6. Product Spaces

In the following, I is a non-empty set.

Definition 50 *Let $(\Omega_i)_{i\in I}$ be a family of sets, indexed by a non-empty set I. We call* **Cartesian product** *of the family $(\Omega_i)_{i\in I}$ the set, denoted $\Pi_{i\in I}\Omega_i$, and defined by:*

$$\prod_{i\in I}\Omega_i \stackrel{\triangle}{=} \{\omega : I \to \cup_{i\in I}\Omega_i \ , \ \omega(i) \in \Omega_i \ , \ \forall i \in I\}$$

In other words, $\Pi_{i\in I}\Omega_i$ is the set of all maps ω defined on I, with values in $\cup_{i\in I}\Omega_i$, such that $\omega(i) \in \Omega_i$ for all $i \in I$.

Theorem 25 (Axiom of choice) *Let $(\Omega_i)_{i\in I}$ be a family of sets, indexed by a non-empty set I. Then, $\Pi_{i\in I}\Omega_i$ is non-empty, if and only if Ω_i is non-empty for all $i \in I$*[1].

[1] When I is finite, this theorem is traditionally derived from other axioms.

Exercise 1.

1. Let Ω be a set and suppose that $\Omega_i = \Omega, \forall i \in I$. We use the notation Ω^I instead of $\Pi_{i\in I}\Omega_i$. Show that Ω^I is the set of all maps $\omega : I \to \Omega$.

2. What are the sets $\mathbf{R}^{\mathbf{R}^+}$, $\mathbf{R}^{\mathbf{N}}$, $[0,1]^{\mathbf{N}}$, $\bar{\mathbf{R}}^{\mathbf{R}}$?

3. Suppose $I = \mathbf{N}^*$. We sometimes use the notation $\Pi_{n=1}^{+\infty}\Omega_n$ instead of $\Pi_{n\in\mathbf{N}^*}\Omega_n$. Let $\mathcal{S}$ be the set of all sequences $(x_n)_{n\geq 1}$ such that $x_n \in \Omega_n$ for all $n \geq 1$. Is $\mathcal{S}$ the same thing as the product $\Pi_{n=1}^{+\infty}\Omega_n$?

4. Suppose $I = \mathbf{N}_n = \{1,\ldots,n\}$, $n \geq 1$. We use the notation $\Omega_1 \times \ldots \times \Omega_n$ instead of $\Pi_{i\in\{1,\ldots,n\}}\Omega_i$. For $\omega \in \Omega_1 \times \ldots \times \Omega_n$, it is customary to write $(\omega_1,\ldots,\omega_n)$ instead of ω, where we have $\omega_i = \omega(i)$. What is your guess for the definition of sets such as $\mathbf{R}^n, \bar{\mathbf{R}}^n, \mathbf{Q}^n, \mathbf{C}^n$.

5. Let E, F, G be three sets. Define $E \times F \times G$.

Definition 51 *Let I be a non-empty set. We say that a family of sets $(I_\lambda)_{\lambda\in\Lambda}$, where $\Lambda \neq \emptyset$, is a* **partition** *of I, if and only if:*

$$\begin{array}{rl} (i) & \forall\lambda\in\Lambda\ ,\ I_\lambda \neq \emptyset \\ (ii) & \forall\lambda,\lambda'\in\Lambda\ ,\ \lambda\neq\lambda' \Rightarrow I_\lambda\cap I_{\lambda'} = \emptyset \\ (iii) & I = \cup_{\lambda\in\Lambda} I_\lambda \end{array}$$

EXERCISE 2. Let $(\Omega_i)_{i\in I}$ be a family of sets indexed by I, and $(I_\lambda)_{\lambda\in\Lambda}$ be a partition of the set I.

1. For each $\lambda\in\Lambda$, recall the definition of $\Pi_{i\in I_\lambda}\Omega_i$.
2. Recall the definition of $\Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}\Omega_i)$.
3. Define a *natural* bijection $\Phi : \Pi_{i\in I}\Omega_i \to \Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}\Omega_i)$.
4. Define a *natural* bijection $\psi : \mathbf{R}^p\times\mathbf{R}^n \to \mathbf{R}^{p+n}$, for all $n, p \geq 1$.

Definition 52 *Let $(\Omega_i)_{i\in I}$ be a family of sets, indexed by a non-empty set I. For all $i \in I$, let $\mathcal{E}_i$ be a set of subsets of Ω_i. We define a* **rectangle** *of the family $(\mathcal{E}_i)_{i\in I}$, as any subset A of $\Pi_{i\in I}\Omega_i$, of the form $A = \Pi_{i\in I} A_i$ where $A_i \in \mathcal{E}_i \cup \{\Omega_i\}$ for all $i \in I$, and such that $A_i = \Omega_i$ except for a finite number of indices $i \in I$. Consequently, the set of all rectangles, denoted $\amalg_{i\in I}\mathcal{E}_i$, is defined as:*

$$\coprod_{i\in I} \mathcal{E}_i \stackrel{\triangle}{=} \left\{ \prod_{i\in I} A_i : A_i \in \mathcal{E}_i \cup \{\Omega_i\} \ , \ A_i \neq \Omega_i \text{ for finitely many } i \in I \right\}$$

EXERCISE 3. $(\Omega_i)_{i\in I}$ and $(\mathcal{E}_i)_{i\in I}$ being as above:

1. Show that if $I = \mathbf{N}_n$ and $\Omega_i \in \mathcal{E}_i$ for all $i = 1, \ldots, n$, then $\mathcal{E}_1 \amalg \ldots \amalg \mathcal{E}_n = \{A_1 \times \ldots \times A_n \ : \ A_i \in \mathcal{E}_i \ , \ \forall i \in I\}$.

2. Let A be a rectangle. Show that there exists a finite subset J of I such that: $A = \{\omega \in \Pi_{i\in I}\Omega_i \ : \ \omega(j) \in A_j \ , \ \forall j \in J\}$ for some A_j's such that $A_j \in \mathcal{E}_j$, for all $j \in J$.

Definition 53 *Let* $(\Omega_i, \mathcal{F}_i)_{i\in I}$ *be a family of measurable spaces, indexed by a non-empty set* I. *We call* **measurable rectangle** *, any rectangle of the family* $(\mathcal{F}_i)_{i\in I}$. *The set of all measurable rectangles is given by* [2]*:*

$$\coprod_{i\in I}\mathcal{F}_i \stackrel{\triangle}{=} \left\{\prod_{i\in I} A_i \ : \ A_i \in \mathcal{F}_i \ , \ A_i \neq \Omega_i \textit{ for finitely many } i \in I\right\}$$

Definition 54 *Let* $(\Omega_i, \mathcal{F}_i)_{i\in I}$ *be a family of measurable spaces, indexed by a non-empty set* I. *We define the* **product** σ**-algebra** *of* $(\mathcal{F}_i)_{i\in I}$*, as the* σ*-algebra on* $\Pi_{i\in I}\Omega_i$*, denoted* $\otimes_{i\in I}\mathcal{F}_i$*, and generated by all measurable rectangles, i.e.*

$$\bigotimes_{i\in I}\mathcal{F}_i \stackrel{\triangle}{=} \sigma\left(\coprod_{i\in I}\mathcal{F}_i\right)$$

[2]Note that $\Omega_i \in \mathcal{F}_i$ for all $i \in I$.

Exercise 4.

1. Suppose $I = \mathbf{N}_n$. Show that $\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$ is generated by all sets of the form $A_1 \times \ldots \times A_n$, where $A_i \in \mathcal{F}_i$ for all $i = 1, \ldots, n$.

2. Show that $\mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})$ is generated by sets of the form $A \times B \times C$ where $A, B, C \in \mathcal{B}(\mathbf{R})$.

3. Show that if $(\Omega, \mathcal{F})$ is a measurable space, $\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}$ is the σ-algebra on $\mathbf{R}^+ \times \Omega$ generated by sets of the form $B \times F$ where $B \in \mathcal{B}(\mathbf{R}^+)$ and $F \in \mathcal{F}$.

Exercise 5. Let $(\Omega_i)_{i \in I}$ be a family of non-empty sets and $\mathcal{E}_i$ be a subset of the power set $\mathcal{P}(\Omega_i)$ for all $i \in I$.

1. Give a generator of the σ-algebra $\otimes_{i \in I} \sigma(\mathcal{E}_i)$ on $\Pi_{i \in I} \Omega_i$.

2. Show that:

$$\sigma\left(\coprod_{i \in I} \mathcal{E}_i\right) \subseteq \bigotimes_{i \in I} \sigma(\mathcal{E}_i)$$

3. Let A be a rectangle of the family $(\sigma(\mathcal{E}_i))_{i\in I}$. Show that if A is not empty, then the representation $A = \Pi_{i\in I} A_i$ with $A_i \in \sigma(\mathcal{E}_i)$ is unique. Define $J_A = \{i \in I : A_i \neq \Omega_i\}$. Explain why J_A is a well-defined finite subset of I.

4. If $A \in \amalg_{i\in I}\sigma(\mathcal{E}_i)$, Show that if $A = \emptyset$, or $A \neq \emptyset$ and $J_A = \emptyset$, then $A \in \sigma(\amalg_{i\in I}\mathcal{E}_i)$.

Exercise 6. Everything being as before, Let $n \geq 0$. We assume that the following induction hypothesis has been proved:

$$A \in \coprod_{i\in I}\sigma(\mathcal{E}_i), A \neq \emptyset, \mathrm{card} J_A = n \;\Rightarrow\; A \in \sigma\left(\coprod_{i\in I}\mathcal{E}_i\right)$$

We assume that A is a non empty measurable rectangle of $(\sigma(\mathcal{E}_i))_{i\in I}$ with $\mathrm{card} J_A = n+1$. Let $J_A = \{i_1, \ldots, i_{n+1}\}$ be an extension of J_A.

For all $B \subseteq \Omega_{i_1}$, we define:

$$A^B \stackrel{\triangle}{=} \prod_{i \in I} \bar{A}_i$$

where each $\bar{A}_i$ is equal to A_i except $\bar{A}_{i_1} = B$. We define the set:

$$\Gamma \stackrel{\triangle}{=} \left\{ B \subseteq \Omega_{i_1} \ : \ A^B \in \sigma\left(\coprod_{i \in I} \mathcal{E}_i\right) \right\}$$

1. Show that $A^{\Omega_{i_1}} \neq \emptyset$, $\mathrm{card} J_{A^{\Omega_{i_1}}} = n$ and that $A^{\Omega_{i_1}} \in \amalg_{i \in I} \sigma(\mathcal{E}_i)$.
2. Show that $\Omega_{i_1} \in \Gamma$.
3. Show that for all $B \subseteq \Omega_{i_1}$, we have $A^{\Omega_{i_1} \setminus B} = A^{\Omega_{i_1}} \setminus A^B$.
4. Show that $B \in \Gamma \Rightarrow \Omega_{i_1} \setminus B \in \Gamma$.
5. Let $B_n \subseteq \Omega_{i_1}$, $n \geq 1$. Show that $A^{\cup B_n} = \cup_{n \geq 1} A^{B_n}$.
6. Show that Γ is a σ-algebra on Ω_{i_1}.

7. Let $B \in \mathcal{E}_{i_1}$, and for $i \in I$ define $\bar{B}_i = \Omega_i$ for all i's except $\bar{B}_{i_1} = B$. Show that $A^B = A^{\Omega_{i_1}} \cap (\Pi_{i\in I}\bar{B}_i)$.

8. Show that $\sigma(\mathcal{E}_{i_1}) \subseteq \Gamma$.

9. Show that $A = A^{A_{i_1}}$ and $A \in \sigma(\amalg_{i\in I}\mathcal{E}_i)$.

10. Show that $\amalg_{i\in I}\sigma(\mathcal{E}_i) \subseteq \sigma(\amalg_{i\in I}\mathcal{E}_i)$.

11. Show that $\sigma(\amalg_{i\in I}\mathcal{E}_i) = \otimes_{i\in I}\sigma(\mathcal{E}_i)$.

Theorem 26 *Let $(\Omega_i)_{i\in I}$ be a family of non-empty sets indexed by a non-empty set I. For all $i \in I$, let $\mathcal{E}_i$ be a set of subsets of Ω_i. Then, the product σ-algebra $\otimes_{i\in I}\sigma(\mathcal{E}_i)$ on the Cartesian product $\Pi_{i\in I}\Omega_i$ is generated by the rectangles of $(\mathcal{E}_i)_{i\in I}$, i.e. :*

$$\bigotimes_{i\in I}\sigma(\mathcal{E}_i) = \sigma\left(\coprod_{i\in I}\mathcal{E}_i\right)$$

Exercise 7. Let $\mathcal{T}_{\mathbf{R}}$ denote the usual topology in $\mathbf{R}$. Let $n \geq 1$.

1. Show that $\mathcal{T}_{\mathbf{R}} \amalg \ldots \amalg \mathcal{T}_{\mathbf{R}} = \{A_1 \times \ldots \times A_n \ : \ A_i \in \mathcal{T}_{\mathbf{R}}\}$.

2. Show that $\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R}) = \sigma(\mathcal{T}_{\mathbf{R}} \amalg \ldots \amalg \mathcal{T}_{\mathbf{R}})$.

3. Define $\mathcal{C}_2 = \{]a_1, b_1] \times \ldots \times]a_n, b_n] \ : \ a_i, b_i \in \mathbf{R}\}$. Show that $\mathcal{C}_2 \subseteq \mathcal{S} \amalg \ldots \amalg \mathcal{S}$, where $\mathcal{S} = \{]a, b] \ : \ a, b \in \mathbf{R}\}$, but that the inclusion is strict.

4. Show that $\mathcal{S} \amalg \ldots \amalg \mathcal{S} \subseteq \sigma(\mathcal{C}_2)$.

5. Show that $\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R}) = \sigma(\mathcal{C}_2)$.

Exercise 8. Let Ω and Ω' be two non-empty sets. Let A be a subset of Ω such that $\emptyset \neq A \neq \Omega$. Let $\mathcal{E} = \{A\} \subseteq \mathcal{P}(\Omega)$ and $\mathcal{E}' = \emptyset \subseteq \mathcal{P}(\Omega')$.

1. Show that $\sigma(\mathcal{E}) = \{\emptyset, A, A^c, \Omega\}$.

2. Show that $\sigma(\mathcal{E}') = \{\emptyset, \Omega'\}$.

3. Define $\mathcal{C} = \{E \times F\ ,\ E \in \mathcal{E}, F \in \mathcal{E}'\}$ and show that $\mathcal{C} = \emptyset$.
4. Show that $\mathcal{E} \amalg \mathcal{E}' = \{A \times \Omega', \Omega \times \Omega'\}$.
5. Show that $\sigma(\mathcal{E}) \otimes \sigma(\mathcal{E}') = \{\emptyset, A \times \Omega', A^c \times \Omega', \Omega \times \Omega'\}$.
6. Conclude that $\sigma(\mathcal{E}) \otimes \sigma(\mathcal{E}') \neq \sigma(\mathcal{C}) = \{\emptyset, \Omega \times \Omega'\}$.

Exercise 9. Let $n \geq 1$ and $p \geq 1$ be two positive integers.

1. Define $\mathcal{F} = \underbrace{\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R})}_{n}$, and $\mathcal{G} = \underbrace{\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R})}_{p}$.
   Explain why $\mathcal{F} \otimes \mathcal{G}$ can be viewed as a σ-algebra on $\mathbf{R}^{n+p}$.
2. Show that $\mathcal{F} \otimes \mathcal{G}$ is generated by sets of the form $A_1 \times \ldots \times A_{n+p}$ where $A_i \in \mathcal{B}(\mathbf{R}), i = 1, \ldots, n+p$.

3. Show that:

$$\underbrace{\mathcal{B}(\mathbf{R})\otimes\ldots\otimes\mathcal{B}(\mathbf{R})}_{n+p} = \underbrace{(\mathcal{B}(\mathbf{R})\otimes\ldots\otimes\mathcal{B}(\mathbf{R}))}_{n}\otimes\underbrace{(\mathcal{B}(\mathbf{R})\otimes\ldots\otimes\mathcal{B}(\mathbf{R}))}_{p}$$

Exercise 10. Let $(\Omega_i, \mathcal{F}_i)_{i\in I}$ be a family of measurable spaces. Let $(I_\lambda)_{\lambda\in\Lambda}$, where $\Lambda \neq \emptyset$, be a partition of I. Let $\Omega = \Pi_{i\in I}\Omega_i$ and $\Omega' = \Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}\Omega_i)$.

1. Define a *natural* bijection between $\mathcal{P}(\Omega)$ and $\mathcal{P}(\Omega')$.

2. Show that through such bijection, $A = \Pi_{i\in I}A_i \subseteq \Omega$, where $A_i \subseteq \Omega_i$, is identified with $A' = \Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}A_i) \subseteq \Omega'$.

3. Show that $\amalg_{i\in I}\mathcal{F}_i = \amalg_{\lambda\in\Lambda}(\amalg_{i\in I_\lambda}\mathcal{F}_i)$.

4. Show that $\otimes_{i\in I}\mathcal{F}_i = \otimes_{\lambda\in\Lambda}(\otimes_{i\in I_\lambda}\mathcal{F}_i)$.

Definition 55 *Let Ω be set and $\mathcal{A}$ be a set of subsets of Ω. We call* **topology generated** *by $\mathcal{A}$, the topology on Ω, denoted $\mathcal{T}(\mathcal{A})$, equal to the intersection of all topologies on Ω, which contain $\mathcal{A}$.*

EXERCISE 11. Let Ω be a set and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$.

1. Explain why $\mathcal{T}(\mathcal{A})$ is indeed a topology on Ω.

2. Show that $\mathcal{T}(\mathcal{A})$ is the smallest topology $\mathcal{T}$ such that $\mathcal{A} \subseteq \mathcal{T}$.

3. Show that the metric topology on a metric space (E, d) is generated by the open balls $\mathcal{A} = \{B(x, \epsilon) : x \in E, \epsilon > 0\}$.

Definition 56 *Let $(\Omega_i, \mathcal{T}_i)_{i \in I}$ be a family of topological spaces, indexed by a non-empty set I. We define the* **product topology** *of $(\mathcal{T}_i)_{i \in I}$, as the topology on $\Pi_{i \in I} \Omega_i$, denoted $\odot_{i \in I} \mathcal{T}_i$, and generated by* *all rectangles of* $(\mathcal{T}_i)_{i\in I}$*, i.e.*

$$\bigodot_{i\in I}\mathcal{T}_i \stackrel{\triangle}{=} \mathcal{T}\left(\coprod_{i\in I}\mathcal{T}_i\right)$$

Exercise 12. Let $(\Omega_i, \mathcal{T}_i)_{i\in I}$ be a family of topological spaces.

1. Show that $U \in \odot_{i\in I}\mathcal{T}_i$, if and only if:

$$\forall x \in U\ ,\ \exists V \in \amalg_{i\in I}\mathcal{T}_i\ ,\ x \in V \subseteq U$$

2. Show that $\amalg_{i\in I}\mathcal{T}_i \subseteq \odot_{i\in I}\mathcal{T}_i$.

3. Show that $\otimes_{i\in I}\mathcal{B}(\Omega_i) = \sigma(\amalg_{i\in I}\mathcal{T}_i)$.

4. Show that $\otimes_{i\in I}\mathcal{B}(\Omega_i) \subseteq \mathcal{B}(\Pi_{i\in I}\Omega_i)$.

Exercise 13. Let $n \geq 1$ be a positive integer. For all $x, y \in \mathbf{R}^n$, let:

$$(x, y) \stackrel{\triangle}{=} \sum_{i=1}^{n} x_i y_i$$

and we put $\|x\| = \sqrt{(x, x)}$.

1. Show that for all $t \in \mathbf{R}$, $\|x + ty\|^2 = \|x\|^2 + t^2\|y\|^2 + 2t(x, y)$.
2. From $\|x + ty\|^2 \geq 0$ for all t, deduce that $|(x, y)| \leq \|x\|.\|y\|$.
3. Conclude that $\|x + y\| \leq \|x\| + \|y\|$.

Exercise 14. Let $(\Omega_1, \mathcal{T}_1), \ldots, (\Omega_n, \mathcal{T}_n)$, $n \geq 1$, be metrizable topological spaces. Let $d_1, \ldots, d_n$ be metrics on $\Omega_1, \ldots, \Omega_n$, inducing the topologies $\mathcal{T}_1, \ldots, \mathcal{T}_n$ respectively. Let $\Omega = \Omega_1 \times \ldots \times \Omega_n$ and $\mathcal{T}$ be the product topology on Ω. For all $x, y \in \Omega$, we define:

$$d(x,y) \stackrel{\triangle}{=} \sqrt{\sum_{i=1}^{n} (d_i(x_i, y_i))^2}$$

1. Show that $d : \Omega \times \Omega \to \mathbf{R}^+$ is a metric on Ω.
2. Show that $U \subseteq \Omega$ is open in Ω, if and only if, for all $x \in U$ there are open sets $U_1, \ldots, U_n$ in $\Omega_1, \ldots, \Omega_n$ respectively, such that:

$$x \in U_1 \times \ldots \times U_n \subseteq U$$

3. Let $U \in \mathcal{T}$ and $x \in U$. Show the existence of $\epsilon > 0$ such that:

$$(\forall i = 1, \ldots, n \ d_i(x_i, y_i) < \epsilon) \ \Rightarrow \ y \in U$$

4. Show that $\mathcal{T} \subseteq \mathcal{T}_\Omega^d$.
5. Let $U \in \mathcal{T}_\Omega^d$ and $x \in U$. Show the existence of $\epsilon > 0$ such that:

$$x \in B(x_1, \epsilon) \times \ldots \times B(x_n, \epsilon) \subseteq U$$

6. Show that $\mathcal{T}^d_\Omega \subseteq \mathcal{T}$.
7. Show that the product topological space $(\Omega, \mathcal{T})$ is metrizable.
8. For all $x, y \in \Omega$, define:

$$\begin{aligned} d'(x,y) &\stackrel{\triangle}{=} \sum_{i=1}^n d_i(x_i, y_i) \\ d''(x,y) &\stackrel{\triangle}{=} \max_{i=1,\ldots,n} d_i(x_i, y_i) \end{aligned}$$

   Show that d', d'' are metrics on Ω.
9. Show the existence of α', β', α'' and $\beta'' > 0$, such that we have $\alpha' d' \leq d \leq \beta' d'$ and $\alpha'' d'' \leq d \leq \beta'' d''$.
10. Show that d' and d'' also induce the product topology on Ω.

Exercise 15. Let $(\Omega_n, \mathcal{T}_n)_{n\geq 1}$ be a sequence of metrizable topological spaces. For all $n \geq 1$, let d_n be a metric on Ω_n inducing the topology

$\mathcal{T}_n$. Let $\Omega = \Pi_{n=1}^{+\infty}\Omega_n$ be the Cartesian product and $\mathcal{T}$ be the product topology on Ω. For all $x, y \in \Omega$, we define:

$$d(x,y) \stackrel{\triangle}{=} \sum_{n=1}^{+\infty} \frac{1}{2^n}(1 \wedge d_n(x_n, y_n))$$

1. Show that for all $a, b \in \mathbf{R}^+$, we have $1 \wedge (a+b) \leq 1 \wedge a + 1 \wedge b$.

2. Show that d is a metric on Ω.

3. Show that $U \subseteq \Omega$ is open in Ω, if and only if, for all $x \in U$, there is an integer $N \geq 1$ and open sets $U_1, \ldots, U_N$ in $\Omega_1, \ldots, \Omega_N$ respectively, such that:

$$x \in U_1 \times \ldots \times U_N \times \prod_{n=N+1}^{+\infty} \Omega_n \subseteq U$$

4. Show that $d(x,y) < 1/2^n \;\Rightarrow\; d_n(x_n, y_n) \leq 2^n d(x,y)$.

5. Show that for all $U \in \mathcal{T}$ and $x \in U$, there exists $\epsilon > 0$ such that $d(x,y) < \epsilon \ \Rightarrow \ y \in U$.

6. Show that $\mathcal{T} \subseteq \mathcal{T}_\Omega^d$.

7. Let $U \in \mathcal{T}_\Omega^d$ and $x \in U$. Show the existence of $\epsilon > 0$ and $N \geq 1$, such that:

$$\sum_{n=1}^{N} \frac{1}{2^n}(1 \wedge d_n(x_n, y_n)) < \epsilon \ \Rightarrow \ y \in U$$

8. Show that for all $U \in \mathcal{T}_\Omega^d$ and $x \in U$, there is $\epsilon > 0$ and $N \geq 1$ such that:

$$x \in B(x_1, \epsilon) \times \ldots \times B(x_N, \epsilon) \times \prod_{n=N+1}^{+\infty} \Omega_n \subseteq U$$

9. Show that $\mathcal{T}_\Omega^d \subseteq \mathcal{T}$.

10. Show that the product topological space $(\Omega, \mathcal{T})$ is metrizable.

Definition 57 *Let $(\Omega, \mathcal{T})$ be a topological space. A subset $\mathcal{H}$ of $\mathcal{T}$ is called a* **countable base** *of $(\Omega, \mathcal{T})$, if and only if $\mathcal{H}$ is at most countable, and has the property:*

$$\forall U \in \mathcal{T}\ ,\ \exists \mathcal{H}' \subseteq \mathcal{H}\ ,\ U = \bigcup_{V \in \mathcal{H}'} V$$

Exercise 16.

1. Show that $\mathcal{H} = \{]r, q[\ :\ r, q \in \mathbf{Q}\}$ is a countable base of $(\mathbf{R}, \mathcal{T}_{\mathbf{R}})$.
2. Show that if $(\Omega, \mathcal{T})$ is a topological space with countable base, and $\Omega' \subseteq \Omega$, then the induced topological space $(\Omega', \mathcal{T}_{|\Omega'})$ also has a countable base.
3. Show that $[-1, 1]$ has a countable base.
4. Show that if $(\Omega, \mathcal{T})$ and $(S, \mathcal{T}_S)$ are homeomorphic, then $(\Omega, \mathcal{T})$ has a countable base if and only if $(S, \mathcal{T}_S)$ has a countable base.

5. Show that $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ has a countable base.

Exercise 17. Let $(\Omega_n, \mathcal{T}_n)_{n\geq 1}$ be a sequence of topological spaces with countable base. For $n \geq 1$, Let $\{V_n^k : k \in I_n\}$ be a countable base of $(\Omega_n, \mathcal{T}_n)$ where I_n is a finite or countable set. Let $\Omega = \Pi_{n=1}^{\infty}\Omega_n$ be the Cartesian product and $\mathcal{T}$ be the product topology on Ω. For all $p \geq 1$, we define:

$$\mathcal{H}^p \triangleq \left\{ V_1^{k_1} \times \ldots \times V_p^{k_p} \times \prod_{n=p+1}^{+\infty} \Omega_n \ : \ (k_1, \ldots, k_p) \in I_1 \times \ldots \times I_p \right\}$$

and we put $\mathcal{H} = \cup_{p\geq 1}\mathcal{H}^p$.

1. Show that for all $p \geq 1$, $\mathcal{H}^p \subseteq \mathcal{T}$.
2. Show that $\mathcal{H} \subseteq \mathcal{T}$.
3. For all $p \geq 1$, show the existence of an injection $j_p : \mathcal{H}^p \to \mathbf{N}^p$.

4. Show the existence of a bijection $\phi_2 : \mathbf{N}^2 \to \mathbf{N}$.

5. For $p \geq 1$, show the existence of an bijection $\phi_p : \mathbf{N}^p \to \mathbf{N}$.

6. Show that $\mathcal{H}^p$ is at most countable for all $p \geq 1$.

7. Show the existence of an injection $j : \mathcal{H} \to \mathbf{N}^2$.

8. Show that $\mathcal{H}$ is a finite or countable set of open sets in Ω.

9. Let $U \in \mathcal{T}$ and $x \in U$. Show that there is $p \geq 1$ and $U_1, \ldots, U_p$ open sets in $\Omega_1, \ldots, \Omega_p$ such that:

$$x \in U_1 \times \ldots \times U_p \times \prod_{n=p+1}^{+\infty} \Omega_n \subseteq U$$

10. Show the existence of some $V_x \in \mathcal{H}$ such that $x \in V_x \subseteq U$.

11. Show that $\mathcal{H}$ is a countable base of the topological space $(\Omega, \mathcal{T})$.

12. Show that $\otimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n) \subseteq \mathcal{B}(\Omega)$.

13. Show that $\mathcal{H} \subseteq \otimes_{n=1}^{+\infty}\mathcal{B}(\Omega_n)$.

14. Show that $\mathcal{B}(\Omega) = \otimes_{n=1}^{+\infty}\mathcal{B}(\Omega_n)$

Theorem 27 *Let $(\Omega_n, \mathcal{T}_n)_{n\geq 1}$ be a sequence of topological spaces with countable base. Then, the product space $(\Pi_{n=1}^{+\infty}\Omega_n, \odot_{n=1}^{+\infty}\mathcal{T}_n)$ has a countable base and:*

$$\mathcal{B}\left(\prod_{n=1}^{+\infty}\Omega_n\right) = \bigotimes_{n=1}^{+\infty}\mathcal{B}(\Omega_n)$$

Exercise 18.

1. Show that if $(\Omega, \mathcal{T})$ has a countable base and $n \geq 1$:

$$\mathcal{B}(\Omega^n) = \underbrace{\mathcal{B}(\Omega) \otimes \ldots \otimes \mathcal{B}(\Omega)}_{n}$$

2. Show that $\mathcal{B}(\bar{\mathbf{R}}^n) = \mathcal{B}(\bar{\mathbf{R}}) \otimes \ldots \otimes \mathcal{B}(\bar{\mathbf{R}})$.
3. Show that $\mathcal{B}(\mathbf{C}) = \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})$.

Definition 58 *We say that a metric space (E, d) is* **separable**, *if and only if there exists a finite or countable dense subset of E, i.e. a finite or countable subset A of E such that $E = \bar{A}$, where $\bar{A}$ is the closure of A in E.*

EXERCISE 19. Let (E, d) be a metric space.

1. Suppose that (E, d) is separable. Let $\mathcal{H} = \{B(x_n, \frac{1}{p}) : n, p \geq 1\}$, where $\{x_n : n \geq 1\}$ is a countable dense subset in E. Show that $\mathcal{H}$ is a countable base of the metric topological space $(E, \mathcal{T}_E^d)$.
2. Suppose conversely that $(E, \mathcal{T}_E^d)$ has a countable base $\mathcal{H}$. For all $V \in \mathcal{H}$ such that $V \neq \emptyset$, take $x_V \in V$. Show that the set $\{x_V \ : \ V \in \mathcal{H} \ , \ V \neq \emptyset\}$ is at most countable and dense in E.

3. For all $x, y, x', y' \in E$, show that:

$$|d(x,y) - d(x',y')| \leq d(x,x') + d(y,y')$$

4. Let $\mathcal{T}_{E\times E}$ be the product topology on $E \times E$. Show that the map $d : (E \times E, \mathcal{T}_{E\times E}) \to (\mathbf{R}^+, \mathcal{T}_{\mathbf{R}^+})$ is continuous.

5. Show that $d : (E \times E, \mathcal{B}(E \times E)) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

6. Show that $d : (E \times E, \mathcal{B}(E) \otimes \mathcal{B}(E)) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, whenever (E, d) is a separable metric space.

7. Let $(\Omega, \mathcal{F})$ be a measurable space and $f, g : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ be measurable maps. Show that $\Phi : (\Omega, \mathcal{F}) \to E \times E$ defined by $\Phi(\omega) = (f(\omega), g(\omega))$ is measurable with respect to the product σ-algebra $\mathcal{B}(E) \otimes \mathcal{B}(E)$.

8. Show that if (E, d) is separable, then $\Psi : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ defined by $\Psi(\omega) = d(f(\omega), g(\omega))$ is measurable.

9. Show that if (E, d) is separable then $\{f = g\} \in \mathcal{F}$.

10. Let $(E_n, d_n)_{n\geq 1}$ be a sequence of separable metric spaces. Show that the product space $\Pi_{n=1}^{+\infty} E_n$ is metrizable and separable.

Exercise 20. Prove the following theorem.

Theorem 28 *Let $(\Omega_i, \mathcal{F}_i)_{i\in I}$ be a family of measurable spaces and $(\Omega, \mathcal{F})$ be a measurable space. For all $i \in I$, let $f_i : \Omega \to \Omega_i$ be a map, and define $f : \Omega \to \Pi_{i\in I}\Omega_i$ by $f(\omega) = (f_i(\omega))_{i\in I}$. Then, the map:*

$$f : (\Omega, \mathcal{F}) \to \left(\prod_{i\in I} \Omega_i, \bigotimes_{i\in I} \mathcal{F}_i\right)$$

is measurable, if and only if each $f_i : (\Omega, \mathcal{F}) \to (\Omega_i, \mathcal{F}_i)$ is measurable.

Exercise 21.

1. Let $\phi, \psi : \mathbf{R}^2 \to \mathbf{R}$ with $\phi(x, y) = x + y$ and $\psi(x, y) = x.y$. Show that both ϕ and ψ are continuous.

2. Show that $\phi, \psi : (\mathbf{R}^2, \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})) \rightarrow (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are measurable.

3. Let $(\Omega, \mathcal{F})$ be a measurable space, and $f, g : (\Omega, \mathcal{F}) \rightarrow (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ be measurable maps. Using the previous results, show that $f+g$ and $f.g$ are measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\mathbf{R})$.