1 \times \Omega_2$.

3. Show that $\mu = \mu_1 \otimes \mu_2$.

4. Show that $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$ is a σ-finite measure space.

5. Show that for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, we have:

$$\mu_1 \otimes \mu_2(E) = \int_{\Omega_2} \left(\int_{\Omega_1} 1_E(x,y) d\mu_1(x) \right) d\mu_2(y)$$

Exercise 8. Let $(\Omega_1, \mathcal{F}_1, \mu_1), \ldots, (\Omega_n, \mathcal{F}_n, \mu_n)$ be n σ-finite measure spaces, $n \geq 2$. Let $i_0 \in \{1, \ldots, n\}$ and put $E_1 = \Omega_{i_0}$, $E_2 = \Pi_{i \neq i_0} \Omega_i$, $\mathcal{E}_1 = \mathcal{F}_{i_0}$ and $\mathcal{E}_2 = \otimes_{i \neq i_0} \mathcal{F}_i$. Put $\nu_1 = \mu_{i_0}$, and suppose that ν_2 is a σ-finite measure on $(E_2, \mathcal{E}_2)$ such that for all measurable rectangle $\Pi_{i \neq i_0} A_i \in \amalg_{i \neq i_0} \mathcal{F}_i$, we have $\nu_2 (\Pi_{i \neq i_0} A_i) = \Pi_{i \neq i_0} \mu_i(A_i)$.

1. Show that $\nu_1 \otimes \nu_2$ is a σ-finite measure on the measure space $(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)$ such that for all measurable rectangles $A_1 \times \ldots \times A_n$, we have:

$$\nu_1 \otimes \nu_2(A_1 \times \ldots \times A_n) = \mu_1(A_1) \ldots \mu_n(A_n)$$

2. Show by induction the existence of a measure μ on $\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$,

such that for all measurable rectangles $A_1 \times \ldots \times A_n$, we have:

$$\mu(A_1 \times \ldots \times A_n) = \mu_1(A_1) \ldots \mu_n(A_n)$$

3. Show the uniqueness of such measure, denoted $\mu_1 \otimes \ldots \otimes \mu_n$.
4. Show that $\mu_1 \otimes \ldots \otimes \mu_n$ is σ-finite.
5. Let $i_0 \in \{1, \ldots, n\}$. Show that $\mu_{i_0} \otimes (\otimes_{i \neq i_0} \mu_i) = \mu_1 \otimes \ldots \otimes \mu_n$.

Definition 62 *Let $(\Omega_1, \mathcal{F}_1, \mu_1), \ldots, (\Omega_n, \mathcal{F}_n, \mu_n)$ be n σ-finite measure spaces, with $n \geq 2$. We call* **product measure** *of $\mu_1, \ldots, \mu_n$, the unique measure on $\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$, denoted $\mu_1 \otimes \ldots \otimes \mu_n$, such that for all measurable rectangles $A_1 \times \ldots \times A_n$ in $\mathcal{F}_1 \amalg \ldots \amalg \mathcal{F}_n$, we have:*

$$\mu_1 \otimes \ldots \otimes \mu_n(A_1 \times \ldots \times A_n) = \mu_1(A_1) \ldots \mu_n(A_n)$$

This measure is itself σ-finite.

Exercise 9. Prove that the following definition is legitimate:

Definition 63 *We call* **Lebesgue measure** *in* $\mathbf{R}^n$, $n \geq 1$, *the unique measure on* $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, *denoted* dx, dx^n *or* $dx_1 \ldots dx_n$, *such that for all* $a_i \leq b_i$, $i = 1, \ldots, n$, *we have:*

$$dx([a_1, b_1] \times \ldots \times [a_n, b_n]) = \prod_{i=1}^{n} (b_i - a_i)$$

Exercise 10.

1. Show that $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx^n)$ is a σ-finite measure space.
2. For $n, p \geq 1$, show that $dx^{n+p} = dx^n \otimes dx^p$.

Exercise 11. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be σ-finite.

1. Let s be a simple function on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$. Show that:

$$\int_{\Omega_1 \times \Omega_2} s d\mu_1 \otimes \mu_2 = \int_{\Omega_1} \left(\int_{\Omega_2} s d\mu_2 \right) d\mu_1 = \int_{\Omega_2} \left(\int_{\Omega_1} s d\mu_1 \right) d\mu_2$$

2. Show the following:

Theorem 31 (Fubini) *Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two σ-finite measure spaces. Let $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to [0, +\infty]$ be a non-negative and measurable map. Then:*

$$\int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2 = \int_{\Omega_1} \left(\int_{\Omega_2} f d\mu_2 \right) d\mu_1 = \int_{\Omega_2} \left(\int_{\Omega_1} f d\mu_1 \right) d\mu_2$$

Exercise 12. Let $(\Omega_1, \mathcal{F}_1, \mu_1), \ldots, (\Omega_n, \mathcal{F}_n, \mu_n)$ be n σ-finite measure spaces, $n \geq 2$. Let $f : (\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n) \to [0, +\infty]$ be a non-negative, measurable map. Let σ be a permutation of $\mathbf{N}_n$, i.e. a bijection from $\mathbf{N}_n$ to itself.

1. For all $\omega \in \Pi_{i \neq \sigma(1)} \Omega_i$, define:

$$J_1(\omega) \stackrel{\triangle}{=} \int_{\Omega_{\sigma(1)}} f(\omega, x) d\mu_{\sigma(1)}(x)$$

Explain why $J_1 : (\Pi_{i \neq \sigma(1)} \Omega_i, \otimes_{i \neq \sigma(1)} \mathcal{F}_i) \to [0, +\infty]$ is a well defined, non-negative and measurable map.

2. Suppose $J_k : (\Pi_{i \notin \{\sigma(1), \ldots, \sigma(k)\}} \Omega_i, \otimes_{i \notin \{\sigma(1), \ldots, \sigma(k)\}} \mathcal{F}_i) \to [0, +\infty]$ is a non-negative, measurable map, for $1 \leq k < n - 2$. Define:

$$J_{k+1}(\omega) \stackrel{\triangle}{=} \int_{\Omega_{\sigma(k+1)}} J_k(\omega, x) d\mu_{\sigma(k+1)}(x)$$

and show that:

$$J_{k+1} : (\Pi_{i \notin \{\sigma(1), \ldots, \sigma(k+1)\}} \Omega_i, \otimes_{i \notin \{\sigma(1), \ldots, \sigma(k+1)\}} \mathcal{F}_i) \to [0, +\infty]$$

is also well-defined, non-negative and measurable.

3. Propose a rigorous definition for the following notation:

$$\int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$

Exercise 13. Further to ex. (12), Let $(f_p)_{p\geq 1}$ be a sequence of non-negative and measurable maps:

$$f_p : (\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n) \to [0, +\infty]$$

such that $f_p \uparrow f$. Define similarly:

$$\begin{aligned} J_1^p(\omega) &\stackrel{\triangle}{=} \int_{\Omega_{\sigma(1)}} f_p(\omega, x) d\mu_{\sigma(1)}(x) \\ J_{k+1}^p(\omega) &\stackrel{\triangle}{=} \int_{\Omega_{\sigma(k+1)}} J_k^p(\omega, x) d\mu_{\sigma(k+1)}(x) \ , \ 1 \leq k < n-2 \end{aligned}$$

1. Show that $J_1^p \uparrow J_1$.
2. Show that if $J_k^p \uparrow J_k$, then $J_{k+1}^p \uparrow J_{k+1}$, $1 \leq k < n-2$.

3. Show that:

$$\int_{\Omega_{\sigma(n)}}\dots\int_{\Omega_{\sigma(1)}} f_p d\mu_{\sigma(1)}\dots d\mu_{\sigma(n)} \ \uparrow\ \int_{\Omega_{\sigma(n)}}\dots\int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)}\dots d\mu_{\sigma(n)}$$

4. Show that the map $\mu : \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n \to [0, +\infty]$, defined by:

$$\mu(E) = \int_{\Omega_{\sigma(n)}}\dots\int_{\Omega_{\sigma(1)}} 1_E d\mu_{\sigma(1)}\dots d\mu_{\sigma(n)}$$

is a measure on $\mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n$.

5. Show that for all $E \in \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n$, we have:

$$\mu_1 \otimes \dots \otimes \mu_n(E) = \int_{\Omega_{\sigma(n)}}\dots\int_{\Omega_{\sigma(1)}} 1_E d\mu_{\sigma(1)}\dots d\mu_{\sigma(n)}$$

6. Show the following:

Theorem 32 *Let $(\Omega_1, \mathcal{F}_1, \mu_1), \ldots, (\Omega_n, \mathcal{F}_n, \mu_n)$ be n σ-finite measure spaces, with $n \geq 2$. Let $f : (\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n) \to [0, +\infty]$ be a non-negative and measurable map. let σ be a permutation of $\mathbf{N}_n$. Then:*

$$\int_{\Omega_1 \times \ldots \times \Omega_n} f d\mu_1 \otimes \ldots \otimes \mu_n = \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$

EXERCISE 14. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Define:

$$L^1 \stackrel{\triangle}{=} \{f : \Omega \to \bar{\mathbf{R}} \ , \ \exists g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \ , \ f = g \ \mu\text{-a.s.}\}$$

1. Show that if $f \in L^1$, then $|f| < +\infty$, μ-a.s.

2. Suppose there exists $A \subseteq \Omega$, such that $A \notin \mathcal{F}$ and $A \subseteq N$ for some $N \in \mathcal{F}$ with $\mu(N) = 0$. Show that $1_A \in L^1$ and 1_A is not measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\bar{\mathbf{R}})$.

3. Explain why if $f \in L^1$, the integrals $\int |f| d\mu$ and $\int f d\mu$ may not be well defined.

4. Suppose that $f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is a measurable map with $\int |f| d\mu < +\infty$. Show that $f \in L^1$.

5. Show that if $f \in L^1$ and $f = f_1$ μ-a.s. then $f_1 \in L^1$.

6. Suppose that $f \in L^1$ and $g_1, g_2 \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ are such that $f = g_1$ μ-a.s. and $f = g_2$ μ-a.s.. Show that $\int g_1 d\mu = \int g_2 d\mu$.

7. Propose a definition of the integral $\int f d\mu$ for $f \in L^1$ which extends the integral defined on $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

Exercise 15. Further to ex. (14), Let $(f_n)_{n\geq 1}$ be a sequence in L^1, and $f, h \in L^1$, with $f_n \to f$ μ-a.s. and for all $n \geq 1$, $|f_n| \leq h$ μ-a.s..

1. Show the existence of $N_1 \in \mathcal{F}, \mu(N_1) = 0$, such that for all $\omega \in N_1^c$, $f_n(\omega) \to f(\omega)$, and for all $n \geq 1$, $|f_n(\omega)| \leq h(\omega)$.

2. Show the existence of $g_n, g, h_1 \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ and $N_2 \in \mathcal{F}$, $\mu(N_2) = 0$, such that for all $\omega \in N_2^c$, $g(\omega) = f(\omega)$, $h(\omega) = h_1(\omega)$, and for all $n \geq 1$, $g_n(\omega) = f_n(\omega)$.

3. Show the existence of $N \in \mathcal{F}$, $\mu(N) = 0$, such that for all $\omega \in N^c$, $g_n(\omega) \to g(\omega)$, and for all $n \geq 1$, $|g_n(\omega)| \leq h_1(\omega)$.

4. Show that the Dominated Convergence Theorem can be applied to $g_n 1_{N^c}, g 1_{N^c}$ and $h_1 1_{N^c}$.

5. Recall the definition of $\int |f_n - f| d\mu$ when $f, f_n \in L^1$.

6. Show that $\int |f_n - f| d\mu \to 0$.

Exercise 16. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two σ-finite measure spaces. Let f be an element of $L^1_{\mathbf{R}}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$. Let $\theta : (\Omega_2 \times \Omega_1, \mathcal{F}_2 \otimes \mathcal{F}_1) \to (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ be the map defined by $\theta(\omega_2, \omega_1) = (\omega_1, \omega_2)$ for all $(\omega_2, \omega_1) \in \Omega_2 \times \Omega_1$.

1. Let $A = \{\omega_1 \in \Omega_1 \ : \ \int_{\Omega_2} |f(\omega_1, x)| d\mu_2(x) < +\infty\}$. Show that $A \in \mathcal{F}_1$ and $\mu_1(A^c) = 0$.

2. Show that $f(\omega_1, .) \in L^1_{\mathbf{R}}(\Omega_2, \mathcal{F}_2, \mu_2)$ for all $\omega_1 \in A$.

3. Show that $\bar{I}(\omega_1) = \int_{\Omega_2} f(\omega_1, x) d\mu_2(x)$ is well defined for all $\omega_1 \in A$. Let I be an arbitrary extension of $\bar{I}$, on Ω_1.

4. Define $J = I 1_A$. Show that:

$$J(\omega) = 1_A(\omega) \int_{\Omega_2} f^+(\omega, x) d\mu_2(x) - 1_A(\omega) \int_{\Omega_2} f^-(\omega, x) d\mu_2(x)$$

5. Show that J is $\mathcal{F}_1$-measurable and $\mathbf{R}$-valued.

6. Show that $J \in L^1_{\mathbf{R}}(\Omega_1, \mathcal{F}_1, \mu_1)$ and that $J = I$ μ_1-a.s.

7. Propose a definition for the integral:

$$\int_{\Omega_1} \left(\int_{\Omega_2} f(x, y) d\mu_2(y) \right) d\mu_1(x)$$

8. Show that $\int_{\Omega_1} (1_A \int_{\Omega_2} f^+ d\mu_2) d\mu_1 = \int_{\Omega_1 \times \Omega_2} f^+ d\mu_1 \otimes \mu_2$.

9. Show that:

$$\int_{\Omega_1}\left(\int_{\Omega_2} f(x,y)d\mu_2(y)\right)d\mu_1(x) = \int_{\Omega_1\times\Omega_2} f d\mu_1\otimes\mu_2 \qquad (1)$$

10. Show that if $f \in L^1_{\mathbf{C}}(\Omega_1\times\Omega_2, \mathcal{F}_1\otimes\mathcal{F}_2, \mu_1\otimes\mu_2)$, then the map $\omega_1 \to \int_{\Omega_2} f(\omega_1,y)d\mu_2(y)$ is μ_1-almost surely equal to an element of $L^1_{\mathbf{C}}(\Omega_1,\mathcal{F}_1,\mu_1)$, and furthermore that (1) is still valid.

11. Show that if $f : (\Omega_1\times\Omega_2, \mathcal{F}_1\otimes\mathcal{F}_2) \to [0,+\infty]$ is non-negative and measurable, then $f\circ\theta$ is non-negative and measurable, and:

$$\int_{\Omega_2\times\Omega_1} f\circ\theta d\mu_2\otimes\mu_1 = \int_{\Omega_1\times\Omega_2} f d\mu_1\otimes\mu_2$$

12. Show that if $f \in L^1_{\mathbf{C}}(\Omega_1\times\Omega_2, \mathcal{F}_1\otimes\mathcal{F}_2, \mu_1\otimes\mu_2)$, then $f\circ\theta$ is an element of $L^1_{\mathbf{C}}(\Omega_2\times\Omega_1, \mathcal{F}_2\otimes\mathcal{F}_1, \mu_2\otimes\mu_1)$, and:

$$\int_{\Omega_2\times\Omega_1} f\circ\theta d\mu_2\otimes\mu_1 = \int_{\Omega_1\times\Omega_2} f d\mu_1\otimes\mu_2$$

13. Show that if $f \in L^1_{\mathbf{C}}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$, then the map $\omega_2 \to \int_{\Omega_1} f(x, \omega_2) d\mu_1(x)$ is μ_2-almost surely equal to an element of $L^1_{\mathbf{C}}(\Omega_2, \mathcal{F}_2, \mu_2)$, and furthermore:

$$\int_{\Omega_2} \left(\int_{\Omega_1} f(x,y) d\mu_1(x) \right) d\mu_2(y) = \int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2$$

Theorem 33 *Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two σ-finite measure spaces. Let $f \in L^1_{\mathbf{C}}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$. Then, the map:*

$$\omega_1 \to \int_{\Omega_2} f(\omega_1, x) d\mu_2(x)$$

is μ_1-almost surely equal to an element of $L^1_{\mathbf{C}}(\Omega_1, \mathcal{F}_1, \mu_1)$ and:

$$\int_{\Omega_1} \left(\int_{\Omega_2} f(x,y) d\mu_2(y) \right) d\mu_1(x) = \int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2$$

Furthermore, the map:

$$\omega_2 \to \int_{\Omega_1} f(x, \omega_2) d\mu_1(x)$$

is μ_2*-almost surely equal to an element of* $L^1_{\mathbf{C}}(\Omega_2, \mathcal{F}_2, \mu_2)$ *and:*

$$\int_{\Omega_2} \left(\int_{\Omega_1} f(x, y) d\mu_1(x) \right) d\mu_2(y) = \int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2$$

Exercise 17. Let $(\Omega_1, \mathcal{F}_1, \mu_1), \ldots, (\Omega_n, \mathcal{F}_n, \mu_n)$ be n σ-finite measure spaces, $n \geq 2$. Let $f \in L^1_{\mathbf{C}}(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n, \mu_1 \otimes \ldots \otimes \mu_n)$. Let σ be a permutation of $\mathbf{N}_n$.

1. For all $\omega \in \Pi_{i \neq \sigma(1)} \Omega_i$, define:

$$J_1(\omega) \stackrel{\triangle}{=} \int_{\Omega_{\sigma(1)}} f(\omega, x) d\mu_{\sigma(1)}(x)$$

   Explain why J_1 is well defined and equal to an element of $L^1_{\mathbf{C}}(\Pi_{i \neq \sigma(1)} \Omega_i, \otimes_{i \neq \sigma(1)} \mathcal{F}_i, \otimes_{i \neq \sigma(1)} \mu_i)$, $\otimes_{i \neq \sigma(1)} \mu_i$-almost surely.

2. Suppose $1 \leq k < n-2$ and that $\bar{J}_k$ is well defined and equal to an element of:

$$L^1_{\mathbf{C}}(\Pi_{i \notin \{\sigma(1),\ldots,\sigma(k)\}}\Omega_i, \otimes_{i \notin \{\sigma(1),\ldots,\sigma(k)\}}\mathcal{F}_i, \otimes_{i \notin \{\sigma(1),\ldots,\sigma(k)\}}\mu_i)$$

$\otimes_{i \notin \{\sigma(1),\ldots,\sigma(k)\}}\mu_i$-almost surely. Define:

$$J_{k+1}(\omega) \stackrel{\triangle}{=} \int_{\Omega_{\sigma(k+1)}} \bar{J}_k(\omega, x) d\mu_{\sigma(k+1)}(x)$$

What can you say about J_{k+1}.

3. Show that:

$$\int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$

is a well defined complex number. (Propose a definition for it).

4. Show that:

$$\int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)} = \int_{\Omega_1 \times \ldots \times \Omega_n} f d\mu_1 \otimes \ldots \otimes \mu_n$$