3. Stieltjes-Lebesgue Measure

Definition 12 *Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ and $\mu : \mathcal{A} \to [0, +\infty]$ be a map. We say that μ is* **finitely additive** *if and only if, given $n \geq 1$:*

$$A \in \mathcal{A}, A_i \in \mathcal{A}, A = \biguplus_{i=1}^{n} A_i \;\Rightarrow\; \mu(A) = \sum_{i=1}^{n} \mu(A_i)$$

We say that μ is **finitely sub-additive** *if and only if, given $n \geq 1$:*

$$A \in \mathcal{A}, A_i \in \mathcal{A}, A \subseteq \bigcup_{i=1}^{n} A_i \;\Rightarrow\; \mu(A) \leq \sum_{i=1}^{n} \mu(A_i)$$

Exercise 1. Let $\mathcal{S} \stackrel{\triangle}{=} \{]a, b] \;,\; a, b \in \mathbf{R}\}$ be the set of all intervals $]a, b]$, defined as $]a, b] = \{x \in \mathbf{R}, a < x \leq b\}$.

1. Show that $]a, b] \cap]c, d] =]a \vee c, b \wedge d]$
2. Show that $]a, b] \setminus]c, d] =]a, b \wedge c] \cup]a \vee d, b]$

3. Show that $c \leq d \;\Rightarrow\; b \wedge c \leq a \vee d$.

4. Show that $\mathcal{S}$ is a semi-ring on $\mathbf{R}$.

Exercise 2. Suppose $\mathcal{S}$ is a semi-ring in Ω and $\mu : \mathcal{S} \to [0, +\infty]$ is finitely additive. Show that μ can be extended to a finitely additive map $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$, with $\bar{\mu}_{|\mathcal{S}} = \mu$.

Exercise 3. Everything being as before, Let $A \in \mathcal{R}(\mathcal{S})$, $A_i \in \mathcal{R}(\mathcal{S})$, $A \subseteq \cup_{i=1}^n A_i$ where $n \geq 1$. Define $B_1 = A_1 \cap A$ and for $i = 1, \ldots, n-1$:

$$B_{i+1} \stackrel{\triangle}{=} (A_{i+1} \cap A) \setminus ((A_1 \cap A) \cup \ldots \cup (A_i \cap A))$$

1. Show that $B_1, \ldots, B_n$ are pairwise disjoint elements of $\mathcal{R}(\mathcal{S})$ such that $A = \uplus_{i=1}^n B_i$.

2. Show that for all $i = 1, \ldots, n$, we have $\bar{\mu}(B_i) \leq \bar{\mu}(A_i)$.

3. Show that $\bar{\mu}$ is finitely sub-additive.

4. Show that μ is finitely sub-additive.

EXERCISE 4. Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. Let $\mathcal{S}$ be the semi-ring on $\mathbf{R}$, $\mathcal{S} = \{]a, b] \ , \ a, b \in \mathbf{R}\}$. Define the map $\mu : \mathcal{S} \to [0, +\infty]$ by $\mu(\emptyset) = 0$, and:

$$\forall a \leq b \ , \ \mu(]a, b]) \stackrel{\triangle}{=} F(b) - F(a) \tag{1}$$

Let $a < b$ and $a_i < b_i$ for $i = 1, \ldots, n$ and $n \geq 1$, with :

$$]a, b] = \biguplus_{i=1}^{n}]a_i, b_i]$$

1. Show that there is $i_1 \in \{1, \ldots, n\}$ such that $a_{i_1} = a$.

2. Show that $]b_{i_1}, b] = \uplus_{i \in \{1,\ldots,n\} \setminus \{i_1\}}]a_i, b_i]$

3. Show the existence of a permutation $(i_1, \ldots, i_n)$ of $\{1, \ldots, n\}$ such that $a = a_{i_1} < b_{i_1} = a_{i_2} < \ldots < b_{i_n} = b$.

4. Show that μ is finitely additive and finitely sub-additive.

EXERCISE 5. μ being defined as before, suppose $a < b$ and $a_n < b_n$ for $n \geq 1$ with:

$$]a,b] = \biguplus_{n=1}^{+\infty}]a_n, b_n]$$

Given $N \geq 1$, let $(i_1, \ldots, i_N)$ be a permutation of $\{1, \ldots, N\}$ with:

$$a \leq a_{i_1} < b_{i_1} \leq a_{i_2} < \ldots < b_{i_N} \leq b$$

1. Show that $\sum_{k=1}^{N} F(b_{i_k}) - F(a_{i_k}) \leq F(b) - F(a)$.

2. Show that $\sum_{n=1}^{+\infty} \mu(]a_n, b_n]) \leq \mu(]a,b])$

3. Given $\epsilon > 0$, show that there is $\eta \in]0, b-a[$ such that:

$$0 \leq F(a+\eta) - F(a) \leq \epsilon$$

4. For $n \geq 1$, show that there is $\eta_n > 0$ such that:

$$0 \leq F(b_n + \eta_n) - F(b_n) \leq \frac{\epsilon}{2^n}$$

5. Show that $[a + \eta, b] \subseteq \cup_{n=1}^{+\infty}]a_n, b_n + \eta_n[$.

6. Explain why there exist $p \geq 1$ and integers $n_1, \ldots, n_p$ such that:

$$]a + \eta, b] \subseteq \cup_{k=1}^{p}]a_{n_k}, b_{n_k} + \eta_{n_k}]$$

7. Show that $F(b) - F(a) \leq 2\epsilon + \sum_{n=1}^{+\infty} F(b_n) - F(a_n)$

8. Show that $\mu : \mathcal{S} \to [0, +\infty]$ is a measure.

Definition 13 *A* **topology** *on* Ω *is a subset* $\mathcal{T}$ *of the power set* $\mathcal{P}(\Omega)$*, with the following properties:*

$$\begin{array}{rl} (i) & \Omega, \emptyset \in \mathcal{T} \\ (ii) & A, B \in \mathcal{T} \ \Rightarrow \ A \cap B \in \mathcal{T} \\ (iii) & A_i \in \mathcal{T}, \forall i \in I \ \Rightarrow \ \bigcup_{i \in I} A_i \in \mathcal{T} \end{array}$$

Property (iii) of definition (13) can be translated as: *for any family* $(A_i)_{i \in I}$ *of elements of* $\mathcal{T}$*, the union* $\cup_{i \in I} A_i$ *is still an element of* $\mathcal{T}$*.* Hence, a topology on Ω, is a set of subsets of Ω containing Ω and the empty set, which is closed under finite intersection and arbitrary union.

Definition 14 *A* **topological space** *is an ordered pair* $(\Omega, \mathcal{T})$*, where* Ω *is a set and* $\mathcal{T}$ *is a topology on* Ω*.*

Definition 15 *Let $(\Omega, \mathcal{T})$ be a topological space. We say that $A \subseteq \Omega$ is an* **open set** *in Ω, if and only if it is an element of the topology $\mathcal{T}$. We say that $A \subseteq \Omega$ is a* **closed set** *in Ω, if and only if its complement A^c is an open set in Ω.*

Definition 16 *Let $(\Omega, \mathcal{T})$ be a topological space. We define the* **Borel σ-algebra** *on Ω, denoted $\mathcal{B}(\Omega)$, as the σ-algebra on Ω, generated by the topology $\mathcal{T}$. In other words, $\mathcal{B}(\Omega) = \sigma(\mathcal{T})$*

Definition 17 *We define the* **usual topology** *on $\mathbf{R}$, denoted $\mathcal{T}_{\mathbf{R}}$, as the set of all $U \subseteq \mathbf{R}$ such that:*

$$\forall x \in U \ , \ \exists \epsilon > 0 \ , \ ]x - \epsilon, x + \epsilon[\subseteq U$$

Exercise 6. Show that $\mathcal{T}_{\mathbf{R}}$ is indeed a topology on $\mathbf{R}$.

Exercise 7. Consider the semi-ring $\mathcal{S} \stackrel{\triangle}{=} \{]a, b] \ , \ a, b \in \mathbf{R}\}$. Let $\mathcal{T}_{\mathbf{R}}$ be the usual topology on $\mathbf{R}$, and $\mathcal{B}(\mathbf{R})$ be the Borel σ-algebra on $\mathbf{R}$.

1. Let $a \leq b$. Show that $]a, b] = \cap_{n=1}^{+\infty}]a, b + 1/n[$.

2. Show that $\sigma(\mathcal{S}) \subseteq \mathcal{B}(\mathbf{R})$.

3. Let U be an open subset of $\mathbf{R}$. Show that for all $x \in U$, there exist $a_x, b_x \in \mathbf{Q}$ such that $x \in]a_x, b_x] \subseteq U$.

4. Show that $U = \cup_{x \in U}]a_x, b_x]$.

5. Show that the set $I \stackrel{\triangle}{=} \{]a_x, b_x]\ ,\ x \in U\}$ is countable.

6. Show that U can be written $U = \cup_{i \in I} A_i$ with $A_i \in \mathcal{S}$.

7. Show that $\sigma(\mathcal{S}) = \mathcal{B}(\mathbf{R})$.

Theorem 6 *Let $\mathcal{S}$ be the semi-ring $\mathcal{S} = \{]a, b]\ ,\ a, b \in \mathbf{R}\}$. Then, the Borel σ-algebra $\mathcal{B}(\mathbf{R})$ on $\mathbf{R}$, is generated by $\mathcal{S}$, i.e. $\mathcal{B}(\mathbf{R}) = \sigma(\mathcal{S})$.*

Definition 18 *A* **measurable space** *is an ordered pair $(\Omega, \mathcal{F})$ where Ω is a set and $\mathcal{F}$ is a σ-algebra on Ω.*

Definition 19 *A* **measure space** *is a triple* $(\Omega, \mathcal{F}, \mu)$ *where* $(\Omega, \mathcal{F})$ *is a measurable space and* $\mu : \mathcal{F} \to [0, +\infty]$ *is a measure on* $\mathcal{F}$*.*

Exercise 8. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $(A_n)_{n\geq 1}$ be a sequence of elements of $\mathcal{F}$ such that $A_n \subseteq A_{n+1}$ for all $n \geq 1$, and let $A = \cup_{n=1}^{+\infty} A_n$ (we write $A_n \uparrow A$). Define $B_1 = A_1$ and for all $n \geq 1$, $B_{n+1} = A_{n+1} \setminus A_n$.

1. Show that (B_n) is a sequence of pairwise disjoint elements of $\mathcal{F}$ such that $A = \uplus_{n=1}^{+\infty} B_n$.
2. Given $N \geq 1$ show that $A_N = \uplus_{n=1}^{N} B_n$.
3. Show that $\mu(A_N) \to \mu(A)$ as $N \to +\infty$
4. Show that $\mu(A_n) \leq \mu(A_{n+1})$ for all $n \geq 1$.

Theorem 7 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Then if $(A_n)_{n\geq 1}$ is a sequence of elements of $\mathcal{F}$, such that $A_n \uparrow A$, we have $\mu(A_n) \uparrow \mu(A)$*[1].

EXERCISE 9. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $(A_n)_{n\geq 1}$ be a sequence of elements of $\mathcal{F}$ such that $A_{n+1} \subseteq A_n$ for all $n \geq 1$, and let $A = \cap_{n=1}^{+\infty} A_n$ (we write $A_n \downarrow A$). We assume that $\mu(A_1) < +\infty$.

1. Define $B_n \stackrel{\triangle}{=} A_1 \setminus A_n$ and show that $B_n \in \mathcal{F}, B_n \uparrow A_1 \setminus A$.
2. Show that $\mu(B_n) \uparrow \mu(A_1 \setminus A)$
3. Show that $\mu(A_n) = \mu(A_1) - \mu(A_1 \setminus A_n)$
4. Show that $\mu(A) = \mu(A_1) - \mu(A_1 \setminus A)$
5. Why is $\mu(A_1) < +\infty$ important in deriving those equalities.
6. Show that $\mu(A_n) \to \mu(A)$ as $n \to +\infty$

[1] i.e. the sequence $(\mu(A_n))_{n\geq 1}$ is non-decreasing and converges to $\mu(A)$.

7. Show that $\mu(A_{n+1}) \leq \mu(A_n)$ for all $n \geq 1$.

Theorem 8 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Then if $(A_n)_{n\geq 1}$ is a sequence of elements of $\mathcal{F}$, such that $A_n \downarrow A$ and $\mu(A_1) < +\infty$, we have $\mu(A_n) \downarrow \mu(A)$.*

EXERCISE 10. Take $\Omega = \mathbf{R}$ and $\mathcal{F} = \mathcal{B}(\mathbf{R})$. Suppose μ is a measure on $\mathcal{B}(\mathbf{R})$ such that $\mu(]a,b]) = b - a$, for $a < b$. Take $A_n =]n, +\infty[$.

1. Show that $A_n \downarrow \emptyset$.
2. Show that $\mu(A_n) = +\infty$, for all $n \geq 1$.
3. Conclude that $\mu(A_n) \downarrow \mu(\emptyset)$ fails to be true.

EXERCISE 11. Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. Show the existence of a measure $\mu : \mathcal{B}(\mathbf{R}) \to [0, +\infty]$ such that:

$$\forall a, b \in \mathbf{R} \ , \ a \leq b \ , \ \mu(]a,b]) = F(b) - F(a) \tag{2}$$

Exercise 12. Let μ_1, μ_2 be two measures on $\mathcal{B}(\mathbf{R})$ with property (2). For $n \geq 1$, we define:

$$\mathcal{D}_n \stackrel{\triangle}{=} \{B \in \mathcal{B}(\mathbf{R})\ ,\ \mu_1(B \cap]-n,n]) = \mu_2(B \cap]-n,n])\}$$

1. Show that $\mathcal{D}_n$ is a Dynkin system on $\mathbf{R}$.
2. Explain why $\mu_1(]-n,n]) < +\infty$ and $\mu_2(]-n,n]) < +\infty$ is needed when proving 1.
3. Show that $\mathcal{S} \stackrel{\triangle}{=} \{]a,b]\ ,\ a,b \in \mathbf{R}\} \subseteq \mathcal{D}_n$.
4. Show that $\mathcal{B}(\mathbf{R}) \subseteq \mathcal{D}_n$.
5. Show that $\mu_1 = \mu_2$.
6. Prove the following theorem.

Theorem 9 *Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. There exists a unique measure $\mu : \mathcal{B}(\mathbf{R}) \to [0, +\infty]$ such that:*

$$\forall a, b \in \mathbf{R}\ ,\ a \leq b\ ,\ \mu(]a, b]) = F(b) - F(a)$$

Definition 20 *Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. We call* **Stieltjes measure** *on $\mathbf{R}$ associated with F, the unique measure on $\mathcal{B}(\mathbf{R})$, denoted dF, such that:*

$$\forall a, b \in \mathbf{R}\ ,\ a \leq b\ ,\ dF(]a, b]) = F(b) - F(a)$$

Definition 21 *We call* **Lebesgue measure** *on $\mathbf{R}$, the unique measure on $\mathcal{B}(\mathbf{R})$, denoted dx, such that:*

$$\forall a, b \in \mathbf{R}\ ,\ a \leq b\ ,\ dx(]a, b]) = b - a$$

Exercise 13. Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. Let $x_0 \in \mathbf{R}$.

1. Show that the limit $F(x_0-) = \lim_{x<x_0, x\to x_0} F(x)$ exists and is an element of $\mathbf{R}$.

2. Show that $\{x_0\} = \cap_{n=1}^{+\infty}]x_0 - 1/n, x_0]$.

3. Show that $\{x_0\} \in \mathcal{B}(\mathbf{R})$

4. Show that $dF(\{x_0\}) = F(x_0) - F(x_0-)$

EXERCISE 14.Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. Let $a \leq b$.

1. Show that $]a, b] \in \mathcal{B}(\mathbf{R})$ and $dF(]a, b]) = F(b) - F(a)$

2. Show that $[a, b] \in \mathcal{B}(\mathbf{R})$ and $dF([a, b]) = F(b) - F(a-)$

3. Show that $]a, b[\in \mathcal{B}(\mathbf{R})$ and $dF(]a, b[) = F(b-) - F(a)$

4. Show that $[a, b[\in \mathcal{B}(\mathbf{R})$ and $dF([a, b[) = F(b-) - F(a-)$

Exercise 15. Let $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. Let $\Omega' \subseteq \Omega$. Define:

$$\mathcal{A}_{|\Omega'} \stackrel{\triangle}{=} \{A \cap \Omega' \ , \ A \in \mathcal{A}\}$$

1. Show that if $\mathcal{A}$ is a topology on Ω, $\mathcal{A}_{|\Omega'}$ is a topology on Ω'.
2. Show that if $\mathcal{A}$ is a σ-algebra on Ω, $\mathcal{A}_{|\Omega'}$ is a σ-algebra on Ω'.

Definition 22 *Let Ω be a set, and $\Omega' \subseteq \Omega$. Let $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. We call* **trace** *of $\mathcal{A}$ on Ω', the subset $\mathcal{A}_{|\Omega'}$ of the power set $\mathcal{P}(\Omega')$ defined by:*

$$\mathcal{A}_{|\Omega'} \stackrel{\triangle}{=} \{A \cap \Omega' \, , \, A \in \mathcal{A}\}$$

Definition 23 *Let $(\Omega, \mathcal{T})$ be a topological space and $\Omega' \subseteq \Omega$. We call* **induced topology** *on Ω', denoted $\mathcal{T}_{|\Omega'}$, the topology on Ω' defined by:*

$$\mathcal{T}_{|\Omega'} \stackrel{\triangle}{=} \{A \cap \Omega' \ , \ A \in \mathcal{T}\}$$

In other words, the induced topology $\mathcal{T}_{|\Omega'}$ is the trace of $\mathcal{T}$ on Ω'.

Exercise 16. Let $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. Let $\Omega' \subseteq \Omega$, and $\mathcal{A}_{|\Omega'}$ be the trace of $\mathcal{A}$ on Ω'. Define:

$$\Gamma \stackrel{\triangle}{=} \{A \in \sigma(\mathcal{A}) \ , \ A \cap \Omega' \in \sigma(\mathcal{A}_{|\Omega'})\}$$

where $\sigma(\mathcal{A}_{|\Omega'})$ refers to the σ-algebra generated by $\mathcal{A}_{|\Omega'}$ on Ω'.

1. Explain why the notation $\sigma(\mathcal{A}_{|\Omega'})$ by itself is ambiguous.

2. Show that $\mathcal{A} \subseteq \Gamma$.

3. Show that Γ is a σ-algebra on Ω.

4. Show that $\sigma(\mathcal{A}_{|\Omega'}) = \sigma(\mathcal{A})_{|\Omega'}$

Theorem 10 *Let $\Omega' \subseteq \Omega$ and $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. Then, the trace on Ω' of the σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$, is equal to the σ-algebra on Ω' generated by the trace of $\mathcal{A}$ on Ω'. In other words, $\sigma(\mathcal{A})_{|\Omega'} = \sigma(\mathcal{A}_{|\Omega'})$.*

Exercise 17. Let $(\Omega, \mathcal{T})$ be a topological space and $\Omega' \subseteq \Omega$ with its induced topology $\mathcal{T}_{|\Omega'}$.

1. Show that $\mathcal{B}(\Omega)_{|\Omega'} = \mathcal{B}(\Omega')$.
2. Show that if $\Omega' \in \mathcal{B}(\Omega)$ then $\mathcal{B}(\Omega') \subseteq \mathcal{B}(\Omega)$.
3. Show that $\mathcal{B}(\mathbf{R}^+) = \{A \cap \mathbf{R}^+ \ , \ A \in \mathcal{B}(\mathbf{R})\}$.
4. Show that $\mathcal{B}(\mathbf{R}^+) \subseteq \mathcal{B}(\mathbf{R})$.

Exercise 18. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\Omega' \subseteq \Omega$

1. Show that $(\Omega', \mathcal{F}_{|\Omega'})$ is a measurable space.
2. If $\Omega' \in \mathcal{F}$, show that $\mathcal{F}_{|\Omega'} \subseteq \mathcal{F}$.
3. If $\Omega' \in \mathcal{F}$, show that $(\Omega', \mathcal{F}_{|\Omega'}, \mu_{|\Omega'})$ is a measure space, where $\mu_{|\Omega'}$ is defined as $\mu_{|\Omega'} = \mu_{|(\mathcal{F}_{|\Omega'})}$.

Exercise 19. Let $F : \mathbf{R}^+ \to \mathbf{R}$ be a right-continuous, non-decreasing map with $F(0) \geq 0$. Define:

$$\bar{F}(x) \stackrel{\triangle}{=} \begin{cases} 0 & \text{if} \quad x < 0 \\ F(x) & \text{if} \quad x \geq 0 \end{cases}$$

1. Show that $\bar{F} : \mathbf{R} \to \mathbf{R}$ is right-continuous and non-decreasing.
2. Show that $\mu : \mathcal{B}(\mathbf{R}^+) \to [0, +\infty]$ defined by $\mu = d\bar{F}_{|\mathcal{B}(\mathbf{R}^+)}$, is a measure on $\mathcal{B}(\mathbf{R}^+)$ with the properties:

$$\begin{array}{ll} (i) & \mu(\{0\}) = F(0) \\ (ii) & \forall 0 \leq a \leq b \ , \ \mu(]a, b]) = F(b) - F(a) \end{array}$$

Exercise 20. Define: $\mathcal{C} = \{\{0\}\} \cup \{]a, b]\ ,\ 0 \leq a \leq b\}$

1. Show that $\mathcal{C} \subseteq \mathcal{B}(\mathbf{R}^+)$
2. Let U be open in $\mathbf{R}^+$. Show that U is of the form:
$$U = \bigcup_{i \in I} (\mathbf{R}^+ \cap]a_i, b_i])$$
where I is a countable set and $a_i, b_i \in \mathbf{R}$ with $a_i \leq b_i$.
3. For all $i \in I$, show that $\mathbf{R}^+ \cap]a_i, b_i] \in \sigma(\mathcal{C})$.
4. Show that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbf{R}^+)$

Exercise 21. Let μ_1 and μ_2 be two measures on $\mathcal{B}(\mathbf{R}^+)$ with:

$$(i) \qquad \mu_1(\{0\}) = \mu_2(\{0\}) = F(0)$$
$$(ii) \qquad \mu_1(]a, b]) = \mu_2(]a, b]) = F(b) - F(a)$$

for all $0 \leq a \leq b$. For $n \geq 1$, we define:

$$\mathcal{D}_n = \{B \in \mathcal{B}(\mathbf{R}^+)\ ,\ \mu_1(B \cap [0,n]) = \mu_2(B \cap [0,n])\}$$

1. Show that $\mathcal{D}_n$ is a Dynkin system on $\mathbf{R}^+$ with $\mathcal{C} \subseteq \mathcal{D}_n$, where the set $\mathcal{C}$ is defined as in exercise (20).
2. Explain why $\mu_1([0,n]) < +\infty$ and $\mu_2([0,n]) < +\infty$ is important when proving 1.
3. Show that $\mu_1 = \mu_2$.
4. Prove the following theorem.

Theorem 11 *Let $F : \mathbf{R}^+ \to \mathbf{R}$ be a right-continuous, non-decreasing map with $F(0) \geq 0$. There exists a unique $\mu : \mathcal{B}(\mathbf{R}^+) \to [0,+\infty]$ measure on $\mathcal{B}(\mathbf{R}^+)$ such that:*

$$\begin{array}{ll} (i) & \mu(\{0\}) = F(0) \\ (ii) & \forall 0 \leq a \leq b\ ,\ \mu(]a,b]) = F(b) - F(a) \end{array}$$

Definition 24 *Let* $F : \mathbf{R}^+ \to \mathbf{R}$ *be a right-continuous, non-decreasing map with* $F(0) \geq 0$*. We call* **Stieltjes measure** *on* $\mathbf{R}^+$ *associated with* F*, the unique measure on* $\mathcal{B}(\mathbf{R}^+)$*, denoted* dF*, such that:*

$$(i) \qquad dF(\{0\}) = F(0)$$

$$(ii) \qquad \forall 0 \leq a \leq b \ , \ dF(]a, b]) = F(b) - F(a)$$