8. Jensen inequality

Definition 64 *Let $a, b \in \bar{\mathbf{R}}$, with $a < b$. Let $\phi :]a, b[\to \mathbf{R}$ be an $\mathbf{R}$-valued function. We say that ϕ is a* **convex function***, if and only if, for all $x, y \in]a, b[$ and $t \in [0, 1]$, we have:*

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$

Exercise 1. Let $a, b \in \bar{\mathbf{R}}$, with $a < b$. Let $\phi :]a, b[\to \mathbf{R}$ be a map.

1. Show that $\phi :]a, b[\to \mathbf{R}$ is convex, if and only if for all $x_1, \ldots, x_n$ in $]a, b[$ and $\alpha_1, \ldots, \alpha_n$ in $\mathbf{R}^+$ with $\alpha_1 + \ldots + \alpha_n = 1$, $n \geq 1$, we have:

$$\phi(\alpha_1 x_1 + \ldots + \alpha_n x_n) \leq \alpha_1 \phi(x_1) + \ldots \alpha_n \phi(x_n)$$

2. Show that $\phi :]a, b[\to \mathbf{R}$ is convex, if and only if for all x, y, z with $a < x < y < z < b$ we have:

$$\phi(y) \leq \frac{z-y}{z-x}\phi(x) + \frac{y-x}{z-x}\phi(z)$$

3. Show that $\phi :]a, b[\to \mathbf{R}$ is convex if and only if for all x, y, z with $a < x < y < z < b$, we have:

$$\frac{\phi(y) - \phi(x)}{y - x} \leq \frac{\phi(z) - \phi(y)}{z - y}$$

4. Let $\phi :]a, b[\to \mathbf{R}$ be convex. Let $x_0 \in]a, b[$, and $u, u', v, v' \in]a, b[$ be such that $u < u' < x_0 < v < v'$. Show that for all $x \in]x_0, v[$:

$$\frac{\phi(u') - \phi(u)}{u' - u} \leq \frac{\phi(x) - \phi(x_0)}{x - x_0} \leq \frac{\phi(v') - \phi(v)}{v' - v}$$

and deduce that $\lim_{x\downarrow\downarrow x_0} \phi(x) = \phi(x_0)$

5. Show that if $\phi :]a, b[\to \mathbf{R}$ is convex, then ϕ is continuous.

6. Define $\phi : [0, 1] \to \mathbf{R}$ by $\phi(0) = 1$ and $\phi(x) = 0$ for all $x \in]0, 1]$. Show that $\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$, $\forall x, y, t \in [0, 1]$, but that ϕ fails to be continuous on $[0, 1]$.

Definition 65 *Let $(\Omega, \mathcal{T})$ be a topological space. We say that $(\Omega, \mathcal{T})$ is a* **compact topological space** *if and only if, for all family $(V_i)_{i\in I}$ of open sets in Ω, such that $\Omega = \cup_{i\in I} V_i$, there exists a finite subset $\{i_1, \ldots, i_n\}$ of I such that $\Omega = V_{i_1} \cup \ldots \cup V_{i_n}$.*

In short, we say that $(\Omega, \mathcal{T})$ is compact if and only if, from any open covering of Ω, one can extract a finite sub-covering.

Definition 66 *Let $(\Omega, \mathcal{T})$ be a topological space, and $K \subseteq \Omega$. We say that K is a* **compact subset** *of Ω, if and only if the induced topological space $(K, \mathcal{T}_{|K})$ is a compact topological space.*

Exercise 2. Let $(\Omega, \mathcal{T})$ be a topological space.

1. Show that if $(\Omega, \mathcal{T})$ is compact, it is a compact subset of itself.
2. Show that $\emptyset$ is a compact subset of Ω.
3. Show that if $\Omega' \subseteq \Omega$ and K is a compact subset of Ω', then K is also a compact subset of Ω.

4. Show that if $(V_i)_{i\in I}$ is a family of open sets in Ω such that $K \subseteq \cup_{i\in I} V_i$, then $K = \cup_{i\in I}(V_i \cap K)$ and $V_i \cap K$ is open in K for all $i \in I$.

5. Show that $K \subseteq \Omega$ is a compact subset of Ω, if and only if for any family $(V_i)_{i\in I}$ of open sets in Ω such that $K \subseteq \cup_{i\in I} V_i$, there is a finite subset $\{i_1, \ldots, i_n\}$ of I such that $K \subseteq V_{i_1} \cup \ldots \cup V_{i_n}$.

6. Show that if $(\Omega, \mathcal{T})$ is compact and K is closed in Ω, then K is a compact subset of Ω.

Exercise 3. Let $a, b \in \mathbf{R}$, $a < b$. Let $(V_i)_{i\in I}$ be a family of open sets in $\mathbf{R}$ such that $[a, b] \subseteq \cup_{i\in I} V_i$. We define A as the set of all $x \in [a, b]$ such that $[a, x]$ can be covered by a finite number of V_i's. Let $c = \sup A$.

1. Show that $a \in A$.

2. Show that there is $\epsilon > 0$ such that $a + \epsilon \in A$.

3. Show that $a < c \leq b$.

4. Show the existence of $i_0 \in I$ and c', c'' with $a < c' < c < c''$, such that $]c', c''] \subseteq V_{i_0}$.

5. Show that $[a, c']$ can be covered by a finite number of V_i's.

6. Show that $[a, c'']$ can be covered by a finite number of V_i's.

7. Show that $b \wedge c'' \leq c$ and conclude that $c = b$.

8. Show that $[a, b]$ is a compact subset of $\mathbf{R}$.

Theorem 34 *Let $a, b \in \mathbf{R}$, $a < b$. The closed interval $[a, b]$ is a compact subset of $\mathbf{R}$.*

Definition 67 *Let $(\Omega, \mathcal{T})$ be a topological space. We say that $(\Omega, \mathcal{T})$ is a* **Hausdorff topological space***, if and only if for all $x, y \in \Omega$ with $x \neq y$, there exists open sets U and V in Ω, such that:*

$$x \in U \ , \ y \in V \ , \ U \cap V = \emptyset$$

Exercise 4. Let $(\Omega, \mathcal{T})$ be a topological space.

1. Show that if $(\Omega, \mathcal{T})$ is Hausdorff and $\Omega' \subseteq \Omega$, then the induced topological space $(\Omega', \mathcal{T}_{|\Omega'})$ is itself Hausdorff.

2. Show that if $(\Omega, \mathcal{T})$ is metrizable, then it is Hausdorff.

3. Show that any subset of $\bar{\mathbf{R}}$ is Hausdorff.

4. Let $(\Omega_i, \mathcal{T}_i)_{i \in I}$ be a family of Hausdorff topological spaces. Show that the product topological space $\Pi_{i \in I} \Omega_i$ is Hausdorff.

Exercise 5. Let $(\Omega, \mathcal{T})$ be a Hausdorff topological space. Let K be a compact subset of Ω and suppose there exists $y \in K^c$.

1. Show that for all $x \in K$, there are open sets V_x, W_x in Ω, such that $y \in V_x, x \in W_x$ and $V_x \cap W_x = \emptyset$.

2. Show that there exists a finite subset $\{x_1, \ldots, x_n\}$ of K such that $K \subseteq W^y$ where $W^y = W_{x_1} \cup \ldots \cup W_{x_n}$.

3. Let $V^y = V_{x_1} \cap \ldots \cap V_{x_n}$. Show that V^y is open and $V^y \cap W^y = \emptyset$.

4. Show that $y \in V^y \subseteq K^c$.

5. Show that $K^c = \cup_{y \in K^c} V^y$

6. Show that K is closed in Ω.

Theorem 35 *Let $(\Omega, \mathcal{T})$ be a Hausdorff topological space. For all $K \subseteq \Omega$, if K is a compact subset, then it is closed.*

Definition 68 *Let (E,d) be a metric space. For all $A \subseteq E$, we call* **diameter** *of A with respect to d, the element of $\bar{\mathbf{R}}$ denoted $\delta(A)$, defined as $\delta(A) = \sup\{d(x,y) \ : \ x,y \in A\}$, with the convention that $\delta(\emptyset) = -\infty$.*

Definition 69 *Let (E,d) be a metric space, and $A \subseteq E$. We say that A is* **bounded***, if and only if $\delta(A) < +\infty$.*

Exercise 6. Let (E,d) be a metric space. Let $A \subseteq E$.

1. Show that $\delta(A) = 0$ if and only if $A = \{x\}$ for some $x \in E$.

2. Let $\phi : \mathbf{R} \to]-1,1[$ be an increasing homeomorphism. Define $d''(x,y) = |x-y|$ and $d'(x,y) = |\phi(x) - \phi(y)|$, for all $x,y \in \mathbf{R}$. Show that d' is a metric on $\mathbf{R}$ inducing the usual topology on $\mathbf{R}$. Show that $\mathbf{R}$ is bounded with respect to d' but not with respect to d''.

3. Show that if $K \subseteq E$ is a compact subset of E, for all $\epsilon > 0$, there is a finite subset $\{x_1, \ldots, x_n\}$ of K such that:

$$K \subseteq B(x_1, \epsilon) \cup \ldots \cup B(x_n, \epsilon)$$

4. Show that any compact subset of any metrizable topological space $(\Omega, \mathcal{T})$, is bounded with respect to any metric inducing the topology $\mathcal{T}$.

EXERCISE 7. Suppose K is a closed subset of $\mathbf{R}$ which is bounded with respect to the usual metric on $\mathbf{R}$.

1. Show that there exists $M \in \mathbf{R}^+$ such that $K \subseteq [-M, M]$.

2. Show that K is also closed in $[-M, M]$.

3. Show that K is a compact subset of $[-M, M]$.

4. Show that K is a compact subset of $\mathbf{R}$.

5. Show that any compact subset of $\mathbf{R}$ is closed and bounded.

6. Show the following:

Theorem 36 *A subset of $\mathbf{R}$ is compact if and only if it is closed, and bounded with respect to the usual metric on $\mathbf{R}$.*

Exercise 8. Let $(\Omega, \mathcal{T})$ and $(S, \mathcal{T}_S)$ be two topological spaces. Let $f : (\Omega, \mathcal{T}) \to (S, \mathcal{T}_S)$ be a continuous map.

1. Show that if $(W_i)_{i \in I}$ is an open covering of $f(\Omega)$, then the family $(f^{-1}(W_i))_{i \in I}$ is an open covering of Ω.

2. Show that if $(\Omega, \mathcal{T})$ is a compact topological space, then $f(\Omega)$ is a compact subset of $(S, \mathcal{T}_S)$.

Exercise 9.

1. Show that $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is a compact topological space.

2. Show that any compact subset of $\mathbf{R}$ is a compact subset of $\bar{\mathbf{R}}$.

3. Show that a subset of $\bar{\mathbf{R}}$ is compact if and only if it is closed.

4. Let A be a non-empty subset of $\bar{\mathbf{R}}$, and let $\alpha = \sup A$. Show that if $\alpha \neq -\infty$, then for all $U \in \mathcal{T}_{\bar{\mathbf{R}}}$ with $\alpha \in U$, there exists $\beta \in \mathbf{R}$ with $\beta < \alpha$ and $]\beta, \alpha] \subseteq U$. Conclude that $\alpha \in \bar{A}$.

5. Show that if A is a non-empty closed subset of $\bar{\mathbf{R}}$, then we have $\sup A \in A$ and $\inf A \in A$.

6. Consider $A = \{x \in \mathbf{R}\ ,\ \sin(x) = 0\}$. Show that A is closed in $\mathbf{R}$, but that $\sup A \notin A$ and $\inf A \notin A$.

7. Show that if A is a non-empty, closed and bounded subset of $\mathbf{R}$, then $\sup A \in A$ and $\inf A \in A$.

Exercise 10. Let $(\Omega, \mathcal{T})$ be a compact, non-empty topological space. Let $f : (\Omega, \mathcal{T}) \to (\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ be a continuous map.

1. Show that if $f(\Omega) \subseteq \mathbf{R}$, the continuity of f with respect to $\mathcal{T}_{\bar{\mathbf{R}}}$ is equivalent to the continuity of f with respect to $\mathcal{T}_{\mathbf{R}}$.

2. Show the following:

Theorem 37 *Let $f : (\Omega, \mathcal{T}) \to (\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ be a continuous map, where $(\Omega, \mathcal{T})$ is a non-empty topological space. Then, if $(\Omega, \mathcal{T})$ is compact, f attains its maximum and minimum, i.e. there exist $x_m, x_M \in \Omega$, such that:*

$$f(x_m) = \inf_{x \in \Omega} f(x) \ , \ f(x_M) = \sup_{x \in \Omega} f(x)$$

Exercise 11. Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$, and differentiable on $]a, b[$, with $f(a) = f(b)$.

1. Show that if $c \in]a,b[$ and $f(c) = \sup_{x\in[a,b]} f(x)$, then $f'(c) = 0$.

2. Show the following:

Theorem 38 (Rolle) *Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a,b] \to \mathbf{R}$ be continuous on $[a,b]$, and differentiable on $]a,b[$, with $f(a) = f(b)$. Then, there exists $c \in]a,b[$ such that $f'(c) = 0$.*

Exercise 12. Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a,b] \to \mathbf{R}$ be continuous on $[a,b]$ and differentiable on $]a,b[$. Define:

$$h(x) \stackrel{\triangle}{=} f(x) - (x-a)\frac{f(b)-f(a)}{b-a}$$

1. Show that h is continuous on $[a,b]$ and differentiable on $]a,b[$.

2. Show the existence of $c \in]a,b[$ such that:

$$f(b) - f(a) = (b-a)f'(c)$$

Exercise 13. Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a, b] \to \mathbf{R}$ be a map. Let $n \geq 0$. We assume that f is of class C^n on $[a, b]$, and that $f^{(n+1)}$ exists on $]a, b[$. Define:

$$h(x) \stackrel{\triangle}{=} f(b) - f(x) - \sum_{k=1}^{n} \frac{(b-x)^k}{k!} f^{(k)}(x) - \alpha \frac{(b-x)^{n+1}}{(n+1)!}$$

where α is chosen such that $h(a) = 0$.

1. Show that h is continuous on $[a, b]$ and differentiable on $]a, b[$.

2. Show that for all $x \in]a, b[$:

$$h'(x) = \frac{(b-x)^n}{n!} (\alpha - f^{(n+1)}(x))$$

3. Prove the following:

Theorem 39 (Taylor-Lagrange) *Let* $a, b \in \mathbf{R}$, $a < b$, *and* $n \geq 0$. *Let* $f : [a, b] \to \mathbf{R}$ *be a map of class* C^n *on* $[a, b]$ *such that* $f^{(n+1)}$ *exists on* $]a, b[$. *Then, there exists* $c \in]a, b[$ *such that:*

$$f(b) - f(a) = \sum_{k=1}^{n} \frac{(b-a)^k}{k!} f^{(k)}(a) + \frac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

Exercise 14. Let $a, b \in \bar{\mathbf{R}}$, $a < b$ and $\phi :]a, b[\to \mathbf{R}$ be differentiable.

1. Show that if ϕ is convex, then for all $x, y \in]a, b[$, $x < y$, we have:

$$\phi'(x) \leq \phi'(y)$$

2. Show that if $x, y, z \in]a, b[$ with $x < y < z$, there are $c_1, c_2 \in]a, b[$, with $c_1 < c_2$ and:

$$\begin{aligned} \phi(y) - \phi(x) &= \phi'(c_1)(y - x) \\ \phi(z) - \phi(y) &= \phi'(c_2)(z - y) \end{aligned}$$

3. Show conversely that if ϕ' is non-decreasing, then ϕ is convex.

4. Show that $x \to e^x$ is convex on $\mathbf{R}$.

5. Show that $x \to -\ln(x)$ is convex on $]0, +\infty[$.

Definition 70 *we say that a finite measure space* $(\Omega, \mathcal{F}, P)$ *is a* **probability space**, *if and only if* $P(\Omega) = 1$.

Definition 71 *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space, and* (S, Σ) *be a measurable space. We call* **random variable** *w.r. to* (S, Σ), *any measurable map* $X : (\Omega, \mathcal{F}) \to (S, \Sigma)$.

Definition 72 *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space. Let* X *be a non-negative random variable, or an element of* $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, P)$. *We call* **expectation** *of* X, *denoted* $E[X]$, *the integral:*

$$E[X] \stackrel{\triangle}{=} \int_\Omega X dP$$

Exercise 15. Let $a, b \in \bar{\mathbf{R}}$, $a < b$ and $\phi :]a, b[\to \mathbf{R}$ be a convex map. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$ be such that $X(\Omega) \subseteq]a, b[$.

1. Show that $\phi \circ X : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is measurable.

2. Show that $\phi \circ X \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$, if and only if $E[|\phi \circ X|] < +\infty$.

3. Show that if $E[X] = a$, then $a \in \mathbf{R}$ and $X = a$ P-a.s.

4. Show that if $E[X] = b$, then $b \in \mathbf{R}$ and $X = b$ P-a.s.

5. Let $m = E[X]$. Show that $m \in]a, b[$.

6. Define:
$$\beta \stackrel{\triangle}{=} \sup_{x \in]a,m[} \frac{\phi(m) - \phi(x)}{m - x}$$
Show that $\beta \in \mathbf{R}$ and that for all $z \in]m, b[$, we have:
$$\beta \leq \frac{\phi(z) - \phi(m)}{z - m}$$

7. Show that for all $x \in]a, b[$, we have $\phi(m) + \beta(x - m) \leq \phi(x)$.

8. Show that for all $\omega \in \Omega$, $\phi(m) + \beta(X(\omega) - m) \leq \phi(X(\omega))$.

9. Show that if $\phi \circ X \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$ then $\phi(m) \leq E[\phi \circ X]$.

Theorem 40 (Jensen inequality) *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $a, b \in \bar{\mathbf{R}}$, $a < b$ and $\phi :]a, b[\to \mathbf{R}$ be a convex map. Suppose that $X \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$ is such that $X(\Omega) \subseteq]a, b[$ and such that $\phi \circ X \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$. Then:*

$$\phi(E[X]) \leq E[\phi \circ X]$$