15. Stieltjes Integration

Definition 112 *$b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation. The* **Stieltjes $\mathbf{L^1}$-spaces** *associated with b are defined as:*

$$L^1_{\mathbf{C}}(b) \stackrel{\triangle}{=} \left\{ f : \mathbf{R}^+ \to \mathbf{C} \text{ measurable}, \int |f| d|b| < +\infty \right\}$$

$$L^{1,loc}_{\mathbf{C}}(b) \stackrel{\triangle}{=} \left\{ f : \mathbf{R}^+ \to \mathbf{C} \text{ measurable}, \int_0^t |f| d|b| < +\infty, \forall t \in \mathbf{R}^+ \right\}$$

where the notation $|f|$ refers to the modulus map $t \to |f(t)|$.

Warning : In these tutorials, $\int_0^t \ldots$ refers to $\int_{[0,t]} \ldots$, i.e. the domain of integration is always $[0, t]$, not $]0, t]$, $[0, t[$, or $]0, t[$.

EXERCISE 1. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation.

1. Propose a definition for $L^1_{\mathbf{R}}(b)$ and $L^{1,\mathrm{loc}}_{\mathbf{R}}(b)$.
2. Is $L^1_{\mathbf{C}}(b)$ the same thing as $L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), d|b|)$?

3. Is it meaningful to speak of $L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$?

4. Show that $L^1_{\mathbf{C}}(b) = L^1_{\mathbf{C}}(|b|)$ and $L^{1,\text{loc}}_{\mathbf{C}}(b) = L^{1,\text{loc}}_{\mathbf{C}}(|b|)$.

5. Show that $L^1_{\mathbf{C}}(b) \subseteq L^{1,\text{loc}}_{\mathbf{C}}(b)$.

Exercise 2. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. For all $f \in L^{1,\text{loc}}_{\mathbf{C}}(a)$, we define $f.a : \mathbf{R}^+ \to \mathbf{C}$ as:

$$f.a(t) \stackrel{\triangle}{=} \int_0^t f da \ , \ \forall t \in \mathbf{R}^+$$

1. Explain why $f.a : \mathbf{R}^+ \to \mathbf{C}$ is a well-defined map.

2. Let $t \in \mathbf{R}^+$, $(t_n)_{n\geq 1}$ be a sequence in $\mathbf{R}^+$ with $t_n \downarrow\downarrow t$. Show:

$$\lim_{n\to+\infty} \int f 1_{[0,t_n]} da = \int f 1_{[0,t]} da$$

3. Show that $f.a$ is right-continuous.

4. Let $t \in \mathbf{R}^+$ and $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[0,t]$. Show:

$$\sum_{i=1}^{n} |f.a(t_i) - f.a(t_{i-1})| \leq \int_{]0,t]} |f| da$$

5. Show that $f.a$ is a map of finite variation with:

$$|f.a|(t) \leq \int_0^t |f| da \ , \ \forall t \in \mathbf{R}^+$$

Exercise 3. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f \in L^1_{\mathbf{C}}(a)$.

1. Show that $f.a$ is a right-continuous map of bounded variation.

2. Show $d(f.a)([0,t]) = \nu([0,t])$, for all $t \in \mathbf{R}^+$, where $\nu = \int f da$.

3. Prove the following:

Theorem 86 *Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f \in L^1_{\mathbf{C}}(a)$. The map $f.a : \mathbf{R}^+ \to \mathbf{C}$ defined by:*

$$f.a(t) \stackrel{\triangle}{=} \int_0^t f da \ , \ \forall t \in \mathbf{R}^+$$

is a right-continuous map of bounded variation, and its associated complex Stieltjes measure is given by $d(f.a) = \int f da$, i.e.

$$d(f.a)(B) = \int_B f da \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

Exercise 4. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f \in L^{1,\text{loc}}_{\mathbf{R}}(a)$, $f \geq 0$.

1. Show $f.a$ is right-continuous, non-decreasing with $f.a(0) \geq 0$.
2. Show $d(f.a)([0,t]) = \mu([0,t])$, for all $t \in \mathbf{R}^+$, where $\mu = \int f da$.

3. Prove that $d(f.a)([0,T]\cap\cdot\,) = \mu([0,T]\cap\cdot\,)$, for all $T\in\mathbf{R}^+$.

4. Prove with the following:

Theorem 87 *Let $a:\mathbf{R}^+\to\mathbf{R}^+$ be right-continuous, non-decreasing with $a(0)\geq 0$. Let $f\in L^{1,loc}_{\mathbf{R}}(a)$, $f\geq 0$. The map $f.a:\mathbf{R}^+\to\mathbf{R}^+$ defined by:*

$$f.a(t)\stackrel{\triangle}{=}\int_0^t f da\ ,\ \forall t\in\mathbf{R}^+$$

is right-continuous, non-decreasing with $(f.a)(0)\geq 0$, and its associated Stieltjes measure is given by $d(f.a)=\int f da$, i.e.

$$d(f.a)(B)=\int_B f da\ ,\ \forall B\in\mathcal{B}(\mathbf{R}^+)$$

Exercise 5. Let $a:\mathbf{R}^+\to\mathbf{R}^+$ be right-continuous, non-decreasing with $a(0)\geq 0$. Let $f\in L^{1,\mathrm{loc}}_{\mathbf{C}}(a)$ and $T\in\mathbf{R}^+$.

1. Show that $\int |f|1_{[0,T]}da = \int |f|da^{[0,T]} = \int |f|da^T$.

2. Show that $f1_{[0,T]} \in L^1_{\mathbf{C}}(a)$ and $f \in L^1_{\mathbf{C}}(a^T)$.

3. Show that $(f.a)^T = f.(a^T) = (f1_{[0,T]}).a$.

4. Show that for all $B \in \mathcal{B}(\mathbf{R}^+)$:

$$d(f.a)^T(B) = \int_B f da^T = \int_B f1_{[0,T]}da$$

5. Explain why it does not in general make sense to write:

$$d(f.a)^T = d(f.a)([0,T] \cap \cdot\,)$$

6. Show that for all $B \in \mathcal{B}(\mathbf{R}^+)$:

$$|d(f.a)^T|(B) = \int_B |f|1_{[0,T]}da$$

7. Show that $|d(f.a)^T| = d|f.a|([0,T] \cap \cdot\,)$

8. Show that for all $t \in \mathbf{R}^+$
$$|f.a|(t) = (|f|.a)(t) = \int_0^t |f|da$$

9. Show that $f.a$ is of bounded variation if and only if $f \in L^1_{\mathbf{C}}(a)$.

10. Show that $\Delta(f.a)(0) = f(0)\Delta a(0)$.

11. Let $t > 0$, $(t_n)_{n\geq 1}$ be a sequence in $\mathbf{R}^+$ with $t_n \uparrow\uparrow t$. Show:
$$\lim_{n\to+\infty} \int f1_{[0,t_n]}da = \int f1_{[0,t[}da$$

12. Show that $\Delta(f.a)(t) = f(t)\Delta a(t)$ for all $t \in \mathbf{R}^+$.

13. Show that if a is continuous with $a(0) = 0$, then $f.a$ is itself continuous with $(f.a)(0) = 0$.

14. Prove with the following:

Theorem 88 *Let* $a : \mathbf{R}^+ \to \mathbf{R}^+$ *be right-continuous, non-decreasing with* $a(0) \geq 0$. *Let* $f \in L^{1,loc}_{\mathbf{C}}(a)$. *The map* $f.a : \mathbf{R}^+ \to \mathbf{C}$ *defined by:*

$$f.a(t) \stackrel{\triangle}{=} \int_0^t f da \ , \ \forall t \in \mathbf{R}^+$$

is right-continuous of finite variation, and we have $|f.a| = |f|.a$, *i.e.*

$$|f.a|(t) = \int_0^t |f| da \ , \ \forall t \in \mathbf{R}^+$$

In particular, $f.a$ *is of bounded variation if and only if* $f \in L^1_{\mathbf{C}}(a)$. *Furthermore, we have* $\Delta(f.a) = f\Delta a$.

Exercise 6. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation.

1. Prove the equivalence between the following:

$$(i) \qquad d|b| << da$$

$$(ii) \qquad |db^T| << da \ , \ \forall T \in \mathbf{R}^+$$
$$(iii) \qquad db^T << da \ , \ \forall T \in \mathbf{R}^+$$

2. Does it make sense in general to write $db << da$?

Definition 113 *Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. We say that b is* **absolutely continuous** *with respect to a, and we write $b << a$, if and only if, one of the following holds:*

$$(i) \qquad d|b| << da$$
$$(ii) \qquad |db^T| << da \ , \ \forall T \in \mathbf{R}^+$$
$$(iii) \qquad db^T << da \ , \ \forall T \in \mathbf{R}^+$$

In other words, b is absolutely continuous w.r. to a, if and only if the Stieltjes measure associated with the total variation of b is absolutely continuous w.r. to the Stieltjes measure associated with a.

Exercise 7. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation, absolutely continuous w.r. to a, i.e. with $b << a$.

1. Show that for all $T \in \mathbf{R}^+$, there exits $f_T \in L^1_{\mathbf{C}}(a)$ such that:

$$db^T(B) = \int_B f_T da \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

2. Suppose that $T, T' \in \mathbf{R}^+$ and $T \leq T'$. Show that:

$$\int_B f_T da = \int_{B \cap [0,T]} f_{T'} da \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

3. Show that $f_T = f_{T'} 1_{[0,T]}$ da-a.s.

4. Show the existence of a sequence $(f_n)_{n \geq 1}$ in $L^1_{\mathbf{C}}(a)$, such that for all $1 \leq n \leq p$, $f_n = f_p 1_{[0,n]}$ and:

$$\forall n \geq 1 \ , \ db^n(B) = \int_B f_n da \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

5. We define $f : (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ by:

$$\forall t \in \mathbf{R}^+ \ , \ f(t) \stackrel{\triangle}{=} f_n(t) \text{ for any } n \geq 1 : \ t \in [0, n]$$

   Explain why f is unambiguously defined.

6. Show that for all $B \in \mathcal{B}(\mathbf{C})$, $\{f \in B\} = \cup_{n=1}^{+\infty} [0, n] \cap \{f_n \in B\}$.

7. Show that $f : (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is measurable.

8. Show that $f \in L^{1,\text{loc}}_{\mathbf{C}}(a)$ and that we have:

$$b(t) = \int_0^t f da \ , \ \forall t \in \mathbf{R}^+$$

9. Prove the following:

Theorem 89 *Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of finite variation. Then, b is absolutely continuous w.r. to a, i.e. $d|b| << da$, if and only if there exists $f \in L^{1,loc}_{\mathbf{C}}(a)$ such that $b = f.a$, i.e.*

$$b(t) = \int_0^t f da \ , \ \forall t \in \mathbf{R}^+$$

If b is $\mathbf{R}$-valued, we can assume that $f \in L^{1,loc}_{\mathbf{R}}(a)$.
If b is non-decreasing with $b(0) \geq 0$, we can assume that $f \geq 0$.

Exercise 8. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f, g \in L^{1,\mathrm{loc}}_{\mathbf{C}}(a)$ be such that $f.a = g.a$, i.e.:

$$\int_0^t f da = \int_0^t g da \ , \ \forall t \in \mathbf{R}^+$$

1. Show that for all $T \in \mathbf{R}^+$ and $B \in \mathcal{B}(\mathbf{R}^+)$:

$$d(f.a)^T(B) = \int_B f1_{[0,T]}da = \int_B g1_{[0,T]}da$$

2. Show that for all $T \in \mathbf{R}^+$, $f1_{[0,T]} = g1_{[0,T]}$ da-a.s.

3. Show that $f = g$ da-a.s.

Exercise 9. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation.

1. Show the existence of $h \in L^{1,\text{loc}}_{\mathbf{C}}(|b|)$ such that $b = h.|b|$.

2. Show that for all $B \in \mathcal{B}(\mathbf{R}^+)$ and $T \in \mathbf{R}^+$:

$$db^T(B) = \int_B hd|b|^T = \int_B h|db^T|$$

3. Show that $|h| = 1$ $|db^T|$-a.s. for all $T \in \mathbf{R}^+$.

4. Show that for all $T \in \mathbf{R}^+$, $d|b|([0,T] \cap \{|h| \neq 1\}) = 0$.

5. Show that $|h| = 1$ $d|b|$-a.s.

6. Prove the following:

Theorem 90 *Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. There exists $h \in L^{1,loc}_{\mathbf{C}}(|b|)$ such that $|h| = 1$ and $b = h.|b|$, i.e.*

$$b(t) = \int_0^t h d|b| \ , \ \forall t \in \mathbf{R}^+$$

Definition 114 *$b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation. For all $f \in L^1_{\mathbf{C}}(b)$, the* **Stieltjes integral** *of f with respect to b, is defined as:*

$$\int f db \stackrel{\triangle}{=} \int f h d|b|$$

where $h \in L^{1,loc}_{\mathbf{C}}(|b|)$ is such that $|h| = 1$ and $b = h.|b|$.

Warning : the notation $\int f db$ of definition (114) is controversial and potentially confusing: 'db' is not in general a complex measure on $\mathbf{R}^+$, unless b is of bounded variation.

EXERCISE 10. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation.

1. Show that if $f \in L^1_{\mathbf{C}}(b)$, then $\int fhd|b|$ is well-defined.
2. Explain why, given $f \in L^1_{\mathbf{C}}(b)$, $\int f db$ is unambiguously defined.
3. Show that if b is right-continuous, non-decreasing with $b(0) \geq 0$, definition (114) of $\int f db$ coincides with that of an integral w.r. to the Stieltjes measure db.
4. Show that if b is a right-continuous map of bounded variation, definition (114) of $\int f db$ coincides with that of an integral w.r. to the complex Stieltjes measure db.

EXERCISE 11. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of finite variation. For all $f \in L^{1,\text{loc}}_{\mathbf{C}}(b)$, we define $f.b : \mathbf{R}^+ \to \mathbf{C}$ as:

$$f.b(t) \stackrel{\triangle}{=} \int_0^t f db \stackrel{\triangle}{=} \int f 1_{[0,t]} db \ , \ \forall t \in \mathbf{R}^+$$

1. Explain why $f.b : \mathbf{R}^+ \to \mathbf{C}$ is a well-defined map.
2. If b is right-continuous, non-decreasing with $b(0) \geq 0$, show this definition of $f.b$ coincides with that of theorem (88).
3. Show $f.b = (fh).|b|$, where $h \in L^{1,\text{loc}}_{\mathbf{C}}(|b|)$, $|h| = 1$, $b = h.|b|$.
4. Show that $f.b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation, with $|f.b| = |f|.|b|$, i.e.

$$|f.b|(t) = \int_0^t |f|d|b| \ , \ \forall t \in \mathbf{R}^+$$

5. Show that $f.b$ is of bounded variation if and only if $f \in L^1_{\mathbf{C}}(b)$.
6. Show that $\Delta(f.b) = f\Delta b$.

7. Show that if b is continuous with $b(0) = 0$, then $f.b$ is itself continuous with $(f.b)(0) = 0$.

8. Prove the following:

Theorem 91 *Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. For all $f \in L^{1,loc}_{\mathbf{C}}(b)$, the map $f.b : \mathbf{R}^+ \to \mathbf{C}$ defined by:*

$$f.b(t) \stackrel{\triangle}{=} \int_0^t f db \ , \ \forall t \in \mathbf{R}^+$$

is right-continuous of finite variation, and we have $|f.b| = |f|.|b|$, i.e.

$$|f.b|(t) = \int_0^t |f|d|b| \ , \ \forall t \in \mathbf{R}^+$$

In particular, $f.b$ is of bounded variation if and only if $f \in L^1_{\mathbf{C}}(b)$. Furthermore, we have $\Delta(f.b) = f\Delta b$.

Exercise 12. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. Let $f \in L^{1,\text{loc}}_{\mathbf{C}}(b)$ and $T \in \mathbf{R}^+$.

1. Show that $\int |f| 1_{[0,T]} d|b| = \int |f| d|b|^{[0,T]} = \int |f| d|b^T|$.
2. Show that $f1_{[0,T]} \in L^1_{\mathbf{C}}(b)$ and $f \in L^1_{\mathbf{C}}(b^T)$.
3. Show $b^T = h.|b^T|$, where $h \in L^{1,\text{loc}}_{\mathbf{C}}(|b|)$, $|h| = 1$, $b = h.|b|$.
4. Show that $(f.b)^T = f.(b^T) = (f1_{[0,T]}).b$
5. Show that $d|f.b|(B) = \int_B |f| d|b|$ for all $B \in \mathcal{B}(\mathbf{R}^+)$.
6. Let $g : \mathbf{R}^+ \to \mathbf{C}$ be a measurable map. Show the equivalence:
$$g \in L^{1,\text{loc}}_{\mathbf{C}}(f.b) \;\Leftrightarrow\; gf \in L^{1,\text{loc}}_{\mathbf{C}}(b)$$
7. Show that $d(f.b)^T(B) = \int_B fh d|b^T|$ for all $B \in \mathcal{B}(\mathbf{R}^+)$.

8. Show that $db^T = \int hd|b^T|$ and conclude that:

$$d(f.b)^T(B) = \int_B f db^T \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

9. Let $g \in L^{1,\text{loc}}_{\mathbf{C}}(f.b)$. Show that $g \in L^1_{\mathbf{C}}((f.b)^T)$ and:

$$\int g1_{[0,t]}d(f.b)^T = \int gf1_{[0,t]}db^T \ , \ \forall t \in \mathbf{R}^+$$

10. Show that $g.\left((f.b)^T\right) = (gf).(b^T)$.

11. Show that $(g.(f.b))^T = ((gf).b)^T$.

12. Show that $g.(f.b) = (gf).b$

13. Prove the following:

Theorem 92 *Let* $b : \mathbf{R}^+ \to \mathbf{C}$ *be right-continuous of finite variation. For all* $f \in L^{1,loc}_{\mathbf{C}}(b)$ *and* $g : (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ *measurable map, we have the equivalence:*

$$g \in L^{1,loc}_{\mathbf{C}}(f.b) \;\Leftrightarrow\; gf \in L^{1,loc}_{\mathbf{C}}(b)$$

and when such condition is satisfied, $g.(f.b) = (fg).b$*, i.e.*

$$\int_0^t g d(f.b) = \int_0^t gf db \;,\; \forall t \in \mathbf{R}^+$$

Exercise 13. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. let $f, g \in L^{1,\mathrm{loc}}_{\mathbf{C}}(b)$ and $\alpha \in \mathbf{C}$. Show that $f + \alpha g \in L^{1,\mathrm{loc}}_{\mathbf{C}}(b)$, and:

$$(f + \alpha g).b = f.b + \alpha(g.b)$$

Exercise 14. Let $b, c : \mathbf{R}^+ \to \mathbf{C}$ be two right-continuous maps of finite variations. Let $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(b) \cap L^{1,\mathrm{loc}}_{\mathbf{C}}(c)$ and $\alpha \in \mathbf{C}$.

1. Show that for all $T \in \mathbf{R}^+$, $d(b + \alpha c)^T = db^T + \alpha dc^T$.
2. Show that for all $T \in \mathbf{R}^+$, $d|b + \alpha c|^T \leq d|b|^T + |\alpha|d|c|^T$.
3. Show that $d|b + \alpha c| \leq d|b| + |\alpha|d|c|$.
4. Show that $f \in L^{1,\text{loc}}_{\mathbf{C}}(b + \alpha c)$.
5. Show $d(f.(b + \alpha c))^T(B) = \int_B f d(b + \alpha c)^T$ for all $B \in \mathcal{B}(\mathbf{R}^+)$.
6. Show that $d(f.(b + \alpha c))^T = d(f.b)^T + \alpha d(f.c)^T$.
7. Show that $(f.(b + \alpha c))^T = (f.b)^T + \alpha(f.c)^T$
8. Show that $f.(b + \alpha c) = f.b + \alpha(f.c)$.

EXERCISE 15. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation.

1. Show that $d|b| \leq d|b_1| + d|b_2|$, where $b_1 = Re(b)$ and $b_2 = Im(b)$.

2. Show that $d|b_1| \leq d|b|$ and $d|b_2| \leq d|b|$.

3. Show that $f \in L^{1,\text{loc}}_{\mathbf{C}}(b)$, if and only if:

$$f \in L^{1,\text{loc}}_{\mathbf{C}}(|b_1|^+) \cap L^{1,\text{loc}}_{\mathbf{C}}(|b_1|^-) \cap L^{1,\text{loc}}_{\mathbf{C}}(|b_2|^+) \cap L^{1,\text{loc}}_{\mathbf{C}}(|b_2|^-)$$

4. Show that if $f \in L^{1,\text{loc}}_{\mathbf{C}}(b)$, for all $t \in \mathbf{R}^+$:

$$\int_0^t f db = \int_0^t f d|b_1|^+ - \int_0^t f d|b_1|^- + i \left(\int_0^t f d|b_2|^+ - \int_0^t f d|b_2|^- \right)$$

Exercise 16. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. We define $c : \mathbf{R}^+ \to [0, +\infty]$ as:

$$c(t) \stackrel{\triangle}{=} \inf\{s \in \mathbf{R}^+ : \ t < a(s)\} \ , \ \forall t \in \mathbf{R}^+$$

where it is understood that $\inf \emptyset = +\infty$. Let $s, t \in \mathbf{R}^+$.

1. Show that $t < a(s) \ \Rightarrow \ c(t) \leq s$.

2. Show that $c(t) < s \ \Rightarrow\ t < a(s)$.

3. Show that $c(t) \leq s \ \Rightarrow\ t < a(s+\epsilon)\ ,\ \forall\epsilon > 0$.

4. Show that $c(t) \leq s \ \Rightarrow\ t \leq a(s)$.

5. Show that $c(t) < +\infty \ \Leftrightarrow\ t < a(\infty)$.

6. Show that c is non-decreasing.

7. Show that if $t_0 \in [a(\infty), +\infty[$, c is right-continuous at t_0.

8. Suppose $t_0 \in [0, a(\infty)[$. Given $\epsilon > 0$, show the existence of $s \in \mathbf{R}^+$, such that $c(t_0) \leq s < c(t_0) + \epsilon$ and $t_0 < a(s)$.

9. Show that $t \in [t_0, a(s)[\ \Rightarrow\ c(t_0) \leq c(t) \leq c(t_0) + \epsilon$.

10. Show that c is right-continuous.

11. Show that if $a(\infty) = +\infty$, then c is a map $c : \mathbf{R}^+ \to \mathbf{R}^+$ which is right-continuous, non-decreasing with $c(0) \geq 0$.

12. We define $\bar{a}(s) = \inf\{t \in \mathbf{R}^+ : \ s < c(t)\}$ for all $s \in \mathbf{R}^+$. Show that for all $s, t \in \mathbf{R}^+$, $s < c(t) \ \Rightarrow \ a(s) \leq t$.

13. Show that $a \leq \bar{a}$.

14. Show that for all $s, t \in \mathbf{R}^+$ and $\epsilon > 0$:

$$a(s+\epsilon) \leq t \ \Rightarrow \ s < s+\epsilon \leq c(t)$$

15. Show that for all $s, t \in \mathbf{R}^+$ and $\epsilon > 0$, $a(s+\epsilon) \leq t \ \Rightarrow \ \bar{a}(s) \leq t$.

16. Show that $\bar{a} \leq a$ and conclude that:

$$a(s) = \inf\{t \in \mathbf{R}^+ : \ s < c(t)\} \ , \ \forall s \in \mathbf{R}^+$$

Exercise 17. Let $f : \mathbf{R}^+ \to \bar{\mathbf{R}}$ be a non-decreasing map. Let $\alpha \in \mathbf{R}$. We define:

$$x_0 \stackrel{\triangle}{=} \sup\{x \in \mathbf{R}^+ : \ f(x) \leq \alpha\}$$

1. Explain why $x_0 = -\infty$ if and only if $\{f \leq \alpha\} = \emptyset$.

2. Show that $x_0 = +\infty$ if and only if $\{f \leq \alpha\} = \mathbf{R}^+$.

3. We assume from now on that $x_0 \neq \pm\infty$. Show that $x_0 \in \mathbf{R}^+$.

4. Show that if $f(x_0) \leq \alpha$ then $\{f \leq \alpha\} = [0, x_0]$.

5. Show that if $\alpha < f(x_0)$ then $\{f \leq \alpha\} = [0, x_0[$.

6. Conclude that $f : (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

Exercise 18. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. We define $c : \mathbf{R}^+ \to [0, +\infty]$ as:

$$c(t) \stackrel{\triangle}{=} \inf\{s \in \mathbf{R}^+ : \ t < a(s)\} \ , \ \forall t \in \mathbf{R}^+$$

1. Let $f : \mathbf{R}^+ \to [0, +\infty]$ be non-negative and measurable. Show $(f \circ c)1_{\{c<+\infty\}}$ is well-defined, non-negative and measurable.

2. Let $t, u \in \mathbf{R}^+$, and ds be the Lebesgue measure on $\mathbf{R}^+$. Show:

$$\int_0^{a(t)} (1_{[0,u]} \circ c) 1_{\{c<+\infty\}} ds \leq \int 1_{[0,a(t\wedge u)]} 1_{\{c<+\infty\}} ds$$

3. Show that:

$$\int_0^{a(t)} (1_{[0,u]} \circ c) 1_{\{c<+\infty\}} ds \leq a(t \wedge u)$$

4. Show that:

$$a(t \wedge u) = \int_0^{a(t)} 1_{[0,a(u)[} ds = \int_0^{a(t)} 1_{[0,a(u)[} 1_{\{c<+\infty\}} ds$$

5. Show that:

$$a(t \wedge u) \leq \int_0^{a(t)} (1_{[0,u]} \circ c) 1_{\{c<+\infty\}} ds$$

6. Show that:

$$\int_0^t 1_{[0,u]} da = \int_0^{a(t)} (1_{[0,u]} \circ c) 1_{\{c<+\infty\}} ds$$

7. Define:

$$\mathcal{D}_t \stackrel{\triangle}{=} \left\{ B \in \mathcal{B}(\mathbf{R}^+) : \ \int_0^t 1_B da = \int_0^{a(t)} (1_B \circ c) 1_{\{c<+\infty\}} ds \right\}$$

Show that $\mathcal{D}_t$ is a Dynkin system on $\mathbf{R}^+$, and $\mathcal{D}_t = \mathcal{B}(\mathbf{R}^+)$.

8. Show that if $f : \mathbf{R}^+ \to [0, +\infty]$ is non-negative measurable:

$$\int_0^t f da = \int_0^{a(t)} (f \circ c) 1_{\{c<+\infty\}} ds \ , \ \forall t \in \mathbf{R}^+$$

9. Let $f : \mathbf{R}^+ \to \mathbf{C}$ be measurable. Show that $(f \circ c) 1_{\{c<+\infty\}}$ is itself well-defined and measurable.

10. Show that if $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(a)$, then for all $t \in \mathbf{R}^+$, we have:

$$(f \circ c)1_{\{c<+\infty\}}1_{[0,a(t)]} \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), ds)$$

and furthermore:

$$\int_0^t f da = \int_0^{a(t)} (f \circ c)1_{\{c<+\infty\}} ds$$

11. Show that we also have:

$$\int_0^t f da = \int (f \circ c)1_{[0,a(t)[} ds$$

12. Conclude with the following:

Theorem 93 *Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. We define $c : \mathbf{R}^+ \to [0, +\infty]$ as:*

$$c(t) \stackrel{\triangle}{=} \inf\{s \in \mathbf{R}^+ : \ t < a(s)\} \ , \ \forall t \in \mathbf{R}^+$$

Then, for all $f \in L^{1,loc}_{\mathbf{C}}(a)$, we have:

$$\int_0^t f da = \int_0^{a(t)} ((f \circ c) 1_{\{c<+\infty\}})(s) ds \ , \ \forall t \in \mathbf{R}^+$$