18. The Jacobian Formula

In the following, $\mathbf{K}$ denotes $\mathbf{R}$ or $\mathbf{C}$.

Definition 125 *We call* $\mathbf{K}$-**normed space**, *an ordered pair* (E, N), *where* E *is a* $\mathbf{K}$*-vector space, and* $N : E \to \mathbf{R}^+$ *is a norm on* E.

See definition (89) for *vector space*, and definition (95) for *norm*.

EXERCISE 1. Let $\langle \cdot, \cdot \rangle$ be an inner-product on a $\mathbf{K}$-vector space $\mathcal{H}$.

1. Show that $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$ is a norm on $\mathcal{H}$.
2. Show that $(\mathcal{H}, \| \cdot \|)$ is a $\mathbf{K}$-normed space.

EXERCISE 2. Let $(E, \| \cdot \|)$ be a $\mathbf{K}$-normed space:

1. Show that $d(x, y) = \|x - y\|$ defines a metric on E.
2. Show that for all $x, y \in E$, we have $|\, \|x\| - \|y\| \,| \leq \|x - y\|$.

Definition 126 *Let $(E, \|\cdot\|)$ be a $\mathbf{K}$-normed space, and d be the metric defined by $d(x,y) = \|x-y\|$. We call* **norm topology** *on E, denoted $\mathcal{T}_{\|\cdot\|}$, the topology on E associated with d.*

Note that this definition is consistent with definition (82) of the norm topology associated with an inner-product.

EXERCISE 3. Let E, F be two $\mathbf{K}$-normed spaces, and $l : E \to F$ be a linear map. Show that the following are equivalent:

$$
\begin{array}{rl}
(i) & l \text{ is continuous (w.r. to the norm topologies)} \\
(ii) & l \text{ is continuous at } x = 0. \\
(iii) & \exists K \in \mathbf{R}^+ \ , \ \forall x \in E \ , \ \|l(x)\| \leq K\|x\| \\
(iv) & \sup\{\|l(x)\| : \ x \in E \ , \ \|x\| = 1\} < +\infty
\end{array}
$$

Definition 127 *Let E, F be $\mathbf{K}$-normed spaces. The $\mathbf{K}$-vector space of all* **continuous linear maps** *$l : E \to F$ is denoted $\mathcal{L}_{\mathbf{K}}(E,F)$.*

Exercise 4. Show that $\mathcal{L}_{\mathbf{K}}(E,F)$ is indeed a **K**-vector space.

Exercise 5. Let E, F be **K**-normed spaces. Given $l \in \mathcal{L}_{\mathbf{K}}(E,F)$, let:

$$\|l\| \stackrel{\triangle}{=} \sup\{\|l(x)\| : \ x \in E \ , \ \|x\| = 1\} < +\infty$$

1. Show that:

$$\|l\| = \sup\{\|l(x)\| : \ x \in E \ , \ \|x\| \leq 1\}$$

2. Show that:

$$\|l\| = \sup\left\{\frac{\|l(x)\|}{\|x\|} : \ x \in E \ , \ x \neq 0\right\}$$

3. Show that $\|l(x)\| \leq \|l\|.\|x\|$, for all $x \in E$.

4. Show that $\|l\|$ is the smallest $K \in \mathbf{R}^+$, such that:

$$\forall x \in E \ , \ \|l(x)\| \leq K\|x\|$$

5. Show that $l \rightarrow \|l\|$ is a norm on $\mathcal{L}_{\mathbf{K}}(E,F)$.

6. Show that $(\mathcal{L}_{\mathbf{K}}(E,F), \|\cdot\|)$ is a **K**-normed space.

Definition 128 *Let E,F be $\mathbf{R}$-normed spaces and U be an open subset of E. We say that a map $\phi : U \to F$ is* **differentiable** *at some $a \in U$, if and only if there exists $l \in \mathcal{L}_{\mathbf{R}}(E,F)$ such that, for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $h \in E$:*

$$\|h\| \leq \delta \ \Rightarrow \ a + h \in U \ \text{and} \ \|\phi(a+h) - \phi(a) - l(h)\| \leq \epsilon\|h\|$$

Exercise 6. Let E,F be two $\mathbf{R}$-normed spaces, and U be open in E. Let $\phi : U \to F$ be a map and $a \in U$.

1. Suppose that $\phi : U \to F$ is differentiable at $a \in U$, and that $l_1, l_2 \in \mathcal{L}_{\mathbf{R}}(E,F)$ satisfy the requirement of definition (128). Show that for all $\epsilon > 0$, there exists $\delta > 0$ such that:

$$\forall h \in E \ , \ \|h\| \leq \delta \ \Rightarrow \ \|l_1(h) - l_2(h)\| \leq \epsilon\|h\|$$

2. Conclude that $\|l_1 - l_2\| = 0$ and finally that $l_1 = l_2$.

Definition 129 *Let E, F be $\mathbf{R}$-normed spaces and U be an open subset of E. Let $\phi : U \to F$ be a map and $a \in U$. If ϕ is differentiable at a, we call* **differential** *of ϕ at a, the unique element of $\mathcal{L}_{\mathbf{R}}(E, F)$, denoted $d\phi(a)$, satisfying the requirement of definition (128). If ϕ is differentiable at all points of U, the map $d\phi : U \to \mathcal{L}_{\mathbf{R}}(E, F)$ is also called the differential of ϕ.*

Definition 130 *Let E, F be $\mathbf{R}$-normed spaces and U be an open subset of E. A map $\phi : U \to F$ is said to be of* **class** C^1, *if and only if $d\phi(a)$ exists for all $a \in U$, and the differential $d\phi : U \to \mathcal{L}_{\mathbf{R}}(E, F)$ is a continuous map.*

Exercise 7. Let E, F be two $\mathbf{R}$-normed spaces and U be open in E. Let $\phi : U \to F$ be a map, and $a \in U$.

1. Show that ϕ differentiable at $a \Rightarrow \phi$ continuous at a.

2. If ϕ is of class C^1, explain with respect to which topologies the differential $d\phi : U \to \mathcal{L}_{\mathbf{R}}(E, F)$ is said to be continuous.

3. Show that if ϕ is of class C^1, then ϕ is continuous.

4. Suppose that $E = \mathbf{R}$. Show that for all $a \in U$, ϕ is differentiable at $a \in U$, if and only if the derivative:

$$\phi'(a) \stackrel{\triangle}{=} \lim_{t \neq 0, t \to 0} \frac{\phi(a+t) - \phi(a)}{t}$$

exists in F, in which case $d\phi(a) \in \mathcal{L}_{\mathbf{R}}(\mathbf{R}, F)$ is given by:

$$\forall t \in \mathbf{R} \ , \ d\phi(a)(t) = t.\phi'(a)$$

In particular, $\phi'(a) = d\phi(a)(1)$.

Exercise 8. Let E, F, G be three $\mathbf{R}$-normed spaces. Let U be open in E and V be open in F. Let $\phi : U \to F$ and $\psi : V \to G$ be two maps such that $\phi(U) \subseteq V$. We assume that ϕ is differentiable at $a \in U$, and we put:

$$l_1 \stackrel{\triangle}{=} d\phi(a) \in \mathcal{L}_{\mathbf{R}}(E, F)$$

We assume that ψ is differentiable at $\phi(a) \in V$, and we put:

$$l_2 \stackrel{\triangle}{=} d\psi(\phi(a)) \in \mathcal{L}_{\mathbf{R}}(F, G)$$

1. Explain why $\psi \circ \phi : U \to G$ is a well-defined map.

2. Given $\epsilon > 0$, show the existence of $\eta > 0$ such that:

$$\eta(\eta + \|l_1\| + \|l_2\|) \leq \epsilon$$

3. Show the existence of $\delta_2 > 0$ such that for all $h_2 \in F$ with $\|h_2\| \leq \delta_2$, we have $\phi(a) + h_2 \in V$ and:

$$\|\psi(\phi(a) + h_2) - \psi \circ \phi(a) - l_2(h_2)\| \leq \eta\|h_2\|$$

4. Show that if $h_2 \in F$ and $\|h_2\| \leq \delta_2$, then for all $h \in E$, we have:

$$\|\psi(\phi(a)+h_2)-\psi \circ \phi(a)-l_2 \circ l_1(h)\| \leq \eta\|h_2\|+\|l_2\|.\|h_2-l_1(h)\|$$

5. Show the existence of $\delta > 0$ such that for all $h \in E$ with $\|h\| \leq \delta$, we have $a+h \in U$ and $\|\phi(a+h)-\phi(a)-l_1(h)\| \leq \eta\|h\|$, together with $\|\phi(a+h) - \phi(a)\| \leq \delta_2$.

6. Show that if $h \in E$ is such that $\|h\| \leq \delta$, then $a + h \in U$ and:

$$\begin{aligned}\|\psi \circ \phi(a+h) - \psi \circ \phi(a) - l_2 \circ l_1(h)\| &\leq \eta\|\phi(a+h) - \phi(a)\| + \eta\|l_2\|.\|h\| \\ &\leq \eta(\eta + \|l_1\| + \|l_2\|)\|h\| \\ &\leq \epsilon\|h\|\end{aligned}$$

7. Show that $l_2 \circ l_1 \in \mathcal{L}_{\mathbf{R}}(E, G)$

8. Conclude with the following:

Theorem 110 *Let E, F, G be three $\mathbf{R}$-normed spaces, U be open in E and V be open in F. Let $\phi : U \to F$ and $\psi : V \to G$ be two maps such that $\phi(U) \subseteq V$. Let $a \in U$. Then, if ϕ is differentiable at $a \in U$, and ψ is differentiable at $\phi(a) \in V$, then $\psi \circ \phi$ is differentiable at $a \in U$, and furthermore:*

$$d(\psi \circ \phi)(a) = d\psi(\phi(a)) \circ d\phi(a)$$

EXERCISE 9. Let $(\Omega', \mathcal{T}')$ and $(\Omega, \mathcal{T})$ be two topological spaces, and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be a set of subsets of Ω generating the topology $\mathcal{T}$, i.e. such that $\mathcal{T} = \mathcal{T}(\mathcal{A})$ as defined in (55). Let $f : \Omega' \to \Omega$ be a map, and define:

$$\mathcal{U} \stackrel{\triangle}{=} \{A \subseteq \Omega \ : \ f^{-1}(A) \in \mathcal{T}'\}$$

1. Show that $\mathcal{U}$ is a topology on Ω.

2. Show that $f : (\Omega', \mathcal{T}') \to (\Omega, \mathcal{T})$ is continuous, if and only if:

$$\forall A \in \mathcal{A} \ , \ f^{-1}(A) \in \mathcal{T}'$$

Exercise 10. Let $(\Omega', \mathcal{T}')$ be a topological space, and $(\Omega_i, \mathcal{T}_i)_{i\in I}$ be a family of topological spaces, indexed by a non-empty set I. Let Ω be the Cartesian product $\Omega = \Pi_{i\in I}\Omega_i$ and $\mathcal{T} = \odot_{i\in I}\mathcal{T}_i$ be the product topology on Ω. Let $(f_i)_{i\in I}$ be a family of maps $f_i : \Omega' \to \Omega_i$ indexed by I, and let $f : \Omega' \to \Omega$ be the map defined by $f(\omega) = (f_i(\omega))_{i\in I}$ for all $\omega \in \Omega'$. Let $p_i : \Omega \to \Omega_i$ be the canonical projection mapping.

1. Show that $p_i : (\Omega, \mathcal{T}) \to (\Omega_i, \mathcal{T}_i)$ is continuous for all $i \in I$.

2. Show that $f : (\Omega', \mathcal{T}') \to (\Omega, \mathcal{T})$ is continuous, if and only if each coordinate mapping $f_i : (\Omega', \mathcal{T}') \to (\Omega_i, \mathcal{T}_i)$ is continuous.

Exercise 11. Let E, F, G be three $\mathbf{R}$-normed spaces. Let U be open in E and V be open in F. Let $\phi : U \to F$ and $\psi : V \to G$ be two maps of class C^1 such that $\phi(U) \subseteq V$.

1. For all $(l_1, l_2) \in \mathcal{L}_{\mathbf{R}}(F, G) \times \mathcal{L}_{\mathbf{R}}(E, F)$, we define:

$$N_1(l_1, l_2) \stackrel{\triangle}{=} \|l_1\| + \|l_2\|$$

$$N_2(l_1,l_2) \stackrel{\triangle}{=} \sqrt{\|l_1\|^2+\|l_2\|^2}$$
$$N_\infty(l_1,l_2) \stackrel{\triangle}{=} \max(\|l_1\|,\|l_2\|)$$

Show that N_1, N_2, N_∞ are all norms on $\mathcal{L}_{\mathbf{R}}(F,G)\times\mathcal{L}_{\mathbf{R}}(E,F)$.

2. Show they induce the product topology on $\mathcal{L}_{\mathbf{R}}(F,G)\times\mathcal{L}_{\mathbf{R}}(E,F)$.

3. We define the map $H:\mathcal{L}_{\mathbf{R}}(F,G)\times\mathcal{L}_{\mathbf{R}}(E,F)\to\mathcal{L}_{\mathbf{R}}(E,G)$ by:

$$\forall(l_1,l_2)\in\mathcal{L}_{\mathbf{R}}(F,G)\times\mathcal{L}_{\mathbf{R}}(E,F)\ ,\ H(l_1,l_2)\stackrel{\triangle}{=} l_1\circ l_2$$

Show that $\|H(l_1,l_2)\|\leq\|l_1\|.\|l_2\|$, for all l_1,l_2.

4. Show that H is continuous.

5. We define $K:U\to\mathcal{L}_{\mathbf{R}}(F,G)\times\mathcal{L}_{\mathbf{R}}(E,F)$ by:

$$\forall a\in U\ ,\ K(a)\stackrel{\triangle}{=}(d\psi(\phi(a)),d\phi(a))$$

Show that K is continuous.

6. Show that $\psi \circ \phi$ is differentiable on U.

7. Show that $d(\psi \circ \phi) = H \circ K$.

8. Conclude with the following:

Theorem 111 *Let E, F, G be three $\mathbf{R}$-normed spaces, U be open in E and V be open in F. Let $\phi : U \to F$ and $\psi : V \to G$ be two maps of class C^1 such that $\phi(U) \subseteq V$. Then, $\psi \circ \phi : U \to G$ is of class C^1.*

Exercise 12. Let E be an $\mathbf{R}$-normed space. Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a, b] \to E$ and $g : [a, b] \to \mathbf{R}$ be two continuous maps which are differentiable at every point of $]a, b[$. We assume that:

$$\forall t \in]a, b[\ ,\ \|f'(t)\| \leq g'(t)$$

1. Given $\epsilon > 0$, we define $\phi_\epsilon : [a, b] \to \mathbf{R}$ by:

$$\phi_\epsilon(t) \stackrel{\triangle}{=} \|f(t) - f(a)\| - g(t) + g(a) - \epsilon(t - a)$$

for all $t \in [a,b]$. Show that ϕ_ϵ is continuous.

2. Define $E_\epsilon = \{t \in [a,b] : \phi_\epsilon(t) \leq \epsilon\}$, and $c = \sup E_\epsilon$. Show that:
$$c \in [a,b] \text{ and } \phi_\epsilon(c) \leq \epsilon$$
3. Show the existence of $h > 0$, such that:
$$\forall t \in [a, a+h[\cap[a,b]\ ,\ \phi_\epsilon(t) \leq \epsilon$$
4. Show that $c \in]a,b]$.
5. Suppose that $c \in]a,b[$. Show the existence of $t_0 \in]c,b]$ such that:
$$\left\| \frac{f(t_0) - f(c)}{t_0 - c} \right\| \leq \|f'(c)\| + \epsilon/2 \text{ and } g'(c) \leq \frac{g(t_0) - g(c)}{t_0 - c} + \epsilon/2$$
6. Show that $\|f(t_0) - f(c)\| \leq g(t_0) - g(c) + \epsilon(t_0 - c)$.
7. Show that $\|f(c) - f(a)\| \leq g(c) - g(a) + \epsilon(c - a) + \epsilon$.
8. Show that $\|f(t_0) - f(a)\| \leq g(t_0) - g(a) + \epsilon(t_0 - a) + \epsilon$.

9. Show that $c \in]a, b[$ leads to a contradiction.

10. Show that $\|f(b) - f(a)\| \leq g(b) - g(a) + \epsilon(b - a) + \epsilon$.

11. Conclude with the following:

Theorem 112 *Let E be an $\mathbf{R}$-normed space. Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a, b] \to E$ and $g : [a, b] \to \mathbf{R}$ be two continuous maps which are differentiable at every point of $]a, b[$, and such that:*

$$\forall t \in]a, b[\ ,\ \|f'(t)\| \leq g'(t)$$

Then:

$$\|f(b) - f(a)\| \leq g(b) - g(a)$$

Definition 131 *Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space. For all $i = 1, \ldots, n$, we say that ϕ has an ith* **partial derivative** *at $a \in U$, if and only if the limit:*

$$\frac{\partial \phi}{\partial x_i}(a) \stackrel{\triangle}{=} \lim_{h \neq 0, h \to 0} \frac{\phi(a + he_i) - \phi(a)}{h}$$

exists in E, where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbf{R}^n$.

Exercise 13. Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space.

1. Suppose ϕ is differentiable at $a \in U$. Show that for all $i \in \mathbf{N}_n$:

$$\lim_{h \neq 0, h \to 0} \frac{1}{\|he_i\|} \|\phi(a + he_i) - \phi(a) - d\phi(a)(he_i)\| = 0$$

2. Show that for all $i \in \mathbf{N}_n$, $\frac{\partial \phi}{\partial x_i}(a)$ exists, and:

$$\frac{\partial \phi}{\partial x_i}(a) = d\phi(a)(e_i)$$

3. Conclude with the following:

Theorem 113 *Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space. Then, if ϕ is differentiable at $a \in U$, for all $i = 1, \ldots, n$, $\frac{\partial \phi}{\partial x_i}(a)$ exists and we have:*

$$\forall h \stackrel{\triangle}{=} (h_1, \ldots, h_n) \in \mathbf{R}^n \ , \ d\phi(a)(h) = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(a) h_i$$

Exercise 14. Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space.

1. Show that if ϕ is differentiable at $a, b \in U$, then for all $i \in \mathbf{N}_n$:

$$\left\| \frac{\partial \phi}{\partial x_i}(b) - \frac{\partial \phi}{\partial x_i}(a) \right\| \leq \| d\phi(b) - d\phi(a) \|$$

2. Conclude that if ϕ is of class C^1 on U, then $\frac{\partial\phi}{\partial x_i}$ exists and is continuous on U, for all $i \in \mathbf{N}_n$.

Exercise 15. Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space. We assume that $\frac{\partial\phi}{\partial x_i}$ exists on U, and is continuous at $a \in U$, for all $i \in \mathbf{N}_n$. We define $l : \mathbf{R}^n \to E$:

$$\forall h \stackrel{\triangle}{=} (h_1, \ldots, h_n) \in \mathbf{R}^n \ , \ l(h) \stackrel{\triangle}{=} \sum_{i=1}^{n} \frac{\partial\phi}{\partial x_i}(a)h_i$$

1. Show that $l \in \mathcal{L}_{\mathbf{R}}(\mathbf{R}^n, E)$.

2. Given $\epsilon > 0$, show the existence of $\eta > 0$ such that for all $h \in \mathbf{R}^n$ with $\|h\| < \eta$, we have $a + h \in U$, and:

$$\forall i = 1, \ldots, n \ , \ \left\| \frac{\partial\phi}{\partial x_i}(a+h) - \frac{\partial\phi}{\partial x_i}(a) \right\| \leq \epsilon$$

3. Let $h = (h_1, \ldots, h_n) \in \mathbf{R}^n$ be such that $\|h\| < \eta$. $(e_1, \ldots, e_n)$ being the canonical basis of $\mathbf{R}^n$, we define $k_0 = a$ and for $i \in \mathbf{N}_n$:

$$k_i \stackrel{\triangle}{=} a + \sum_{j=1}^{i} h_j e_j$$

Show that $k_0, \ldots, k_n \in U$, and that we have:

$$\phi(a+h)-\phi(a)-l(h)=\sum_{i=1}^{n}\left(\phi(k_{i-1} + h_i e_i)-\phi(k_{i-1})-h_i\frac{\partial\phi}{\partial x_i}(a)\right)$$

4. Let $i \in \mathbf{N}_n$. Assume that $h_i > 0$. We define $f_i : [0, h_i] \to E$ by:

$$\forall t \in [0, h_i]\ ,\ f_i(t) \stackrel{\triangle}{=} \phi(k_{i-1} + te_i) - \phi(k_{i-1}) - t\frac{\partial\phi}{\partial x_i}(a)$$

Show f_i is well-defined, $f_i'(t)$ exists for all $t \in [0, h_i]$, and:

$$\forall t \in [0, h_i]\ ,\ f_i'(t) = \frac{\partial\phi}{\partial x_i}(k_{i-1} + te_i) - \frac{\partial\phi}{\partial x_i}(a)$$

5. Show f_i is continuous on $[0, h_i]$, differentiable on $]0, h_i[$, with:

$$\forall t \in]0, h_i[\ ,\ \|f_i'(t)\| \leq \epsilon$$

6. Show that:

$$\left\| \phi(k_{i-1} + h_i e_i) - \phi(k_{i-1}) - h_i \frac{\partial \phi}{\partial x_i}(a) \right\| \leq \epsilon |h_i|$$

7. Show that the previous inequality still holds if $h_i \leq 0$.

8. Conclude that for all $h \in \mathbf{R}^n$ with $\|h\| < \eta$, we have:

$$\|\phi(a + h) - \phi(a) - l(h)\| \leq \epsilon \sqrt{n} \|h\|$$

9. Prove the following:

Theorem 114 *Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space. If, for all $i \in \mathbf{N}_n$ $\frac{\partial \phi}{\partial x_i}$ exists on U and is continuous at $a \in U$, then ϕ is differentiable at $a \in U$.*

EXERCISE 16. Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space. We assume that for all $i \in \mathbf{N}_n$, $\frac{\partial \phi}{\partial x_i}$ exists and is continuous on U.

1. Show that ϕ is differentiable on U.

2. Show that for all $a, b \in U$ and $h \in \mathbf{R}^n$:

$$\|(d\phi(b) - d\phi(a))(h)\| \leq \left(\sum_{i=1}^{n} \left\| \frac{\partial \phi}{\partial x_i}(b) - \frac{\partial \phi}{\partial x_i}(a) \right\|^2 \right)^{1/2} \|h\|$$

3. Show that for all $a, b \in U$:

$$\|d\phi(b) - d\phi(a)\| \leq \left(\sum_{i=1}^{n} \left\| \frac{\partial \phi}{\partial x_i}(b) - \frac{\partial \phi}{\partial x_i}(a) \right\|^2 \right)^{1/2}$$

4. Show that $d\phi : U \to \mathcal{L}_{\mathbf{R}}(\mathbf{R}^n, E)$ is continuous.

5. Prove the following:

Theorem 115 *Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space. Then, ϕ is of class C^1 on U, if and only if for all $i = 1, \ldots, n$, $\frac{\partial \phi}{\partial x_i}$ exists and is continuous on U.*

Exercise 17. Let E, F be two $\mathbf{R}$-normed spaces and $l \in \mathcal{L}_{\mathbf{R}}(E, F)$. Let U be open in E and $l_{|U}$ be the restriction of l to U. Show that $l_{|U}$ is of class C^1 on U, and that we have:

$$\forall x \in U \ , \ d(l_{|U})(x) = l$$

Exercise 18. Let $E_1, \ldots, E_n$, $n \geq 1$, be n $\mathbf{K}$-normed spaces. Let $E = E_1 \times \ldots \times E_n$. Let $p \in [1, +\infty[$, and for all $x = (x_1, \ldots, x_n) \in E$:

$$\|x\|_p \stackrel{\triangle}{=} \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p}$$

$$\|x\|_\infty \stackrel{\triangle}{=} \max_{i=1,\ldots,n} \|x_i\|$$

1. Using theorem (43), show that $\|.\|_p$ and $\|.\|_\infty$ are norms on E.
2. Show $\|.\|_p$ and $\|.\|_\infty$ induce the product topology on E.
3. Conclude that E is also an $\mathbf{K}$-normed space, and that the norm topology on E is exactly the product topology on E.

Exercise 19. Let E and F be two $\mathbf{R}$-normed spaces. Let U be open in E and $\phi, \psi : U \to F$ be two maps. We assume that both ϕ and ψ are differentiable at $a \in U$. Given $\alpha \in \mathbf{R}$, show that $\phi + \alpha\psi$ is differentiable at $a \in U$ and:

$$d(\phi + \alpha\psi)(a) = d\phi(a) + \alpha d\psi(a)$$

Exercise 20. Let E and F be $\mathbf{K}$-normed spaces. Let U be open in E and $\phi : U \to F$ be a map. Let N_E and N_F be two norms on E and F, inducing the same topologies as the norm topologies of E and F

respectively. For all $l \in \mathcal{L}_{\mathbf{K}}(E,F)$, we define:

$$N(l) = \sup\{N_F(l(x)) : x \in E, N_E(x) = 1\}$$

1. Explain why the set $\mathcal{L}_{\mathbf{K}}(E,F)$ is unambiguously defined.
2. Show that the identity $id_E : (E, \|\cdot\|) \to (E, N_E)$ is continuous
3. Show the existence of $m_E, M_E > 0$ such that:
$$\forall x \in E\ ,\ m_E\|x\| \leq N_E(x) \leq M_E\|x\|$$
4. Show the existence of $m, M > 0$ such that:
$$\forall l \in \mathcal{L}_{\mathbf{K}}(E,F)\ ,\ m\|l\| \leq N(l) \leq M\|l\|$$
5. Show that $\|\cdot\|$ and N induce the same topology on $\mathcal{L}_{\mathbf{K}}(E,F)$.
6. Show that if $\mathbf{K} = \mathbf{R}$ and ϕ is differentiable at $a \in U$, then ϕ is also differentiable at a with respect to the norms N_E and N_F, and the differential $d\phi(a)$ is unchanged

7. Show that if $\mathbf{K} = \mathbf{R}$ and ϕ is of class C^1 on U, then ϕ is also of class C^1 on U with respect to the norms N_E and N_F.

EXERCISE 21. Let E and $F_1, \ldots, F_p$, $p \geq 1$, be $p+1$ $\mathbf{R}$-normed spaces. Let U be open in E and $F = F_1 \times \ldots \times F_p$. Let $\phi : U \to F$ be a map. For all $i \in \mathbf{N}_p$, we denote $p_i : F \to F_i$ the canonical projection and we define $\phi_i = p_i \circ \phi$. We also consider $u_i : F_i \to F$, defined as:

$$\forall x_i \in F_i \ , \ u_i(x_i) \stackrel{\triangle}{=} (0, \ldots, \overbrace{x_i}^{i}, \ldots, 0)$$

1. Given $i \in \mathbf{N}_p$, show that $p_i \in \mathcal{L}_{\mathbf{R}}(F, F_i)$.

2. Given $i \in \mathbf{N}_p$, show that $u_i \in \mathcal{L}_{\mathbf{R}}(F_i, F)$ and $\phi = \sum_{i=1}^p u_i \circ \phi_i$.

3. Show that if ϕ is differentiable at $a \in U$, then for all $i \in \mathbf{N}_p$, $\phi_i : U \to F_i$ is differentiable at $a \in U$ and $d\phi_i(a) = p_i \circ d\phi(a)$.

4. Show that if ϕ_i is differentiable at $a \in U$ for all $i \in \mathbf{N}_p$, then ϕ is differentiable at $a \in U$ and:

$$d\phi(a) = \sum_{i=1}^{p} u_i \circ d\phi_i(a)$$

5. Suppose that ϕ is differentiable at $a, b \in U$. Let F be given the norm $\|\cdot\|_2$ of exercise (18). Show that for all $i \in \mathbf{N}_p$:

$$\|d\phi_i(b) - d\phi_i(a)\| \leq \|d\phi(b) - d\phi(a)\|$$

6. Show that:

$$\|d\phi(b) - d\phi(a)\| \leq \left(\sum_{i=1}^{p} \|d\phi_i(b) - d\phi_i(a)\|^2\right)^{1/2}$$

7. Show that ϕ is of class $C^1 \Leftrightarrow \phi_i$ is of class C^1 for all $i \in \mathbf{N}_p$.

8. Conclude with theorem (116)

Theorem 116 *Let $E, F_1, \ldots, F_p$, $(p \geq 1)$, be $p+1$ $\mathbf{R}$-normed spaces and U be open in E. Let F be the $\mathbf{R}$-normed space $F = F_1 \times \ldots \times F_p$ and $\phi = (\phi_1, \ldots, \phi_p) : U \to F$ be a map. Then, ϕ is differentiable at $a \in U$, if and only if $d\phi_i(a)$ exists for all $i \in \mathbf{N}_p$, in which case:*

$$\forall h \in E \ , \ d\phi(a)(h) = (d\phi_1(a)(h), \ldots, d\phi_p(a)(h))$$

Also, ϕ is of class C^1 on $U \Leftrightarrow \phi_i$ is of class C^1 on U, for all $i \in \mathbf{N}_p$.

Theorem 117 *Let $\phi = (\phi_1, \ldots, \phi_n) : U \to \mathbf{R}^n$ be a map, where $n \geq 1$ and U is open in $\mathbf{R}^n$. We assume that ϕ is differentiable at $a \in U$. Then, for all $i, j = 1, \ldots, n$, $\frac{\partial \phi_i}{\partial x_j}(a)$ exists, and we have:*

$$d\phi(a) = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1}(a) & \ldots & \frac{\partial \phi_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \frac{\partial \phi_n}{\partial x_1}(a) & \ldots & \frac{\partial \phi_n}{\partial x_n}(a) \end{pmatrix}$$

Moreover, ϕ is of class C^1 on U, if and only if for all $i, j = 1, \ldots, n$, $\frac{\partial \phi_i}{\partial x_j}$ exists and is continuous on U.

Exercise 22. Prove theorem (117)

Definition 132 *Let $\phi = (\phi_1, \ldots, \phi_n) : U \to \mathbf{R}^n$ be a map, where $n \geq 1$ and U is open in $\mathbf{R}^n$. We assume that ϕ is differentiable at $a \in U$. We call* **Jacobian** *of ϕ at a, denoted $J(\phi)(a)$, the determinant of the differential $d\phi(a)$ at a, i.e.*

$$J(\phi)(a) = \det \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1}(a) & \ldots & \frac{\partial \phi_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \frac{\partial \phi_n}{\partial x_1}(a) & \ldots & \frac{\partial \phi_n}{\partial x_n}(a) \end{pmatrix}$$

Definition 133 *Let $n \geq 1$ and Ω, Ω' be open in $\mathbf{R}^n$. A bijection $\phi : \Omega \to \Omega'$ is called a* **C^1-diffeomorphism** *between Ω and Ω', if and only if $\phi : \Omega \to \mathbf{R}^n$ and $\phi^{-1} : \Omega' \to \mathbf{R}^n$ are both of class C^1.*

Exercise 23. Let Ω and Ω' be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism, $\psi = \phi^{-1}$, and I_n be the identity mapping of $\mathbf{R}^n$.

1. Explain why $J(\psi) : \Omega' \to \mathbf{R}$ and $J(\phi) : \Omega \to \mathbf{R}$ are continuous.
2. Show that $d\phi(\psi(x)) \circ d\psi(x) = I_n$, for all $x \in \Omega'$.
3. Show that $d\psi(\phi(x)) \circ d\phi(x) = I_n$, for all $x \in \Omega$.
4. Show that $J(\psi)(x) \neq 0$ for all $x \in \Omega'$.
5. Show that $J(\phi)(x) \neq 0$ for all $x \in \Omega$.
6. Show that $J(\psi) = 1/(J(\phi) \circ \psi)$ and $J(\phi) = 1/(J(\psi) \circ \phi)$.

Definition 134 *Let $n \geq 1$ and $\Omega \in \mathcal{B}(\mathbf{R}^n)$, be a Borel set in $\mathbf{R}^n$. We define the* **Lebesgue measure** *on Ω, denoted $dx_{|\Omega}$, as the restriction to $\mathcal{B}(\Omega)$ of the Lebesgue measure on $\mathbf{R}^n$, i.e the measure on $(\Omega, \mathcal{B}(\Omega))$ defined by:*

$$\forall B \in \mathcal{B}(\Omega)\ ,\ dx_{|\Omega}(B) \stackrel{\triangle}{=} dx(B)$$

Exercise 24. Show that $dx_{|\Omega}$ is a well-defined measure on $(\Omega, \mathcal{B}(\Omega))$.

Exercise 25. Let $n \geq 1$ and $\Omega,\ \Omega'$ be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism and $\psi = \phi^{-1}$. Let $a \in \Omega'$. We assume that $d\psi(a) = I_n$, (identity mapping on $\mathbf{R}^n$), and given $\epsilon > 0$, we denote:

$$B(a,\epsilon) \stackrel{\triangle}{=} \{x \in \mathbf{R}^n :\ \|a - x\| < \epsilon\}$$

where $\|.\|$ is the usual norm in $\mathbf{R}^n$.

1. Why are $dx_{|\Omega'}$, $\phi(dx_{|\Omega})$ well-defined measures on $(\Omega', \mathcal{B}(\Omega'))$.
2. Show that for $\epsilon > 0$ sufficiently small, $B(a,\epsilon) \in \mathcal{B}(\Omega')$.

3. Show that it makes sense to investigate whether the limit:

$$\lim_{\epsilon\downarrow\downarrow 0} \frac{\phi(dx_{|\Omega})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))}$$

does exists in $\mathbf{R}$.

4. Given $r > 0$, show the existence of $\epsilon_1 > 0$ such that for all $h \in \mathbf{R}^n$ with $\|h\| \leq \epsilon_1$, we have $a + h \in \Omega'$, and:

$$\|\psi(a+h) - \psi(a) - h\| \leq r\|h\|$$

5. Show for all $h \in \mathbf{R}^n$ with $\|h\| \leq \epsilon_1$, we have $a + h \in \Omega'$, and:

$$\|\psi(a+h) - \psi(a)\| \leq (1+r)\|h\|$$

6. Show that for all $\epsilon \in]0, \epsilon_1[$, we have $B(a,\epsilon) \subseteq \Omega'$, and:

$$\psi(B(a,\epsilon)) \subseteq B(\psi(a), \epsilon(1+r))$$

7. Show that $d\phi(\psi(a)) = I_n$.

8. Show the existence of $\epsilon_2 > 0$ such that for all $k \in \mathbf{R}^n$ with $\|k\| \leq \epsilon_2$, we have $\psi(a) + k \in \Omega$, and:

$$\|\phi(\psi(a) + k) - a - k\| \leq r\|k\|$$

9. Show for all $k \in \mathbf{R}^n$ with $\|k\| \leq \epsilon_2$, we have $\psi(a) + k \in \Omega$, and:

$$\|\phi(\psi(a) + k) - a\| \leq (1 + r)\|k\|$$

10. Show for all $\epsilon \in]0, \epsilon_2(1 + r)[$, we have $B(\psi(a), \frac{\epsilon}{1+r}) \subseteq \Omega$, and:

$$B(\psi(a), \frac{\epsilon}{1 + r}) \subseteq \{\phi \in B(a, \epsilon)\}$$

11. Show that if $B(a, \epsilon) \subseteq \Omega'$, then $\psi(B(a, \epsilon)) = \{\phi \in B(a, \epsilon)\}$.

12. Show if $0 < \epsilon < \epsilon_0 = \epsilon_1 \wedge \epsilon_2(1 + r)$, then $B(a, \epsilon) \subseteq \Omega'$, and:

$$B(\psi(a), \frac{\epsilon}{1 + r}) \subseteq \{\phi \in B(a, \epsilon)\} \subseteq B(\psi(a), \epsilon(1 + r))$$

13. Show that for all $\epsilon \in]0, \epsilon_0[$:

$$
\begin{array}{ll}
(i) & dx(B(\psi(a), \frac{\epsilon}{1+r})) = (1+r)^{-n} dx_{|\Omega'}(B(a,\epsilon)) \\
(ii) & dx(B(\psi(a), \epsilon(1+r))) = (1+r)^{n} dx_{|\Omega'}(B(a,\epsilon)) \\
(iii) & dx(\{\phi \in B(a,\epsilon)\}) = \phi(dx_{|\Omega})(B(a,\epsilon))
\end{array}
$$

14. Show that for all $\epsilon \in]0, \epsilon_0[$, $B(a,\epsilon) \subseteq \Omega'$, and:

$$
(1+r)^{-n} \leq \frac{\phi(dx_{|\Omega})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))} \leq (1+r)^{n}
$$

15. Conclude that:

$$
\lim_{\epsilon \downarrow\downarrow 0} \frac{\phi(dx_{|\Omega})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))} = 1
$$

Exercise 26. Let $n \geq 1$ and $\Omega,\ \Omega'$ be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism and $\psi = \phi^{-1}$. Let $a \in \Omega'$. We put $A = d\psi(a)$.

1. Show that $A : \mathbf{R}^n \to \mathbf{R}^n$ is a linear bijection.

2. Define $\Omega'' = A^{-1}(\Omega)$. Show that this definition does not depend on whether $A^{-1}(\Omega)$ is viewed as inverse , or direct image.

3. Show that Ω'' is an open subset of $\mathbf{R}^n$.

4. We define $\tilde{\phi} : \Omega'' \to \Omega'$ by $\tilde{\phi}(x) = \phi \circ A(x)$. Show that $\tilde{\phi}$ is a C^1-diffeomorphism with $\tilde{\psi} = \tilde{\phi}^{-1} = A^{-1} \circ \psi$.

5. Show that $d\tilde{\psi}(a) = I_n$.

6. Show that:

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{\tilde{\phi}(dx_{|\Omega''})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))} = 1$$

7. Let $\epsilon > 0$ with $B(a,\epsilon) \subseteq \Omega'$. Justify each of the following steps:

$$\begin{aligned} \tilde{\phi}(dx_{|\Omega''})(B(a,\epsilon)) &= dx_{|\Omega''}(\{\tilde{\phi} \in B(a,\epsilon)\}) \\ &= dx(\{\tilde{\phi} \in B(a,\epsilon)\}) \end{aligned}$$

$$\begin{aligned}
&= dx(\{x \in \Omega'' :\ \phi \circ A(x) \in B(a,\epsilon)\}) \\
&= dx(\{x \in \Omega'' : A(x) \in \phi^{-1}(B(a,\epsilon))\}) \\
&= dx(\{x \in \mathbf{R}^n : A(x) \in \phi^{-1}(B(a,\epsilon))\}) \\
&= A(dx)(\{\phi \in B(a,\epsilon)\}) \\
&= |\det A|^{-1} dx(\{\phi \in B(a,\epsilon)\}) \\
&= |\det A|^{-1} dx_{|\Omega}(\{\phi \in B(a,\epsilon)\}) \\
&= |\det A|^{-1} \phi(dx_{|\Omega})(B(a,\epsilon))
\end{aligned}$$

8. Show that:

$$\lim_{\epsilon\downarrow\downarrow 0} \frac{\phi(dx_{|\Omega})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))} = |\det A|$$

9. Conclude with the following:

Theorem 118 *Let $n \geq 1$ and $\Omega,\ \Omega'$ be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism and $\psi = \phi^{-1}$. Then, for all $a \in \Omega'$, we have:*

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{\phi(dx_{|\Omega})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))} = |J(\psi)(a)|$$

where $J(\psi)(a)$ is the Jacobian of ψ at a, $B(a,\epsilon)$ is the open ball in $\mathbf{R}^n$, and $dx_{|\Omega}$, $dx_{|\Omega'}$ are the Lebesgue measures on Ω and Ω' respectively.

Exercise 27. Let $n \geq 1$ and $\Omega,\ \Omega'$ be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism and $\psi = \phi^{-1}$.

1. Let $K \subseteq \Omega'$ be a non-empty compact subset of Ω' such that $dx_{|\Omega'}(K) = 0$. Given $\epsilon > 0$, show the existence of V open in Ω', such that $K \subseteq V \subseteq \Omega'$, and $dx_{|\Omega'}(V) \leq \epsilon$.

2. Explain why V is also open in $\mathbf{R}^n$.

3. Show that $M \stackrel{\triangle}{=} \sup_{x \in K} \|d\psi(x)\| \in \mathbf{R}^+$.

4. For all $x \in K$, show there is $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subseteq V$, and for all $h \in \mathbf{R}^n$ with $\|h\| \leq 3\epsilon_x$, we have $x + h \in \Omega'$, and:

$$\|\psi(x+h) - \psi(x)\| \leq (M+1)\|h\|$$

5. Show that for all $x \in K$, $B(x, 3\epsilon_x) \subseteq \Omega'$, and:

$$\psi(B(x, 3\epsilon_x)) \subseteq B(\psi(x), 3(M+1)\epsilon_x)$$

6. Show that $\psi(B(x, 3\epsilon_x)) = \{\phi \in B(x, 3\epsilon_x)\}$, for all $x \in K$.

7. Show the existence of $\{x_1, \ldots, x_p\} \subseteq K$, $(p \geq 1)$, such that:

$$K \subseteq B(x_1, \epsilon_{x_1}) \cup \ldots \cup B(x_p, \epsilon_{x_p})$$

8. Show the existence of $S \subseteq \{1, \ldots, p\}$ such that the $B(x_i, \epsilon_{x_i})$'s are pairwise disjoint for $i \in S$, and:

$$K \subseteq \bigcup_{i \in S} B(x_i, 3\epsilon_{x_i})$$

9. Show that $\{\phi \in K\} \subseteq \cup_{i \in S} B(\psi(x_i), 3(M+1)\epsilon_{x_i})$.

10. Show that $\phi(dx_{|\Omega})(K) \leq \sum_{i \in S} 3^n (M+1)^n dx(B(x_i, \epsilon_{x_i}))$.

11. Show that $\phi(dx_{|\Omega})(K) \leq 3^n (M+1)^n dx(V)$.

12. Show that $\phi(dx_{|\Omega})(K) \leq 3^n (M+1)^n \epsilon$.

13. Conclude that $\phi(dx_{|\Omega})(K) = 0$.

14. Show that $\phi(dx_{|\Omega})$ is a locally finite measure on $(\Omega', \mathcal{B}(\Omega'))$.

15. Show that for all $B \in \mathcal{B}(\Omega')$:

$$\phi(dx_{|\Omega})(B) = \sup\{\phi(dx_{|\Omega})(K) : \ K \subseteq B \ , \ K \text{ compact } \}$$

16. Show that for all $B \in \mathcal{B}(\Omega')$:

$$dx_{|\Omega'}(B) = 0 \ \Rightarrow \ \phi(dx_{|\Omega})(B) = 0$$

17. Conclude with the following:

Theorem 119 *Let $n \geq 1$, $\Omega,\ \Omega'$ be open in $\mathbf{R}^n$, and $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism. Then, the image measure $\phi(dx_{|\Omega})$, by ϕ of the Lebesgue measure on Ω, is absolutely continuous with respect to $dx_{|\Omega'}$, the Lebesgue measure on Ω', i.e.:*

$$\phi(dx_{|\Omega}) << dx_{|\Omega'}$$

Exercise 28. Let $n \geq 1$ and $\Omega,\ \Omega'$ be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism and $\psi = \phi^{-1}$.

1. Explain why there exists a sequence $(V_p)_{p\geq 1}$ of open sets in Ω', such that $V_p \uparrow \Omega'$ and for all $p \geq 1$, the closure of V_p in Ω', i.e. $\bar{V}_p^{\Omega'}$, is compact.

2. Show that each V_p is also open in $\mathbf{R}^n$, and that $\bar{V}_p^{\Omega'} = \bar{V}_p$.

3. Show that $\phi(dx_{|\Omega})(V_p) < +\infty$, for all $p \geq 1$.

4. Show that $dx_{|\Omega'}$ and $\phi(dx_{|\Omega})$ are two σ-finite measures on Ω'.

5. Show there is $h : (\Omega', \mathcal{B}(\Omega')) \to (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$ measurable, with:

$$\forall B \in \mathcal{B}(\Omega')\ ,\ \phi(dx_{|\Omega})(B) = \int_B h dx_{|\Omega'}$$

6. For all $p \geq 1$, we define $h_p = h1_{V_p}$, and we put:

$$\forall x \in \mathbf{R}^n\ ,\ \tilde{h}_p(x) \stackrel{\triangle}{=} \left\{ \begin{array}{lll} h_p(x) & \text{if} & x \in \Omega' \\ 0 & \text{if} & x \notin \Omega' \end{array} \right.$$

Show that:

$$\int_{\mathbf{R}^n} \tilde{h}_p dx = \int_{\Omega'} h_p dx_{|\Omega'} = \phi(dx_{|\Omega})(V_p) < +\infty$$

and conclude that $\tilde{h}_p \in L^1_{\mathbf{R}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$.

7. Show the existence of some $N \in \mathcal{B}(\mathbf{R}^n)$, such that $dx(N) = 0$

and for all $x \in N^c$ and $p \geq 1$, we have:

$$\tilde{h}_p(x) = \lim_{\epsilon \downarrow\downarrow 0} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} \tilde{h}_p dx$$

8. Put $N' = N \cap \Omega'$. Show that $N' \in \mathcal{B}(\Omega')$ and $dx_{|\Omega'}(N') = 0$.
9. Let $x \in \Omega'$ and $p \geq 1$ be such that $x \in V_p$. Show that if $\epsilon > 0$ is such that $B(x,\epsilon) \subseteq V_p$, then $dx(B(x,\epsilon)) = dx_{|\Omega'}(B(x,\epsilon))$, and:

$$\int_{B(x,\epsilon)} \tilde{h}_p dx = \int_{\mathbf{R}^n} 1_{B(x,\epsilon)} \tilde{h}_p dx = \int_{\Omega'} 1_{B(x,\epsilon)} h_p dx_{|\Omega'}$$

10. Show that:

$$\int_{\Omega'} 1_{B(x,\epsilon)} h_p dx_{|\Omega'} = \int_{\Omega'} 1_{B(x,\epsilon)} h dx_{|\Omega'} = \phi(dx_{|\Omega})(B(x,\epsilon))$$

11. Show that for all $x \in \Omega' \setminus N'$, we have:

$$h(x) = \lim_{\epsilon \downarrow\downarrow 0} \frac{\phi(dx_{|\Omega})(B(x,\epsilon))}{dx_{|\Omega'}(B(x,\epsilon))}$$

12. Show that $h = |J(\psi)|\ dx_{|\Omega'}$-a.s. and conclude with the following:

Theorem 120 *Let $n \geq 1$ and $\Omega,\ \Omega'$ be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism and $\psi = \phi^{-1}$. Then, the image measure by ϕ of the Lebesgue measure on Ω, is equal to the measure on $(\Omega', \mathcal{B}(\Omega'))$ with density $|J(\psi)|$ with respect to the Lebesgue measure on Ω', i.e.:*

$$\phi(dx_{|\Omega}) = \int |J(\psi)| dx_{|\Omega'}$$

Exercise 29. Prove the following:

Theorem 121 (Jacobian Formula 1) *Let $n \geq 1$ and $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism where $\Omega,\ \Omega'$ are open in $\mathbf{R}^n$. Let $\psi = \phi^{-1}$. Then, for all $f : (\Omega', \mathcal{B}(\Omega')) \to [0, +\infty]$ non-negative and measurable:*

$$\int_\Omega f \circ \phi\, dx_{|\Omega} = \int_{\Omega'} f |J(\psi)| dx_{|\Omega'}$$

and:

$$\int_{\Omega}(f\circ\phi)|J(\phi)|dx_{|\Omega} = \int_{\Omega'} f dx_{|\Omega'}$$

Exercise 30. Prove the following:

Theorem 122 (Jacobian Formula 2) *Let $n \geq 1$ and $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism where Ω, Ω' are open in $\mathbf{R}^n$. Let $\psi = \phi^{-1}$. Then, for all measurable map $f : (\Omega', \mathcal{B}(\Omega')) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$, we have the equivalence:*

$$f\circ\phi \in L^1_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega), dx_{|\Omega}) \ \Leftrightarrow\ f|J(\psi)| \in L^1_{\mathbf{C}}(\Omega', \mathcal{B}(\Omega'), dx_{|\Omega'})$$

in which case:

$$\int_{\Omega} f\circ\phi\, dx_{|\Omega} = \int_{\Omega'} f|J(\psi)|dx_{|\Omega'}$$

and, furthermore:

$$(f\circ\phi)|J(\phi)| \in L^1_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega), dx_{|\Omega}) \ \Leftrightarrow\ f \in L^1_{\mathbf{C}}(\Omega', \mathcal{B}(\Omega'), dx_{|\Omega'})$$

in which case:

$$\int_{\Omega} (f \circ \phi)|J(\phi)|dx_{|\Omega} = \int_{\Omega'} f dx_{|\Omega'}$$

Exercise 31. Let $f : \mathbf{R}^2 \to [0, +\infty]$, with $f(x, y) = \exp(-(x^2 + y^2)/2)$.

1. Show that:

$$\int_{\mathbf{R}^2} f(x, y)dxdy = \left(\int_{-\infty}^{+\infty} e^{-u^2/2} du \right)^2$$

2. Define:

$$\Delta_1 \stackrel{\triangle}{=} \{(x, y) \in \mathbf{R}^2 : \ x > 0 \ , \ y > 0\}$$
$$\Delta_2 \stackrel{\triangle}{=} \{(x, y) \in \mathbf{R}^2 : \ x < 0 \ , \ y > 0\}$$

and let Δ_3 and Δ_4 be the other two open quarters of $\mathbf{R}^2$. Show:

$$\int_{\mathbf{R}^2} f(x, y)dxdy = \int_{\Delta_1 \cup \ldots \cup \Delta_4} f(x, y)dxdy$$

3. Let $Q : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $Q(x,y) = (-x,y)$. Show that:

$$\int_{\Delta_1} f(x,y)dxdy = \int_{\Delta_2} f \circ Q^{-1}(x,y)dxdy$$

4. Show that:

$$\int_{\mathbf{R}^2} f(x,y)dxdy = 4\int_{\Delta_1} f(x,y)dxdy$$

5. Let $D_1 =]0,+\infty[\times]0,\pi/2[\subseteq \mathbf{R}^2$, and define $\phi : D_1 \to \Delta_1$ by:

$$\forall (r,\theta) \in D_1 \ , \ \phi(r,\theta) \stackrel{\triangle}{=} (r\cos\theta, r\sin\theta)$$

Show that ϕ is a bijection and that $\psi = \phi^{-1}$ is given by:

$$\forall (x,y) \in \Delta_1 \ , \ \psi(x,y) = (\sqrt{x^2+y^2}, \arctan(y/x))$$

6. Show that ϕ is a C^1-diffeomorphism, with:

$$\forall (r,\theta) \in D_1 \ , \ d\phi(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

and:

$$\forall (x,y) \in \Delta_1 \ , \ d\psi(x,y) = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix}$$

7. Show that $J(\phi)(r,\theta) = r$, for all $(r,\theta) \in D_1$.
8. Show that $J(\psi)(x,y) = 1/(\sqrt{x^2+y^2})$, for all $(x,y) \in \Delta_1$.
9. Show that:

$$\int_{\Delta_1} f(x,y)dxdy = \frac{\pi}{2}$$

10. Prove the following:

Theorem 123 *We have:*

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-u^2/2} du = 1$$