13. Regular Measure

In the following, $\mathbf{K}$ denotes $\mathbf{R}$ or $\mathbf{C}$.

Definition 99 *Let $(\Omega, \mathcal{F})$ be a measurable space. We say that a map $s : \Omega \to \mathbf{C}$ is a* **complex simple function** *on $(\Omega, \mathcal{F})$, if and only if it is of the form:*

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$

where $n \geq 1$, $\alpha_i \in \mathbf{C}$ and $A_i \in \mathcal{F}$ for all $i \in \mathbf{N}_n$. The set of all complex simple functions on $(\Omega, \mathcal{F})$ is denoted $S_{\mathbf{C}}(\Omega, \mathcal{F})$. The set of all $\mathbf{R}$-valued complex simple functions in $(\Omega, \mathcal{F})$ is denoted $S_{\mathbf{R}}(\Omega, \mathcal{F})$.

Recall that a simple function on $(\Omega, \mathcal{F})$, as defined in (40), is just a non-negative element of $S_{\mathbf{R}}(\Omega, \mathcal{F})$.

Exercise 1. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p \in [1, +\infty[$.

1. Suppose $s : \Omega \to \mathbf{C}$ is of the form
$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$
where $n \geq 1$, $\alpha_i \in \mathbf{C}$, $A_i \in \mathcal{F}$ and $\mu(A_i) < +\infty$ for all $i \in \mathbf{N}_n$. Show that $s \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{C}}(\Omega, \mathcal{F})$.

2. Show that any $s \in S_{\mathbf{C}}(\Omega, \mathcal{F})$ can be written as:
$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$
where $n \geq 1,\ \alpha_i \in \mathbf{C} \setminus \{0\},\ A_i \in \mathcal{F}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$.

3. Show that any $s \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{C}}(\Omega, \mathcal{F})$ is of the form:
$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$
where $n \geq 1$, $\alpha_i \in \mathbf{C}$, $A_i \in \mathcal{F}$ and $\mu(A_i) < +\infty$, for all $i \in \mathbf{N}_n$.

4. Show that $L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{C}}(\Omega, \mathcal{F}) = S_{\mathbf{C}}(\Omega, \mathcal{F})$.

Exercise 2. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p \in [1, +\infty[$. Let f be a non-negative element of $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

1. Show the existence of a sequence $(s_n)_{n\geq 1}$ of non-negative functions in $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{R}}(\Omega, \mathcal{F})$ such that $s_n \uparrow f$.

2. Show that:
$$\lim_{n\to+\infty} \int |s_n - f|^p d\mu = 0$$

3. Show that there exists $s \in L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{R}}(\Omega, \mathcal{F})$ such that $\|f - s\|_p \leq \epsilon$, for all $\epsilon > 0$.

4. Show that $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{K}}(\Omega, \mathcal{F})$ is dense in $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$.

Exercise 3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let f be a non-negative element of $L^{\infty}_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. For all $n \geq 1$, we define:

$$s_n \stackrel{\triangle}{=} \sum_{k=0}^{n2^n-1} \frac{k}{2^n} 1_{\{k/2^n \leq f < (k+1)/2^n\}} + n 1_{\{n \leq f\}}$$

1. Show that for all $n \geq 1$, s_n is a simple function.
2. Show there exists $n_0 \geq 1$ and $N \in \mathcal{F}$ with $\mu(N) = 0$, such that:
$$\forall \omega \in N^c\ ,\ 0 \leq f(\omega) < n_0$$
3. Show that for all $n \geq n_0$ and $\omega \in N^c$, we have:
$$0 \leq f(\omega) - s_n(\omega) < \frac{1}{2^n}$$
4. Conclude that:
$$\lim_{n \to +\infty} \|f - s_n\|_{\infty} = 0$$

5. Show the following:

Theorem 67 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p \in [1, +\infty]$. Then, $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{K}}(\Omega, \mathcal{F})$ is dense in $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$.*

Exercise 4. Let $(\Omega, \mathcal{T})$ be a metrizable topological space, and μ be a finite measure on $(\Omega, \mathcal{B}(\Omega))$. We define Σ as the set of all $B \in \mathcal{B}(\Omega)$ such that for all $\epsilon > 0$, there exist F closed and G open in Ω, with:

$$F \subseteq B \subseteq G \ , \ \mu(G \setminus F) \leq \epsilon$$

Given a metric d on $(\Omega, \mathcal{T})$ inducing the topology $\mathcal{T}$, we define:

$$d(x, A) \stackrel{\triangle}{=} \inf\{d(x, y) : \ y \in A\}$$

for all $A \subseteq \Omega$ and $x \in \Omega$.

1. Show that $x \to d(x, A)$ from Ω to $\bar{\mathbf{R}}$ is continuous for all $A \subseteq \Omega$.

2. Show that if F is closed in Ω, $x \in F$ is equivalent to $d(x, F) = 0$.

Exercise 5. Further to exercise (4), we assume that F is a closed subset of Ω. For all $n \geq 1$, we define:

$$G_n \stackrel{\triangle}{=} \{x \in \Omega : \ d(x, F) < \frac{1}{n}\}$$

1. Show that G_n is open for all $n \geq 1$.
2. Show that $G_n \downarrow F$.
3. Show that $F \in \Sigma$.
4. Was it important to assume that μ is finite?
5. Show that $\Omega \in \Sigma$.
6. Show that if $B \in \Sigma$, then $B^c \in \Sigma$.

Exercise 6. Further to exercise (5), let $(B_n)_{n\geq 1}$ be a sequence in Σ. Define $B = \cup_{n=1}^{+\infty} B_n$ and let $\epsilon > 0$.

1. Show that for all n, there is F_n closed and G_n open in Ω, with:
$$F_n \subseteq B_n \subseteq G_n \ , \ \mu(G_n \setminus F_n) \leq \frac{\epsilon}{2^n}$$
2. Show the existence of some $N \geq 1$ such that:
$$\mu\left(\left(\bigcup_{n=1}^{+\infty} F_n\right) \setminus \left(\bigcup_{n=1}^{N} F_n\right)\right) \leq \epsilon$$
3. Define $G = \cup_{n=1}^{+\infty} G_n$ and $F = \cup_{n=1}^{N} F_n$. Show that F is closed, G is open and $F \subseteq B \subseteq G$.
4. Show that:
$$G \setminus F \subseteq G \setminus \left(\bigcup_{n=1}^{+\infty} F_n\right) \uplus \left(\bigcup_{n=1}^{+\infty} F_n\right) \setminus F$$

5. Show that:
$$G \setminus \left(\bigcup_{n=1}^{+\infty} F_n \right) \subseteq \bigcup_{n=1}^{+\infty} G_n \setminus F_n$$
6. Show that $\mu(G \setminus F) \leq 2\epsilon$.
7. Show that Σ is a σ-algebra on Ω, and conclude that $\Sigma = \mathcal{B}(\Omega)$.

Theorem 68 *Let $(\Omega, \mathcal{T})$ be a metrizable topological space, and μ be a finite measure on $(\Omega, \mathcal{B}(\Omega))$. Then, for all $B \in \mathcal{B}(\Omega)$ and $\epsilon > 0$, there exist F closed and G open in Ω such that:*
$$F \subseteq B \subseteq G \ , \ \mu(G \setminus F) \leq \epsilon$$

Definition 100 *Let $(\Omega, \mathcal{T})$ be a topological space. We denote $C^b_{\mathbf{K}}(\Omega)$ the $\mathbf{K}$-vector space of all* **continuous, bounded** *maps $\phi : \Omega \to \mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$.*

Exercise 7. Let $(\Omega, \mathcal{T})$ be a metrizable topological space with some metric d. Let μ be a finite measure on $(\Omega, \mathcal{B}(\Omega))$ and F be a closed subset of Ω. For all $n \geq 1$, we define $\phi_n : \Omega \to \mathbf{R}$ by:

$$\forall x \in \Omega \ , \ \phi_n(x) \stackrel{\triangle}{=} 1 - 1 \wedge (nd(x, F))$$

1. Show that for all $p \in [1, +\infty]$, we have $C^b_{\mathbf{K}}(\Omega) \subseteq L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$.
2. Show that for all $n \geq 1$, $\phi_n \in C^b_{\mathbf{R}}(\Omega)$.
3. Show that $\phi_n \to 1_F$.
4. Show that for all $p \in [1, +\infty[$, we have:

$$\lim_{n \to +\infty} \int |\phi_n - 1_F|^p d\mu = 0$$

5. Show that for all $p \in [1, +\infty[$ and $\epsilon > 0$, there exists $\phi \in C^b_{\mathbf{R}}(\Omega)$ such that $\|\phi - 1_F\|_p \leq \epsilon$.

6. Let $\nu \in M^1(\Omega, \mathcal{B}(\Omega))$. Show that $C^b_{\mathbf{C}}(\Omega) \subseteq L^1_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega), \nu)$ and:

$$\nu(F) = \lim_{n \to +\infty} \int \phi_n d\nu$$

7. Prove the following:

Theorem 69 *Let $(\Omega, \mathcal{T})$ be a metrizable topological space and μ, ν be two complex measures on $(\Omega, \mathcal{B}(\Omega))$ such that:*

$$\forall \phi \in C^b_{\mathbf{R}}(\Omega) \ , \ \int \phi d\mu = \int \phi d\nu$$

Then $\mu = \nu$.

Exercise 8. Let $(\Omega, \mathcal{T})$ be a metrizable topological space and μ be a finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let $s \in S_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega))$ be a complex simple function:

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$

where $n \geq 1$, $\alpha_i \in \mathbf{C}$, $A_i \in \mathcal{B}(\Omega)$ for all $i \in \mathbf{N}_n$. Let $p \in [1, +\infty[$.

1. Show that given $\epsilon > 0$, for all $i \in \mathbf{N}_n$ there is a closed subset F_i of Ω such that $F_i \subseteq A_i$ and $\mu(A_i \setminus F_i) \leq \epsilon$. Let:

$$s' \stackrel{\triangle}{=} \sum_{i=1}^{n} \alpha_i 1_{F_i}$$

2. Show that:

$$\|s - s'\|_p \leq \left(\sum_{i=1}^{n} |\alpha_i| \right) \epsilon^{\frac{1}{p}}$$

3. Conclude that given $\epsilon > 0$, there exists $\phi \in C^b_{\mathbf{C}}(\Omega)$ such that:

$$\|\phi - s\|_p \leq \epsilon$$

4. Prove the following:

Theorem 70 *Let $(\Omega, \mathcal{T})$ be a metrizable topological space and μ be a finite measure on $(\Omega, \mathcal{B}(\Omega))$. Then, for all $p \in [1, +\infty[$, $C^b_{\mathbf{K}}(\Omega)$ is dense in $L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$.*

Definition 101 *A topological space $(\Omega, \mathcal{T})$ is said to be* **σ-compact** *if and only if, there exists a sequence $(K_n)_{n\geq 1}$ of compact subsets of Ω such that $K_n \uparrow \Omega$.*

EXERCISE 9. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space, with metric d. Let Ω' be open in Ω. For all $n \geq 1$, we define:

$$F_n \stackrel{\triangle}{=} \{x \in \Omega : \ d(x, (\Omega')^c) \geq 1/n\}$$

Let $(K_n)_{n\geq 1}$ be a sequence of compact subsets of Ω such that $K_n \uparrow \Omega$.

1. Show that for all $n \geq 1$, F_n is closed in Ω.

2. Show that $F_n \uparrow \Omega'$.

3. Show that $F_n \cap K_n \uparrow \Omega'$.

4. Show that $F_n \cap K_n$ is closed in K_n for all $n \geq 1$.

5. Show that $F_n \cap K_n$ is a compact subset of Ω' for all $n \geq 1$.

6. Prove the following:

Theorem 71 *Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Then, for all Ω' open subsets of Ω, the induced topological space $(\Omega', \mathcal{T}_{|\Omega'})$ is itself metrizable and σ-compact.*

Definition 102 *Let $(\Omega, \mathcal{T})$ be a topological space and μ be a measure on $(\Omega, \mathcal{B}(\Omega))$. We say that μ is* **locally finite***, if and only if, every $x \in \Omega$ has an open neighborhood of finite μ-measure, i.e.*

$$\forall x \in \Omega \ , \ \exists U \in \mathcal{T} \ , \ x \in U \ , \ \mu(U) < +\infty$$

Definition 103 *If μ is a measure on a Hausdorff topological space Ω: We say that μ is* **inner-regular**, *if and only if, for all $B \in \mathcal{B}(\Omega)$:*

$$\mu(B) = \sup\{\mu(K) : \ K \subseteq B \ , \ K \ \textit{compact}\}$$

We say that μ is **outer-regular**, *if and only if, for all $B \in \mathcal{B}(\Omega)$:*

$$\mu(B) = \inf\{\mu(G) : \ B \subseteq G \ , \ G \ \textit{open}\}$$

We say that μ is **regular** *if it is both inner and outer-regular.*

Exercise 10. Let $(\Omega, \mathcal{T})$ be a Hausdorff topological space, μ a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$, and K a compact subset of Ω.

1. Show the existence of open sets $V_1, \ldots, V_n$ with $\mu(V_i) < +\infty$ for all $i \in \mathbf{N}_n$ and $K \subseteq V_1 \cup \ldots \cup V_n$, where $n \geq 1$.
2. Conclude that $\mu(K) < +\infty$.

Exercise 11. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Let μ be a finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let $(K_n)_{n\geq 1}$ be a sequence of compact subsets of Ω such that $K_n \uparrow \Omega$. Let $B \in \mathcal{B}(\Omega)$. We define $\alpha = \sup\{\mu(K) : K \subseteq B , K \text{ compact}\}$.

1. Show that given $\epsilon > 0$, there exists F closed in Ω such that $F \subseteq B$ and $\mu(B \setminus F) \leq \epsilon$.
2. Show that $F \setminus (K_n \cap F) \downarrow \emptyset$.
3. Show that $K_n \cap F$ is closed in K_n.
4. Show that $K_n \cap F$ is compact.
5. Conclude that given $\epsilon > 0$, there exists K compact subset of Ω such that $K \subseteq F$ and $\mu(F \setminus K) \leq \epsilon$.
6. Show that $\mu(B) \leq \mu(K) + 2\epsilon$.
7. Show that $\mu(B) \leq \alpha$ and conclude that μ is inner-regular.

Exercise 12. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let $(K_n)_{n\geq 1}$ be a sequence of compact subsets of Ω such that $K_n \uparrow \Omega$. Let $B \in \mathcal{B}(\Omega)$, and $\alpha \in \mathbf{R}$ be such that $\alpha < \mu(B)$.

1. Show that $\mu(K_n \cap B) \uparrow \mu(B)$.
2. Show the existence of $K \subseteq \Omega$ compact, with $\alpha < \mu(K \cap B)$.
3. Let $\mu^K = \mu(K \cap \cdot\,)$. Show that μ^K is a finite measure, and conclude that $\mu^K(B) = \sup\{\mu^K(K^*) : K^* \subseteq B\ ,\ K^* \text{ compact}\}$.
4. Show the existence of a compact subset K^* of Ω, such that $K^* \subseteq B$ and $\alpha < \mu(K \cap K^*)$.
5. Show that K^* is closed in Ω.
6. Show that $K \cap K^*$ is closed in K.
7. Show that $K \cap K^*$ is compact.
8. Show that $\alpha < \sup\{\mu(K') : K' \subseteq B\ ,\ K' \text{ compact}\}$.

9. Show that $\mu(B) \leq \sup\{\mu(K') : K' \subseteq B \ , \ K' \text{ compact}\}$.

10. Conclude that μ is inner-regular.

EXERCISE 13. Let $(\Omega, \mathcal{T})$ be a metrizable topological space.

1. Show that $(\Omega, \mathcal{T})$ is separable if and only if it has a countable base.

2. Show that if $(\Omega, \mathcal{T})$ is compact, for all $n \geq 1$, Ω can be covered by a finite number of open balls with radius $1/n$.

3. Show that if $(\Omega, \mathcal{T})$ is compact, then it is separable.

EXERCISE 14. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space with metric d. Let $(K_n)_{n\geq 1}$ be a sequence of compact subsets of Ω such that $K_n \uparrow \Omega$.

1. For all $n \geq 1$, give a metric on K_n inducing the topology $\mathcal{T}_{|K_n}$.
2. Show that $(K_n, \mathcal{T}_{|K_n})$ is separable.
3. Let $(x_n^p)_{p \geq 1}$ be an at most countable sequence of $(K_n, \mathcal{T}_{|K_n})$, which is dense. Show that $(x_n^p)_{n,p \geq 1}$ is an at most countable dense family of $(\Omega, \mathcal{T})$, and conclude with the following:

Theorem 72 *Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Then, $(\Omega, \mathcal{T})$ is separable and has a countable base.*

Exercise 15. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let $\mathcal{H}$ be a countable base of $(\Omega, \mathcal{T})$. We define $\mathcal{H}' = \{V \in \mathcal{H} : \mu(V) < +\infty\}$.

1. Show that for all U open in Ω and $x \in U$, there is U_x open in Ω such that $x \in U_x \subseteq U$ and $\mu(U_x) < +\infty$.

2. Show the existence of $V_x \in \mathcal{H}$ such that $x \in V_x \subseteq U_x$.

3. Conclude that $\mathcal{H}'$ is a countable base of $(\Omega, \mathcal{T})$.

4. Show the existence of a sequence $(V_n)_{n\geq 1}$ of open sets in Ω with:

$$\Omega = \bigcup_{n=1}^{+\infty} V_n \ , \ \mu(V_n) < +\infty \ , \ \forall n \geq 1$$

Exercise 16. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let $(V_n)_{n\geq 1}$ a sequence of open subsets of Ω such that:

$$\Omega = \bigcup_{n=1}^{+\infty} V_n \ , \ \mu(V_n) < +\infty \ , \ \forall n \geq 1$$

Let $B \in \mathcal{B}(\Omega)$ and $\alpha = \inf\{\mu(G) : \ B \subseteq G \ , \ G \text{ open}\}$.

1. Given $\epsilon > 0$, show that there exists G_n open in Ω such that $B \subseteq G_n$ and $\mu^{V_n}(G_n \setminus B) \leq \epsilon/2^n$, where $\mu^{V_n} = \mu(V_n \cap \cdot\,)$.
2. Let $G = \cup_{n=1}^{+\infty}(V_n \cap G_n)$. Show that G is open in Ω, and $B \subseteq G$.
3. Show that $G \setminus B = \cup_{n=1}^{+\infty} V_n \cap (G_n \setminus B)$.
4. Show that $\mu(G) \leq \mu(B) + \epsilon$.
5. Show that $\alpha \leq \mu(B)$.
6. Conclude that is μ outer-regular.
7. Show the following:

Theorem 73 *Let μ be a locally finite measure on a metrizable and σ-compact topological space $(\Omega, \mathcal{T})$. Then, μ is regular, i.e.:*

$$\begin{aligned} \mu(B) &= \sup\{\mu(K) : \ K \subseteq B\ ,\ K \text{ compact}\} \\ &= \inf\{\mu(G) : \ B \subseteq G\ ,\ G \text{ open}\} \end{aligned}$$

for all $B \in \mathcal{B}(\Omega)$.

Exercise 17. Show the following:

Theorem 74 *Let* Ω *be an open subset of* $\mathbf{R}^n$, *where* $n \geq 1$. *Any locally finite measure on* $(\Omega, \mathcal{B}(\Omega))$ *is regular.*

Definition 104 *We call* **strongly σ-compact** *topological space, a topological space* $(\Omega, \mathcal{T})$, *for which there exists a sequence* $(V_n)_{n\geq 1}$ *of open sets with compact closure, such that* $V_n \uparrow \Omega$.

Definition 105 *We call* **locally compact** *topological space, a topological space* $(\Omega, \mathcal{T})$, *for which every* $x \in \Omega$ *has an open neighborhood with compact closure, i.e. such that:*

$$\forall x \in \Omega \ , \ \exists U \in \mathcal{T} : \ x \in U \ , \ \bar{U} \text{ is compact}$$

Exercise 18. Let $(\Omega, \mathcal{T})$ be a σ-compact and locally compact topological space. Let $(K_n)_{n\geq 1}$ be a sequence of compact subsets of Ω such that $K_n \uparrow \Omega$.

1. Show that for all $n \geq 1$, there are open sets $V_1^n, \ldots, V_{p_n}^n$, $p_n \geq 1$, such that $K_n \subseteq V_1^n \cup \ldots \cup V_{p_n}^n$ and $\bar{V}_1^n, \ldots, \bar{V}_{p_n}^n$ are compact subsets of Ω.

2. Define $W_n = V_1^n \cup \ldots \cup V_{p_n}^n$ and $V_n = \cup_{k=1}^n W_k$, for $n \geq 1$. Show that $(V_n)_{n \geq 1}$ is a sequence of open sets in Ω with $V_n \uparrow \Omega$.

3. Show that $\bar{W}_n = \bar{V}_1^n \cup \ldots \cup \bar{V}_{p_n}^n$ for all $n \geq 1$.

4. Show that $\bar{W}_n$ is compact for all $n \geq 1$.

5. Show that $\bar{V}_n$ is compact for all $n \geq 1$

6. Conclude with the following:

Theorem 75 *A topological space $(\Omega, \mathcal{T})$ is strongly σ-compact, if and only if it is σ-compact and locally compact.*

Exercise 19. Let $(\Omega, \mathcal{T})$ be a topological space and Ω' be a subset of Ω. Let $A \subseteq \Omega'$. We denote $\bar{A}^{\Omega'}$ the closure of A in the induced topological space $(\Omega', \mathcal{T}_{|\Omega'})$, and $\bar{A}$ its closure in Ω.

1. Show that $A \subseteq \Omega' \cap \bar{A}$.
2. Show that $\Omega' \cap \bar{A}$ is closed in Ω'.
3. Show that $\bar{A}^{\Omega'} \subseteq \Omega' \cap \bar{A}$.
4. Let $x \in \Omega' \cap \bar{A}$. Show that if $x \in U' \in \mathcal{T}_{|\Omega'}$, then $A \cap U' \neq \emptyset$.
5. Show that $\bar{A}^{\Omega'} = \Omega' \cap \bar{A}$.

Exercise 20. Let (Ω, d) be a metric space.

1. Show that for all $x \in \Omega$ and $\epsilon > 0$, we have:

$$\overline{B(x,\epsilon)} \subseteq \{y \in \Omega : \ d(x,y) \leq \epsilon\}$$

2. Take $\Omega = [0, 1/2[\cup\{1\}$. Show that $B(0,1) = [0, 1/2[$.

3. Show that $[0, 1/2[$ is closed in Ω.

4. Show that $\overline{B(0,1)} = [0, 1/2[$.

5. Conclude that $\overline{B(0,1)} \neq \{y \in \Omega : \ |y| \leq 1\} = \Omega$.

EXERCISE 21. Let (Ω, d) be a locally compact metric space. Let Ω' be an open subset of Ω. Let $x \in \Omega'$.

1. Show there exists U open with compact closure, such that $x \in U$.

2. Show the existence of $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U \cap \Omega'$.

3. Show that $\overline{B(x, \epsilon/2)} \subseteq \bar{U}$.

4. Show that $\overline{B(x, \epsilon/2)}$ is closed in $\bar{U}$.

5. Show that $\overline{B(x, \epsilon/2)}$ is a compact subset of Ω.

6. Show that $\overline{B(x,\epsilon/2)} \subseteq \Omega'$.

7. Let $U' = B(x,\epsilon/2) \cap \Omega' = B(x,\epsilon/2)$. Show $x \in U' \in \mathcal{T}_{|\Omega'}$, and:

$$\bar{U'}^{\Omega'} = \overline{B(x,\epsilon/2)}$$

8. Show that the induced topological space Ω' is locally compact.

9. Prove the following:

Theorem 76 *Let $(\Omega,\mathcal{T})$ be a metrizable and strongly σ-compact topological space. Then, for all Ω' open subsets of Ω, the induced topological space $(\Omega',\mathcal{T}_{|\Omega'})$ is itself metrizable and strongly σ-compact.*

Definition 106 *Let $(\Omega,\mathcal{T})$ be a topological space, and $\phi : \Omega \to \mathbf{C}$ be a map. We call* **support** *of ϕ, the closure of the set $\{\phi \neq 0\}$, i.e.:*

$$supp(\phi) \stackrel{\triangle}{=} \overline{\{\omega \in \Omega :\ \phi(\omega) \neq 0\}}$$

Definition 107 *Let $(\Omega, \mathcal{T})$ be a topological space. We denote $C^c_{\mathbf{K}}(\Omega)$ the $\mathbf{K}$-vector space of all* **continuous** *map with* **compact support** *$\phi : \Omega \to \mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$.*

EXERCISE 22. Let $(\Omega, \mathcal{T})$ be a topological space.

1. Show that $0 \in C^c_{\mathbf{K}}(\Omega)$.
2. Show that $C^c_{\mathbf{K}}(\Omega)$ is indeed a $\mathbf{K}$-vector space.
3. Show that $C^c_{\mathbf{K}}(\Omega) \subseteq C^b_{\mathbf{K}}(\Omega)$.

EXERCISE 23. let (Ω, d) be a locally compact metric space. let K be a compact subset of Ω, and G be open in Ω, with $K \subseteq G$.

1. Show the existence of open sets $V_1, \ldots, V_n$ such that:

$$K \subseteq V_1 \cup \ldots \cup V_n$$

   and $\bar{V}_1, \ldots, \bar{V}_n$ are compact subsets of Ω.

2. Show that $V = (V_1 \cup \ldots \cup V_n) \cap G$ is open in Ω, and $K \subseteq V \subseteq G$.

3. Show that $\bar{V} \subseteq \bar{V}_1 \cup \ldots \cup \bar{V}_n$.

4. Show that $\bar{V}$ is compact.

5. We assume $K \neq \emptyset$ and $V \neq \Omega$, and we define $\phi : \Omega \to \mathbf{R}$ by:

$$\forall x \in \Omega \ , \ \phi(x) \stackrel{\triangle}{=} \frac{d(x, V^c)}{d(x, V^c) + d(x, K)}$$

   Show that ϕ is well-defined and continuous.

6. Show that $\{\phi \neq 0\} = V$.

7. Show that $\phi \in C^c_{\mathbf{R}}(\Omega)$.

8. Show that $1_K \leq \phi \leq 1_G$.

9. Show that if $K = \emptyset$, there is $\phi \in C^c_{\mathbf{R}}(\Omega)$ with $1_K \leq \phi \leq 1_G$.

10. Show that if $V = \Omega$ then Ω is compact.

11. Show that if $V = \Omega$, there $\phi \in C^c_{\mathbf{R}}(\Omega)$ with $1_K \leq \phi \leq 1_G$.

Theorem 77 *Let $(\Omega, \mathcal{T})$ be a metrizable and locally compact topological space. Let K be a compact subset of Ω, and G be an open subset of Ω such that $K \subseteq G$. Then, there exists $\phi \in C^c_{\mathbf{R}}(\Omega)$, real-valued continuous map with compact support, such that:*

$$1_K \leq \phi \leq 1_G$$

Exercise 24. Let $(\Omega, \mathcal{T})$ be a metrizable and strongly σ-compact topological space. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let $B \in \mathcal{B}(\Omega)$ be such that $\mu(B) < +\infty$. Let $p \in [1, +\infty[$.

1. Show that $C^c_{\mathbf{K}}(\Omega) \subseteq L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$.

2. Let $\epsilon > 0$. Show the existence of K compact and G open, with:

$$K \subseteq B \subseteq G \ , \ \mu(G \setminus K) \leq \epsilon$$

3. Where did you use the fact that $\mu(B) < +\infty$?

4. Show the existence of $\phi \in C^c_{\mathbf{R}}(\Omega)$ with $1_K \leq \phi \leq 1_G$.

5. Show that:
$$\int |\phi - 1_B|^p d\mu \leq \mu(G \setminus K)$$

6. Conclude that for all $\epsilon > 0$, there exists $\phi \in C^c_{\mathbf{R}}(\Omega)$ such that:
$$\|\phi - 1_B\|_p \leq \epsilon$$

7. Let $s \in S_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega)) \cap L^p_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega), \mu)$. Show that for all $\epsilon > 0$, there exists $\phi \in C^c_{\mathbf{C}}(\Omega)$ such that $\|\phi - s\|_p \leq \epsilon$.

8. Prove the following:

Theorem 78 *Let $(\Omega, \mathcal{T})$ be a metrizable and strongly σ-compact topological space*[1]*. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. Then, for all $p \in [1, +\infty[$, the space $C^c_{\mathbf{K}}(\Omega)$ of $\mathbf{K}$-valued, continuous maps with compact support, is dense in $L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$.*

EXERCISE 25. Prove the following:

Theorem 79 *Let Ω be an open subset of $\mathbf{R}^n$, where $n \geq 1$. Then, for any locally finite measure μ on $(\Omega, \mathcal{B}(\Omega))$ and $p \in [1, +\infty[$, $C^c_{\mathbf{K}}(\Omega)$ is dense in $L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$.*

[1] i.e. a metrizable topological space for which there exists a sequence $(V_n)_{n\geq 1}$ of open sets with compact closure, such that $V_n \uparrow \Omega$.