17. Image Measure

In the following, $\mathbf{K}$ denotes $\mathbf{R}$ or $\mathbf{C}$. We denote $\mathcal{M}_n(\mathbf{K})$, $n \geq 1$, the set of all $n \times n$-matrices with $\mathbf{K}$-valued entries. We recall that for all $M = (m_{ij}) \in \mathcal{M}_n(\mathbf{K})$, M is identified with the linear map $M : \mathbf{K}^n \to \mathbf{K}^n$ uniquely determined by:

$$\forall j = 1, \ldots, n \ , \ Me_j \stackrel{\triangle}{=} \sum_{i=1}^{n} m_{ij} e_i$$

where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbf{K}^n$, i.e. $e_i \stackrel{\triangle}{=} (0, ., \overbrace{1}^{i}, ., 0)$.

Exercise 1. For all $\alpha \in \mathbf{K}$, let $H_\alpha \in \mathcal{M}_n(\mathbf{K})$ be defined by:

$$H_\alpha \stackrel{\triangle}{=} \begin{pmatrix} \alpha & & & \\ & 1 & 0 & \\ & 0 & \ddots & \\ & & & 1 \end{pmatrix}$$

i.e. by $H_\alpha e_1 = \alpha e_1$, $H_\alpha e_j = e_j$, for all $j \geq 2$. Note that H_α is obtained from the identity matrix, by multiplying the top left entry by α. For $k, l \in \{1, \ldots, n\}$, we define the matrix $\Sigma_{kl} \in \mathcal{M}_n(\mathbf{K})$ by $\Sigma_{kl} e_k = e_l$, $\Sigma_{kl} e_l = e_k$ and $\Sigma_{kl} e_j = e_j$, for all $j \in \{1, \ldots, n\} \setminus \{k, l\}$. Note that Σ_{kl} is obtained from the identity matrix, by interchanging column k and column l. If $n \geq 2$, we define the matrix $U \in \mathcal{M}_n(\mathbf{K})$ by:

$$U \stackrel{\triangle}{=} \begin{pmatrix} 1 & 0 & & \\ 1 & 1 & 0 & \\ & 0 & \ddots & \\ & & & 1 \end{pmatrix}$$

i.e. by $Ue_1 = e_1 + e_2$, $Ue_j = e_j$ for all $j \geq 2$. Note that the matrix U is obtained from the identity matrix, by adding column 2 to column 1. If $n = 1$, we put $U = 1$. We define $\mathcal{N}_n(\mathbf{K}) = \{H_\alpha : \ \alpha \in \mathbf{K}\} \cup \{\Sigma_{kl} : k, l = 1, \ldots, n\} \cup \{U\}$, and $\mathcal{M}'_n(\mathbf{K})$ to be the set of all finite products of elements of $\mathcal{N}_n(\mathbf{K})$:

$$\mathcal{M}'_n(\mathbf{K}) \stackrel{\triangle}{=} \{M \in \mathcal{M}_n(\mathbf{K}) : M = Q_1 \ldots . Q_p \, , \, p \geq 1 \, , \, Q_j \in \mathcal{N}_n(\mathbf{K}) \, , \, \forall j\}$$

We shall prove that $\mathcal{M}_n(\mathbf{K}) = \mathcal{M}'_n(\mathbf{K})$.

1. Show that if $\alpha \in \mathbf{K} \setminus \{0\}$, H_α is non-singular with $H_\alpha^{-1} = H_{1/\alpha}$

2. Show that if $k, l = 1, \ldots, n$, Σ_{kl} is non-singular with $\Sigma_{kl}^{-1} = \Sigma_{kl}$.

3. Show that U is non-singular, and that for $n \geq 2$:

$$U^{-1} = \begin{pmatrix} 1 & 0 & & \\ -1 & 1 & 0 & \\ & 0 & \ddots & \\ & & & 1 \end{pmatrix}$$

4. Let $M = (m_{ij}) \in \mathcal{M}_n(\mathbf{K})$. Let $R_1, \ldots, R_n$ be the rows of M:

$$M \triangleq \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$$

Show that for all $\alpha \in \mathbf{K}$:

$$H_\alpha . M = \begin{pmatrix} \alpha R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$$

Conclude that *multiplying M by H_α from the left, amounts to multiplying the first row of M by α.*

5. Show that *multiplying M by H_α from the right, amounts to multiplying the first column of M by α.*

6. Show that *multiplying M by Σ_{kl} from the left, amounts to interchanging the rows R_l and R_k.*

7. Show that *multiplying M by Σ_{kl} from the right, amounts to interchanging the columns C_l and C_k.*

8. Show that *multiplying M by U^{-1} from the left ($n \geq 2$), amounts to subtracting R_1 from R_2*, i.e.:

$$U^{-1}.\begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 - R_1 \\ \vdots \\ R_n \end{pmatrix}$$

9. Show that *multiplying M by U^{-1} from the right (for $n \geq 2$), amounts to subtracting C_2 from C_1.*

10. Define $U' = \Sigma_{12}.U^{-1}.\Sigma_{12}$, ($n \geq 2$). Show that multiplying M by U' from the right, amounts to subtracting C_1 from C_2.

11. Show that if $n = 1$, then indeed we have $\mathcal{M}_1(\mathbf{K}) = \mathcal{M}'_1(\mathbf{K})$.

Exercise 2. Further to exercise (1), we now assume that $n \geq 2$, and make the induction hypothesis that $\mathcal{M}_{n-1}(\mathbf{K}) = \mathcal{M}'_{n-1}(\mathbf{K})$.

1. Let $O_n \in \mathcal{M}_n(\mathbf{K})$ be the matrix with all entries equal to zero. Show the existence of $Q'_1, \ldots, Q'_p \in \mathcal{N}_{n-1}(\mathbf{K})$, $p \geq 1$, such that:

$$O_{n-1} = Q'_1 \ldots . Q'_p$$

2. For $k = 1, \ldots, p$, we define $Q_k \in \mathcal{M}_n(\mathbf{K})$, by:

$$Q_k \stackrel{\triangle}{=} \left(\begin{array}{cccc} & & & 0 \\ & Q'_k & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{array} \right)$$

Show that $Q_k \in \mathcal{N}_n(\mathbf{K})$, and that we have:

$$\Sigma_{1n}.Q_1.\ldots.Q_p.\Sigma_{1n} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & O_{n-1} & \\ 0 & & & \end{pmatrix}$$

3. Conclude that $O_n \in \mathcal{M}'_n(\mathbf{K})$.

4. We now consider $M = (m_{ij}) \in \mathcal{M}_n(\mathbf{K})$, $M \neq O_n$. We want to show that $M \in \mathcal{M}'_n(\mathbf{K})$. Show that for some $k, l \in \{1, \ldots, n\}$:

$$H^{-1}_{m_{kl}}.\Sigma_{1k}.M.\Sigma_{1l} = \begin{pmatrix} 1 & * & \ldots & * \\ * & & & \\ \vdots & & * & \\ * & & & \end{pmatrix}$$

5. Show that if $H^{-1}_{m_{kl}}.\Sigma_{1k}.M.\Sigma_{1l} \in \mathcal{M}'_n(\mathbf{K})$, then $M \in \mathcal{M}'_n(\mathbf{K})$. Conclude that without loss of generality, in order to prove that

M lies in $\mathcal{M}'_n(\mathbf{K})$ we can assume that $m_{11} = 1$.

6. Let $i = 2, \ldots, n$. Show that if $m_{i1} \neq 0$, we have:

$$H^{-1}_{m_{i1}}.\Sigma_{2i}.U^{-1}.\Sigma_{2i}.H^{-1}_{1/m_{i1}}.M = \begin{pmatrix} 1 & * & \ldots & * \\ * & & & \\ 0 & \leftarrow i & * & \\ * & & & \end{pmatrix}$$

7. Conclude that without loss of generality, we can assume that $m_{i1} = 0$ for all $i \geq 2$, i.e. that M is of the form:

$$M = \begin{pmatrix} 1 & * & \ldots & * \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix}$$

8. Show that in order to prove that $M \in \mathcal{M}'_n(\mathbf{K})$, without loss of

generality, we can assume that M is of the form:

$$M = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & M' & \\ 0 & & & \end{pmatrix}$$

9. Prove that $M \in \mathcal{M}'_n(\mathbf{K})$ and conclude with the following:

Theorem 103 *Given $n \geq 2$, any $n \times n$-matrix with values in $\mathbf{K}$ is a finite product of matrices Q of the following types:*

(i) $Qe_1 = \alpha e_1 \ , \ Qe_j = e_j \ , \ \forall j = 2, \dots, n \ , \ (\alpha \in \mathbf{K})$

(ii) $Qe_l = e_k \ , \ Qe_k = e_l \ , \ Qe_j = e_j \ , \ \forall j \neq k, l \ , \ (k, l \in \mathbf{N}_n)$

(iii) $Qe_1 = e_1 + e_2 \ , \ Qe_j = e_j \ , \ \forall j = 2, \dots, n$

where $(e_1, \dots, e_n)$ is the canonical basis of $\mathbf{K}^n$.

Definition 123 *Let $X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')$ be a measurable map, where $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ are two measurable spaces. Let μ be a (possibly complex) measure on $(\Omega, \mathcal{F})$. Then, we call* **distribution** *of X under μ, or* **image measure** *of μ by X, or even* **law** *of X under μ, the (possibly complex) measure on $(\Omega', \mathcal{F}')$, denoted μ^X, $X(\mu)$ or $\mathcal{L}_\mu(X)$, and defined by:*

$$\forall B \in \mathcal{F}' \ , \ \mu^X(B) \stackrel{\triangle}{=} \mu(\{X \in B\}) = \mu(X^{-1}(B))$$

Exercise 3. Let $X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')$ be a measurable map, where $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ are two measurable spaces.

1. Let $B \in \mathcal{F}'$. Show that if $(B_n)_{n\geq 1}$ is a measurable partition of B, then $(X^{-1}(B_n))_{n\geq 1}$ is a measurable partition of $X^{-1}(B)$.
2. Show that if μ is a measure on $(\Omega, \mathcal{F})$, μ^X is a well-defined measure on $(\Omega', \mathcal{F}')$.
3. Show that if μ is a complex measure on $(\Omega, \mathcal{F})$, μ^X is a well-defined complex measure on $(\Omega', \mathcal{F}')$.

4. Show that if μ is a complex measure on $(\Omega, \mathcal{F})$, then $|\mu^X| \leq |\mu|^X$.

5. Let $Y : (\Omega', \mathcal{F}') \to (\Omega'', \mathcal{F}'')$ be a measurable map, where $(\Omega'', \mathcal{F}'')$ is another measurable space. Show that for all (possibly complex) measure μ on $(\Omega, \mathcal{F})$, we have:

$$Y(X(\mu)) = (Y \circ X)(\mu) = (\mu^X)^Y = \mu^{(Y \circ X)}$$

Definition 124 *Let μ be a (possibly complex) measure on $\mathbf{R}^n$, $n \geq 1$. We say that μ is* **invariant by translation**, *if and only if $\tau_a(\mu) = \mu$ for all $a \in \mathbf{R}^n$, where $\tau_a : \mathbf{R}^n \to \mathbf{R}^n$ is the translation mapping defined by $\tau_a(x) = a + x$, for all $x \in \mathbf{R}^n$.*

EXERCISE 4. Let μ be a (possibly complex) measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$.

1. Show that $\tau_a : (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n)) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ is measurable.

2. Show $\tau_a(\mu)$ is therefore a well-defined (possibly complex) measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, for all $a \in \mathbf{R}^n$.

3. Show that $\tau_a(dx) = dx$ for all $a \in \mathbf{R}^n$.

4. Show the Lebesgue measure on $\mathbf{R}^n$ is invariant by translation.

Exercise 5. Let $k_\alpha : \mathbf{R}^n \to \mathbf{R}^n$ be defined by $k_\alpha(x) = \alpha x$, $\alpha > 0$.

1. Show that $k_\alpha : (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n)) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ is measurable.

2. Show that $k_\alpha(dx) = \alpha^{-n} dx$.

Exercise 6. Show the following:

Theorem 104 (Integral Projection 1) *Let* $X:(\Omega,\mathcal{F})\to(\Omega',\mathcal{F}')$ *be a measurable map, where* $(\Omega,\mathcal{F})$, $(\Omega',\mathcal{F}')$ *are measurable spaces. Let* μ *be a measure on* $(\Omega,\mathcal{F})$. *Then, for all* $f:(\Omega',\mathcal{F}')\to[0,+\infty]$ *non-negative and measurable, we have:*

$$\int_\Omega f\circ X d\mu = \int_{\Omega'} f dX(\mu)$$

Exercise 7. Show the following:

Theorem 105 (Integral Projection 2) *Let* $X:(\Omega,\mathcal{F})\to(\Omega',\mathcal{F}')$ *be a measurable map, where* $(\Omega,\mathcal{F})$, $(\Omega',\mathcal{F}')$ *are measurable spaces. Let* μ *be a measure on* $(\Omega,\mathcal{F})$. *Then, for all* $f:(\Omega',\mathcal{F}')\to(\mathbf{C},\mathcal{B}(\mathbf{C}))$ *measurable, we have the equivalence:*

$$f\circ X\in L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu) \;\Leftrightarrow\; f\in L^1_{\mathbf{C}}(\Omega',\mathcal{F}',X(\mu))$$

in which case, we have:

$$\int_\Omega f\circ X d\mu = \int_{\Omega'} f dX(\mu)$$

Exercise 8. Further to theorem (105), suppose μ is in fact a complex measure on $(\Omega, \mathcal{F})$. Show that:

$$\int_{\Omega'} |f| d|X(\mu)| \leq \int_{\Omega} |f \circ X| d|\mu| \tag{1}$$

Conclude with the following:

Theorem 106 (Integral Projection 3) *Let $X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')$ be a measurable map, where $(\Omega, \mathcal{F})$, $(\Omega', \mathcal{F}')$ are measurable spaces. Let μ be a complex measure on $(\Omega, \mathcal{F})$. Then, for all measurable maps $f : (\Omega', \mathcal{F}') \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$, we have:*

$$f \circ X \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \ \Rightarrow \ f \in L^1_{\mathbf{C}}(\Omega', \mathcal{F}', X(\mu))$$

and when the left-hand side of this implication is satisfied:

$$\int_{\Omega} f \circ X d\mu = \int_{\Omega'} f dX(\mu)$$

Exercise 9. Let $X : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ be a measurable map with distribution $\mu = X(P)$, where $(\Omega, \mathcal{F}, P)$ is a probability space.

1. Show that X is integrable, i.e. $\int |X|dP < +\infty$, if and only if:

$$\int_{-\infty}^{+\infty} |x|d\mu(x) < +\infty$$

2. Show that if X is integrable, then:

$$E[X] = \int_{-\infty}^{+\infty} x d\mu(x)$$

3. Show that:

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 d\mu(x)$$

Exercise 10. Let μ be a locally finite measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, which is invariant by translation. For all $a = (a_1, \ldots, a_n) \in (\mathbf{R}^+)^n$, we define $Q_a = [0, a_1[\times \ldots \times [0, a_n[$, and in particular $Q = Q_{(1,\ldots,1)} = [0, 1[^n$.

1. Show that $\mu(Q_a) < +\infty$ for all $a \in (\mathbf{R}^+)^n$, and $\mu(Q) < +\infty$.

2. Let $p = (p_1, \ldots, p_n)$ where $p_i \geq 1$ is an integer for all i's. Show:

$$Q_p = \biguplus_{\substack{k \in \mathbf{N}^n \\ 0 \leq k_i < p_i}} [k_1, k_1 + 1[\times \ldots \times [k_n, k_n + 1[$$

3. Show that $\mu(Q_p) = p_1 \ldots p_n \mu(Q)$.

4. Let $q_1, \ldots, q_n \geq 1$ be n positive integers. Show that:

$$Q_p = \biguplus_{\substack{k \in \mathbf{N}^n \\ 0 \leq k_i < q_i}} [\frac{k_1 p_1}{q_1}, \frac{(k_1 + 1)p_1}{q_1}[\times \ldots \times [\frac{k_n p_n}{q_n}, \frac{(k_n + 1)p_n}{q_n}[$$

5. Show that $\mu(Q_p) = q_1 \ldots q_n \mu(Q_{(p_1/q_1, \ldots, p_n/q_n)})$

6. Show that $\mu(Q_r) = r_1 \ldots r_n \mu(Q)$, for all $r \in (\mathbf{Q}^+)^n$.

7. Show that $\mu(Q_a) = a_1 \ldots a_n \mu(Q)$, for all $a \in (\mathbf{R}^+)^n$.

8. Show that $\mu(B) = \mu(Q)dx(B)$, for all $B \in \mathcal{C}$, where:

$$\mathcal{C} \stackrel{\triangle}{=} \{[a_1, b_1[\times \ldots \times [a_n, b_n[\ ,\ a_i, b_i \in \mathbf{R}\ ,\ a_i \leq b_i\ ,\ \forall i \in \mathbf{N}^n\}$$

9. Show that $B(\mathbf{R}^n) = \sigma(\mathcal{C})$.

10. Show that $\mu = \mu(Q)dx$, and conclude with the following:

Theorem 107 *Let μ be a locally finite measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$. If μ is invariant by translation, then there exists $\alpha \in \mathbf{R}^+$ such that:*

$$\mu = \alpha dx$$

Exercise 11. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear bijection.

1. Show that T and T^{-1} are continuous.

2. Show that for all $B \subseteq \mathbf{R}^n$, the inverse image $T^{-1}(B) = \{T \in B\}$ coincides with the direct image:

$$T^{-1}(B) \stackrel{\triangle}{=} \{y : \ y = T^{-1}(x) \text{ for some } x \in B\}$$

3. Show that for all $B \subseteq \mathbf{R}^n$, the direct image $T(B)$ coincides with the inverse image $(T^{-1})^{-1}(B) = \{T^{-1} \in B\}$.

4. Let $K \subseteq \mathbf{R}^n$ be compact. Show that $\{T \in K\}$ is compact.

5. Show that $T(dx)$ is a locally finite measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$.

6. Let τ_a be the translation of vector $a \in \mathbf{R}^n$. Show that:

$$T \circ \tau_{T^{-1}(a)} = \tau_a \circ T$$

7. Show that $T(dx)$ is invariant by translation.

8. Show the existence of $\alpha \in \mathbf{R}^+$, such that $T(dx) = \alpha dx$. Show that such constant is unique, and denote it by $\Delta(T)$.

9. Show that $Q = T([0,1]^n) \in \mathcal{B}(\mathbf{R}^n)$ and that we have:

$$\Delta(T)dx(Q) = T(dx)(Q) = 1$$

10. Show that $\Delta(T) \neq 0$.

11. Let $T_1, T_2 : \mathbf{R}^n \to \mathbf{R}^n$ be two linear bijections. Show that:

$$(T_1 \circ T_2)(dx) = \Delta(T_1)\Delta(T_2)dx$$

and conclude that $\Delta(T_1 \circ T_2) = \Delta(T_1)\Delta(T_2)$.

Exercise 12. Let $\alpha \in \mathbf{R} \setminus \{0\}$. Let $H_\alpha : \mathbf{R}^n \to \mathbf{R}^n$ be the linear bijection uniquely defined by $H_\alpha(e_1) = \alpha e_1$, $H_\alpha(e_j) = e_j$ for $j \geq 2$.

1. Show that $H_\alpha(dx)([0,1]^n) = |\alpha|^{-1}$.

2. Conclude that $\Delta(H_\alpha) = |\det H_\alpha|^{-1}$.

Exercise 13. Let $k, l \in \mathbf{N}_n$and $\Sigma : \mathbf{R}^n \to \mathbf{R}^n$ be the linear bijection uniquely defined by $\Sigma(e_k) = e_l$, $\Sigma(e_l) = e_k$, $\Sigma(e_j) = e_j$, for $j \neq k, l$.

1. Show that $\Sigma(dx)([0,1]^n) = 1$.
2. Show that $\Sigma.\Sigma = I_n$. (Identity mapping on $\mathbf{R}^n$).
3. Show that $|\det \Sigma| = 1$.
4. Conclude that $\Delta(\Sigma) = |\det \Sigma|^{-1}$.

Exercise 14. Let $n \geq 2$ and $U : \mathbf{R}^n \to \mathbf{R}^n$ be the linear bijection uniquely defined by $U(e_1) = e_1 + e_2$ and $U(e_j) = e_j$ for $j \geq 2$. Let $Q = [0,1[^n$.

1. Show that:

$$U^{-1}(Q) = \{x \in \mathbf{R}^n : \ 0 \leq x_1 + x_2 < 1 \ , \ 0 \leq x_i < 1 \ , \ \forall i \neq 2\}$$

2. Define:

$$\Omega_1 \stackrel{\triangle}{=} U^{-1}(Q) \cap \{x \in \mathbf{R}^n : \ x_2 \geq 0\}$$
$$\Omega_2 \stackrel{\triangle}{=} U^{-1}(Q) \cap \{x \in \mathbf{R}^n : \ x_2 < 0\}$$

   Show that $\Omega_1, \Omega_2 \in \mathcal{B}(\mathbf{R}^n)$.

3. Let τ_{e_2} be the translation of vector e_2. Draw a picture of Q, Ω_1, Ω_2 and $\tau_{e_2}(\Omega_2)$ in the case when $n = 2$.

4. Show that if $x \in \Omega_1$, then $0 \leq x_2 < 1$.

5. Show that $\Omega_1 \subseteq Q$.

6. Show that if $x \in \tau_{e_2}(\Omega_2)$, then $0 \leq x_2 < 1$.

7. Show that $\tau_{e_2}(\Omega_2) \subseteq Q$.

8. Show that if $x \in Q$ and $x_1 + x_2 < 1$ then $x \in \Omega_1$.

9. Show that if $x \in Q$ and $x_1 + x_2 \geq 1$ then $x \in \tau_{e_2}(\Omega_2)$.

10. Show that if $x \in \tau_{e_2}(\Omega_2)$ then $x_1 + x_2 \geq 1$.

11. Show that $\tau_{e_2}(\Omega_2) \cap \Omega_1 = \emptyset$.

12. Show that $Q = \Omega_1 \uplus \tau_{e_2}(\Omega_2)$.

13. Show that $dx(Q) = dx(U^{-1}(Q))$.

14. Show that $\Delta(U) = 1$.

15. Show that $\Delta(U) = |\det U|^{-1}$.

Exercise 15. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear bijection, $(n \geq 1)$.

1. Show the existence of linear bijections $Q_1, \ldots, Q_p : \mathbf{R}^n \to \mathbf{R}^n$, $p \geq 1$, with $T = Q_1 \circ \ldots \circ Q_p$, $\Delta(Q_i) = |\det Q_i|^{-1}$ for all $i \in \mathbf{N}_p$.

2. Show that $\Delta(T) = |\det T|^{-1}$.

3. Conclude with the following:

Theorem 108 *Let $n \geq 1$ and $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear bijection. Then, the image measure $T(dx)$ of the Lebesgue measure on $\mathbf{R}^n$ is:*

$$T(dx) = |\det T|^{-1} dx$$

Exercise 16. Let $f : (\mathbf{R}^2, \mathcal{B}(\mathbf{R}^2)) \to [0, +\infty]$ be a non-negative and measurable map. Let $a, b, c, d \in \mathbf{R}$ such that $ad - bc \neq 0$. Show that:

$$\int_{\mathbf{R}^2} f(ax + by, cx + dy) dx dy = |ad - bc|^{-1} \int_{\mathbf{R}^2} f(x, y) dx dy$$

Exercise 17. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear bijection. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$, we have $T(B) \in \mathcal{B}(\mathbf{R}^n)$ and:

$$dx(T(B)) = |\det T| dx(B)$$

Exercise 18. Let V be a linear subspace of $\mathbf{R}^n$ and $p = \dim V$. We assume that $1 \leq p \leq n - 1$. Let $u_1, \ldots, u_p$ be an orthonormal basis of

V, and $u_{p+1},\ldots,u_n$ be such that $u_1,\ldots,u_n$ is an orthonormal basis of $\mathbf{R}^n$. For $i\in\mathbf{N}_n$, Let $\phi_i:\mathbf{R}^n\to\mathbf{R}$ be defined by $\phi_i(x)=\langle u_i,x\rangle$.

1. Show that all ϕ_i's are continuous.

2. Show that $V=\bigcap_{j=p+1}^n\phi_j^{-1}(\{0\})$.

3. Show that V is a closed subset of $\mathbf{R}^n$.

4. Let $Q=(q_{ij})\in\mathcal{M}_n(\mathbf{R})$ be the matrix uniquely defined by $Qe_j=u_j$ for all $j\in\mathbf{N}_n$, where $(e_1,\ldots,e_n)$ is the canonical basis of $\mathbf{R}^n$. Show that for all $i,j\in\mathbf{N}_n$:

$$\langle u_i,u_j\rangle=\sum_{k=1}^n q_{ki}q_{kj}$$

5. Show that $Q^t.Q=I_n$ and conclude that $|\det Q|=1$.

6. Show that $dx(\{Q\in V\})=dx(V)$.

7. Show that $\{Q \in V\} = \text{span}(e_1, \ldots, e_p)$.[1]

8. For all $m \geq 1$, we define:

$$E_m \stackrel{\triangle}{=} \overbrace{[-m, m] \times \ldots \times [-m, m]}^{n-1} \times \{0\}$$

   Show that $dx(E_m) = 0$ for all $m \geq 1$.

9. Show that $dx(\text{span}(e_1, \ldots, e_{n-1})) = 0$.

10. Conclude with the following:

Theorem 109 *Let $n \geq 1$. Any linear subspace V of $\mathbf{R}^n$ is a closed subset of $\mathbf{R}^n$. Moreover, if $\dim V \leq n-1$, then $dx(V) = 0$.*

[1] i.e. the linear subspace of $\mathbf{R}^n$ generated by $e_1, \ldots, e_p$.